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**Deformations of N=2 super-conformal algebra**

**and supersymmetric two-component Camassa-Holm equation**

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**Abstract**

This paper is concerned with a link between central extensions of $N = 2$ superconformal algebra and a supersymmetric two-component generalization of the Camassa-Holm equation.

Deformations of superconformal algebra give rise to two compatible bracket structures. One of the bracket structures is derived from the central extension and admits a momentum operator which agrees with the Sobolev norm of a coadjoint orbit element. The momentum operator induces via Lenard relations a chain of conserved hamiltonians of the resulting supersymmetric Camassa-Holm hierarchy.
1 Introduction

The last few years have greatly enhanced our understanding of bihamiltonian structures. Bihamiltonian structure is frequently viewed as one of the key features of integrability and has recently been adopted as the basis for the classification program (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10])

In recent publications [14, 16], a study of deformations of bihamiltonian structures of hydrodynamic type led to a new class of bihamiltonian structure associated with a two-component generalization of Camassa–Holm equation. In this construction the second bracket component of the bihamiltonian structure contained a centerless Virasoro algebra

\[ \{ U(x), U(y) \}_2 = \delta'(x - y) (U(x) + U(y)) = 2U(x)\delta'(x - y) + U'(x)\delta(x - y) \]

(1)
coupled to a spin-one current via :

\[ \{ U(x), v(y) \}_2 = \delta'(x - y) (v(x) + (s - 1)v(y)) ; \quad s = 1 . \]

(2)

Two totally different hierarchies emerged from deformations of bihamiltonian system depending on whether a first or second bracket structure has being deformed by addition of an appropriate central element. Deformation of a second bracket led to a standard AKNS hierarchy while deformation of a first bracket gave rise to a new result: the 2-component Camassa-Holm hierarchy. The Hamiltonians of the bihamiltonian structure were shown to split into two chains, one chain of the positive order inducing the AKNS hierarchy and another of the negative order inducing the 2-component Camassa-Holm hierarchy [15].

Understanding of central extensions of a Poisson structure is crucial for analysis of deformations. Let us now consider a general Poisson bracket with a cocycle and link it to the question of compatibility so essential in the theory of integrable systems. Generally for a given Poisson bracket structure a cocycle of this bracket provides an additional compatible bracket structure. A trivial cocycle routinely agrees with the conventional first bracket structure. For example, shifting \( v \rightarrow v + c \) in (2) leads to a mixture of the second bracket with the first bracket of the type \( \{ U(x), v(y) \}_1 = \delta'(x - y) \). A non-trivial cocycle can serve as a deformation of first or, equally well, second structure. Algebraic properties of central elements ensure compatibility of all three Poisson bracket structures, meaning compatibility of trivial cocycle, non-trivial cocycle and the original bracket structure. In the setting of equation (2) the non-trivial central element is proportional to \( \delta''(x - y) \) and it can be added to either (2) or the first bracket \( \{ U(x), v(y) \}_1 = \delta'(x - y) \) (see equation (25)) without violating compatibility of the underlying bihamiltonian structure. The above arguments constitute an algebraic reasoning behind a trihamiltonian formalism introduced in [17].

In this paper we will apply the above observations to the N=2 superconformal algebra with a goal of building generalization of the 2-component Camassa-Holm model which is invariant under supersymmetric transformations. This approach will not be based on N=2 superspace formalism, like the one used in [18], but instead it will only employ the bihamiltonian method.

We will apply coadjoint orbit method to the \( N = 2 \) superconformal model in order to develop a formalism, which derives equations of motion of the N=2 supersymmetric Camassa-Holm model as Euler-Arnold equations. Misiolek [19], showed that the Camassa-Holm equation can be characterized as a geodesic flow on the Bott-Virasoro group. Guha and Olver
obtained a fermionic (but not supersymmetric) extension of Camassa-Holm equation as a geodesic flow. Here we will fill this gap by providing a Sobolev $H^{(1)}$ inner-product which is compatible with supersymmetric structure inherent in the $N = 2$ superconformal algebra and we will show that in this setting the supersymmetric Camassa-Holm equation follows as the Euler-Arnold equation.

Our formalism provides a direct guide on how to extend the conventional $L_2$ inner product to its deformed version given by the Sobolev $H^{(1)}$ inner-product. In all models we consider the basic algebra defining the model is inducing the second bracket structure. The compatible first bracket structure is then obtained by a central extension containing the deformation parameter in front of the non-trivial cocycle. In this setup the Sobolev $H^{(1)}$ inner-product is derived by demanding that the norm of the generic algebra element is a momentum of the first bracket structure. This norm is then used to produce a chain of Hamiltonians defining the hierarchy via Lenard relations.

This paper is organized as follows: In section 2 we review the construction of the two-component Camassa-Holm equation within the coadjoint orbit formalism emphasizing a general link between deformation of the first bracket and the Sobolev norm defining its momentum operator. In section 3 we construct bihamiltonian structure based on $N = 2$ superconformal algebra with deformation contained in the first bracket structure. The method developed in section 2 leads here to the momentum operator defining generalized two-component Camassa-Holm hierarchy invariant under the $N = 2$ supersymmetry transformations. In section 4 we derive the $N = 2$ supersymmetric Camassa-Holm equations as Euler-Arnold equations. In section 5 we show that the invariance under $N = 2$ supersymmetry transformations extends to the $N = 2$ Camassa-Holm model expressed in the hodographic variables when combined with redefinition of the supersymmetry transformations to ensure a closure of the $N = 2$ algebra in new variables. We conclude the paper by presenting a brief outlook in section 6.

2 Two-component Camassa-Holm model

The semi-direct product $G$ of centerless Virasoro (Witt) and spin-one current algebra is given by

$$ [(f, a), (g, b)] = \left( f g' - f' g, f b' - a' g \right) $$

The brackets (1) and (2) can then be derived using the Poisson bracket structure on a coadjoint orbit $G^*$ induced by algebra commutation relations via :

$$ \{F, G\}_P (\mu) = \left\langle \mu \left| \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right. \right\rangle, $$

where $F, G$ are real functions on $G^*$, $\mu$ is in $G^*$ and $\langle \cdot | \cdot \rangle$ is an inner-product.

Choosing $F$ as $F = \langle (U, v) | (f, a) \rangle$, $G$ as $G = \langle (U, v) | (g, b) \rangle$ and $\mu = (U, v)$ in definition
(4) yields:

\[
\left\{ \langle (U, v) | (f, a) \rangle, \langle (U, v) | (g, b) \rangle \right\}_P = \left\langle (U, v) \mid [(f, a), (g, b)] \right\rangle = \left\langle (U, v) \mid (fg' - f'g, fb' - a'g) \right\rangle
\]

If \( \langle \cdot | \cdot \rangle \) is taken to be the \( L_2 \) inner-product:

\[
\langle (U, v) | (f, a) \rangle_{L_2} = \int [Uf + va] \, dx
\]

then \( \{ \cdot, \cdot \}_P \) in relation (5) reproduces a structure given in (1) and (2) together with

\[
\{v(x), v(y)\}_2 = 0
\]

The Lie algebra (3) is compatible with a central extension:

\[
\left[(f, a, c_1, c_2), (g, b, c_1, c_2)\right] = (0, 0, \omega(a, b), \omega(f, g))
\]

using notation where \( c_1 \) and \( c_2 \) represent two additional central directions and

\[
\omega(a, b) = \frac{1}{2} \langle (0, a'), (0, b) \rangle = \frac{1}{2} \int a' b \, dx,
\]

\[
\omega(f, g) = \frac{1}{2} \langle (f', 0), (g, 0) \rangle = \frac{1}{2} \int (f' g + \beta^2 f'' g') \, dx.
\]

Pairing (6) extends in a straightforward way to incorporate two extra central elements and implies via relation (5) the following new bracket structure:

\[
\left\{ \langle (U, v) | (f, a) \rangle_{L_2}, \langle (U, v) | (g, b) \rangle_{L_2} \right\}_1 = \langle (U, v, c_1, c_2) | (0, 0, \omega(a, b), \omega(f, g)) \rangle_{L_2} = \frac{1}{2} \int [c_1 \left(f' g + \beta^2 f'' g'\right) + c_2 \left(a' b\right)] \, dx.
\]

The above bracket is compatible with a second bracket \( \{ \cdot, \cdot \}_2 \) for all values of \( c_1, c_2 \). For \( c_1 = c_2 = 1 \) the above construction admits the following local first Poisson structure:

\[
\{U(x), U(y)\}_1 = \frac{1}{2} \left(1 - \beta^2 \partial_x^2\right) \delta'(x-y)
\]

\[
\{v(x), U(y)\}_1 = 0
\]

\[
\{v(x), v(y)\}_1 = \frac{1}{2} \delta'(x - y).
\]

The existence of two compatible bracket structures signals integrability and implies presence of infinitely many conserved Hamiltonian structures. To construct these Hamiltonians we
first search for the object which acts as a gradient with respect to the first bracket $\{\cdot, \cdot\}_1$ from (10). We first investigate the $L_2$ norm of $(U, v)$:

$$\langle (U, v)| (U, v) \rangle_{L_2} = \int (U^2 + v^2) \, dx.$$  

For $h = (1/2)\langle (U, v)| (U, v) \rangle_{L_2}$ we find:

$$\{ h, U(x) \}_1 = \frac{1}{2} \left( 1 - \beta^2 \partial_x^2 \right) U'(x), \quad \{ h, v(x) \}_1 = \frac{1}{2} v'(x).$$

By redefining $h$ as follows

$$h \rightarrow \bar{h} = \frac{1}{2} \int \left( U \left( 1 - \beta^2 \partial_x^2 \right)^{-1} U + v^2 \right) \, dx$$

we now obtain desired relations:

$$\{ \bar{h}, U(x) \}_1 = \frac{1}{2} U''(x), \quad \{ h, v(x) \}_1 = \frac{1}{2} v'(x)$$

but for the price of introducing a non-locality in the definition of $\bar{h}$. To remove this non-locality we need to redefine variable $U$ to

$$U \quad \rightarrow \quad m = U - \beta^2 U''$$

Replacing $U$ by $m$ in $\bar{h}$ yields

$$\bar{h} \rightarrow H = \frac{1}{2} \int \left( m \left( 1 - \beta^2 \partial_x^2 \right)^{-1} m + v^2 \right) \, dx$$

$$= \frac{1}{2} \int (Um + v^2) \, dx = \frac{1}{2} \int \left( U^2 + \beta^2 (U')^2 + v^2 \right) \, dx$$

where we assumed periodic boundary conditions. The above Hamiltonian $H$ appears as a norm of $(U, v)$:

$$H = \frac{1}{2} \langle (U, v)| (U, v) \rangle$$  \hspace{1cm} (10)$$

with respect to the $H^{(1)}$ inner product:

$$\langle (U, v)| (f, a) \rangle = \int \left[ Uf + \beta^2 U' f' + va \right] \, dx$$  \hspace{1cm} (11)$$

When the bilinear form $\langle \cdot|\cdot \rangle$ in definition (4) is given by the above $H^{(1)}$ inner product then $\{\cdot, \cdot\}_P$ from relation (5) reproduces a familiar second bracket structure:

$$\{ m(x), m(y) \}_2 = 2m(x)\delta'(x - y) + m'(x)\delta(x - y)$$

$$\{ v(x), m(y) \}_2 = v(x)\delta'(x - y) + v'(x)\delta(x - y)$$

$$\{ v(x), v(y) \}_2 = 0$$  \hspace{1cm} (12)$$
Similarly replacing \( L_2 \) inner-product by \( H^1 \) inner product in relation (5) yields for the first bracket structure:

\[
\{m(x), m(y)\}_1 = \frac{1}{2} (1 - \beta^2 \partial_x^2) \delta'(x-y)
\]

\[
\{v(x), m(y)\}_1 = 0
\]

\[
\{v(x), v(y)\}_1 = \frac{1}{2} \delta'(x-y)
\]

In the remaining part of this section we will use the \( H^{(1)} \) inner product as defined in (11). Thus the canonical bracket structure \( \{\cdot, \cdot\}_P \) agrees with that given by (12).

The coadjoint action is obtained from

\[
\langle \text{ad}^\ast_{(f, a)}(g, b) | (h, c) \rangle = \langle (g, b) | [(f, a), (h, c)] \rangle = \int [g(fh' - f'h) + \beta^2 g'(fh' - f'h)' + b(fc' - a'h)] \, dx.
\]  

(14)

For brevity we introduce the following notation:

\[
\text{ad}^\ast_{(f, a)}(g, b) = (B_1, B_0).
\]

Then the left hand side of eq. (14) equals

\[
\int (B_1 h + \beta^2 B'_1 h' + B_0 c) \, dx = \int (h(1 - \beta^2 \partial_x^2)B_1 + B_0 c) \, dx.
\]

Comparing with the right hand side of eq. (14) we find

\[
(1 - \beta^2 \partial_x^2)B_1 = -f(1 - \beta^2 \partial_x^2)g' - 2f'(1 - \beta^2 \partial_x^2)g - ba'
\]

\[
B_0 = -(bf)'
\]

(15)

Defining Euler equations as

\[
\frac{d}{dt}(U, v) = -\text{ad}^\ast_{(U, v)}(U, v)
\]

(16)

we find from eq. (15) setting \( f = g = U, a = b = v \) that

\[
\frac{d}{dt}(U, v) = -\left( (1 - \beta^2 \partial_x^2)^{-1} \left[ -U(1 - \beta^2 \partial_x^2)U' - 2U'(1 - \beta^2 \partial_x^2)U - vv' \right], -(vU)' \right)
\]

(17)

from which we derive the 2-component Camassa-Holm equations:

\[
\frac{d}{dt}(1 - \beta^2 \partial_x^2)U = 3UU' - 2\beta^2 U'U'' - \beta^2 U''U''' + vv'
\]

\[
\frac{d}{dt}v = (vU)'
\]

(18)

With the Hamiltonian \( H \) from definition (10) and the bracket \( \{\cdot, \cdot\}_2 \) from eq. (12), the Euler equations are derived as Hamiltonian equations through:

\[
\frac{d}{dt}(m, v) = \left( \{m(x), H\}_2, \{v(x), H\}_2 \right) = \left( 2mU' + m'U + vv', (vU)' \right)
\]

(19)
Remarkably, one now can define a third local compatible bracket structure through the two lowest Poisson brackets:

\[ \{m(x), m(y)\}_i = P_i \delta(x-y), \quad i = 1, 2. \]

With the above in mind we can rewrite two compatible bracket structures which connects the first and second bracket structure.

\[ \begin{pmatrix} m(x), m(y) \\ v(x), m(y) \end{pmatrix}_i = \begin{pmatrix} 0 & \frac{1}{2} (1 - \beta^2 \partial^2) \partial \\ v^2 \partial + \partial v^2 & 0 \end{pmatrix} P_i \delta(x-y), \quad i = 1, 2. \]

Then

\[ P_2 = \begin{pmatrix} m \partial + \partial m & v^2 \partial + \partial v^2 \\ v^2 \partial + \partial v^2 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} \frac{1}{2} (1 - \beta^2 \partial^2) \partial & 0 \\ 0 & v^2 \partial + \partial v^2 \end{pmatrix}. \]

Remarkably, one now can define a third local compatible bracket structure through the formula \( P_0 = P_1 P_2^{-1} P_1 \). \( P_0 \) is explicitly given by

\[ P_0 = \begin{pmatrix} 0 & \frac{1}{2} (1 - \beta^2 \partial^2) \partial \\ \frac{1}{2} (1 - \beta^2 \partial^2) \partial & -m \partial - \partial m \end{pmatrix} \]

or

\[ \begin{align*}
\{m(x), m(y)\}_0 &= 0 \\
\{v(x), m(y)\}_0 &= \frac{1}{2} (1 - \beta^2 \partial^2) \delta'(x-y) \\
\{v(x), v(y)\}_0 &= -2m(x) \delta'(x-y) - m'(x) \delta(x-y)
\end{align*} \]

Defining the recurrence operator \( R = P_1 P_0^{-1} \) we can write recurrence relation \( P_2 = RP_1 \), which connects the first and second bracket structure.

It is easy to see that the Hamiltonian \( H \) from eq. \( (10) \) is a Casimir for the bracket structure \( \{\cdot, \cdot\} \) given by \( (21) \). We also note, that \( H \) acts as a gradient operator, \( \{X(x), H\}_1 = X'(x)/2 \) for \( X = m, v \), for the first bracket structure, \( \{\cdot, \cdot\}_1 \), from \( \|3\)\.

For new variables \( w_1, w_2 \) connected to \( m, v \) via

\[ m = \frac{1}{2} (w_1 + \beta w'_1), \quad w_2 = v^2 + \frac{w_1^2}{4} \]

the two lowest Poisson brackets:

\[ \begin{align*}
\{w_1(x), w_1(y)\}_1 &= 2 \delta'(x-y), \\
\{w_1(x), w_2(y)\}_1 &= w_1(x) \delta'(x-y) + w'_1(x) \delta(x-y), \\
\{w_2(x), w_2(y)\}_1 &= 2w_2(x) \delta'(x-y) + w'_2(x) \delta(x-y), \\
\{w_1(x), w_1(y)\}_0 &= \{w_2(x), w_2(y)\}_0 = 0, \\
\{w_1(x), w_2(y)\}_0 &= \delta'(x-y) - \beta \delta''(x-y).
\end{align*} \]

form for \( \beta = 0 \) a standard bihamiltonian structure which contains the second bracket structure from equations \( (11) \) and \( (2) \). For a non-zero value of the deformation parameter, \( \beta \neq 0 \), relations \( (25) \) describe a deformation of a bihamiltonian structure with the deformed first Poisson structure \( \|5\)\). This is that deformation of the bihamiltonian structure which gave rise to the two-component Camassa-Holm equation in \( \|6\,\|4\). In the next section we will focus on the similar deformation of the larger \( N = 2 \) superconformal Poisson algebra.
3 N=2 Superconformal Algebra

A starting point is a general second bracket structure:

\[
\begin{align*}
\{w_1(x), w_1(y)\}_2 &= 2\epsilon \delta'(x - y), \\
\{w_1(x), w_2(y)\}_2 &= w_1(x)\delta'(x - y) + w'_1(x)\delta(x - y) - 2\epsilon\delta''(x - y) + \kappa\delta'(x - y), \\
\{w_2(x), w_2(y)\}_2 &= 2w_2(x)\delta'(x - y) + w'_2(x)\delta(x - y), \\
\{w_1(x), \xi_1(y)\}_2 &= \xi_1(x)\delta(x - y), \\
\{w_1(x), \xi_2(y)\}_2 &= -\xi_2(x)\delta(x - y), \\
\{w_2(x), \xi_1(y)\}_2 &= 2\xi_1(x)\delta'(x - y) + \xi'_1(x)\delta(x - y), \\
\{w_2(x), \xi_2(y)\}_2 &= \xi_2(x)\delta'(x - y), \\
\{\xi_1(x), \xi_2(y)\}_2 &= -\frac{1}{4}w_2(x)\delta(x - y) - \frac{1}{4}w_1(x)\delta'(x - y) - \frac{1}{4}\epsilon\delta''(x - y) - \frac{1}{4}\kappa\delta'(x - y),
\end{align*}
\]

(26)

It distinguishes itself from the second bracket from (1), (2) by presence of additional fermionic variables \(\xi_i, i = 1, 2\) and central elements. The terms with \(\epsilon\) describe deformation of a second bracket which is compatible with Jacobi relations. The terms with a constant \(\kappa\) are trivial cocycle terms which can easily be absorbed by a shift \(w_1 \rightarrow w_1 - \kappa\).

The above second bracket is compatible with the first bracket structure:

\[
\begin{align*}
\{w_1(x), w_1(y)\}_1 &= 2\beta \delta'(x - y) \\
\{w_1(x), w_2(y)\}_1 &= \delta'(x - y) - \beta\delta''(x - y) \\
\{\xi_1(x), \xi_2(y)\}_1 &= -\frac{1}{4}\delta'(x - y) - \frac{\beta}{4}\delta''(x - y),
\end{align*}
\]

(27)

where terms with a constant \(\beta\) describe deformation of a first bracket which is compatible with the second bracket in (26).

Comparing the bracket structures in (26) and (27) we notice that they have identical central elements for \(\epsilon = \beta\) and \(\kappa = 1\).

Next, we define the quantity:

\[
H = \int dx \left[ w_1 (1 + \beta \partial_x)^{-1} w_2 - \beta \left( (1 + \beta \partial_x)^{-1} w_2 \right)^2 + 4\xi_1 (1 - \beta \partial_x)^{-1} \xi_2 \right],
\]

(28)

which acts as a gradient with respect to the first bracket \(\{\cdot, \cdot\}_1\) from (27), meaning that

\[
\{X(x), H\}_1 = X'(x) \quad \text{for} \quad X = w_i, \xi_i, \ i = 1, 2.
\]

The gradient operator \(H\) contains non-localities and as in section 2 we will absorb non-localities within new redefined variables. For this purpose we now introduce variables \(\phi_i, \eta_i, i = 1, 2\) such that they allow rewriting the first bracket (27) as an Heisenberg-like algebra:

\[
\begin{align*}
\{w_i(x), \phi_i(y)\}_1 &= 0, \quad \{w_i(x), \phi_j(y)\}_1 = \delta'(x - y), \ i \neq j, \ i, j = 1, 2 \\
\{\xi_2(x), \eta_1(y)\}_1 &= -\{\xi_1(x), \eta_2(y)\}_1 = \frac{1}{4}\delta'(x - y).
\end{align*}
\]

(29)
It is easy to verify that original variables \( w_i, \xi_i, i = 1, 2 \) must be related to \( \phi_i, \eta_i, i = 1, 2 \) through relations:

\[
\begin{align*}
   w_1 &= 2\beta \phi_2 + \phi_1 - \beta \phi'_1, \\
   w_2 &= \phi_2 + \beta \phi'_2, \\
   \xi_1 &= \eta_1 + \beta \eta'_1, \\
   \xi_2 &= \eta_2 - \beta \eta'_2.
\end{align*}
\]

Now, set

\[
u = \frac{1}{2} \phi_1, \quad v = \frac{1}{2} w_1
\]

and define \( m \) as a linear combination

\[
m = c_1 w_2 + c_2 w_1 + c_3 w'_1.
\]

The coefficients \( c_1, c_2, c_3 \) will be chosen in such a way as to ensure that \( m \) and \( v \) are decoupled, with respect to the first bracket structure, meaning that the first bracket of \( m \) with \( v \) vanishes. In view of equation (29) this requirement implies that \( m \) should not contain \( \phi_2 \) or in different words that \( m \) should be given by

\[
m = c_1 \left( w_2 - \frac{1}{2\beta} w_1 - \frac{1}{2} w'_1 \right)
\]

In the following we set \( c_1 = -\beta \) and accordingly use the definition:

\[
m = u - \beta^2 u_{xx} = \frac{1}{2} (w_1 + \beta w_{1x}) - \beta w_2; \quad \beta \phi_2 = v - u + \beta u_x
\]

Note, that the first bracket (27) reads in terms of \( m, v \) and \( \xi_i, \eta_i, i = 1, 2 \) as

\[
\begin{align*}
   \{m(x), m(y)\}_1 &= -\frac{1}{2} \beta \left( 1 - \beta^2 \partial_x^2 \right) \delta'(x - y) \\
   \{v(x), m(y)\}_1 &= 0 \\
   \{v(x), v(y)\}_1 &= \frac{1}{2} \beta \delta_x (x - y) \\
   \{\xi_i(x), \eta_j(y)\}_1 &= -\epsilon_{ij} \frac{1}{4} \delta'(x - y), \ i, j = 1, 2,
\end{align*}
\]

where \( \epsilon_{12} = -\epsilon_{21} = 1 \) and zero otherwise. Also, note that the first three equations agree with the first bracket structure in (13) up to an overall factor of \( \pm \beta \). The gradient operator (28) becomes in this notation

\[
H = \int [\phi_2 w_1 - \beta \phi_2^2 - 4\eta_2 \xi_1] \, dx = -\frac{1}{\beta} \int dx \left[ um - v^2 - 4\beta \xi_1 \eta_2 \right]
\]

and it holds that \( \{X(x), H\}_1 = X'(x) \) for \( X = m, v, \xi_i, i = 1, 2 \). It is important to point out that \( H \) is invariant under the following \( N = 2 \) supersymmetry transformation:

\[
\begin{align*}
   \delta \xi_1 &= \frac{\epsilon_1 - \epsilon_2}{2\beta} (m - (1 + \beta \partial_x)v), \quad \delta \xi_2 = \frac{\epsilon_1 + \epsilon_2}{2\beta} (-m + (1 + \beta \partial_x)v) \\
   \delta u &= \epsilon_1 (\eta_1 + \eta_2) + \epsilon_2 (\eta_1 - \eta_2), \quad \delta v = \epsilon_1 (\xi_1 + \xi_2) + \epsilon_2 (\xi_1 - \xi_2)
\end{align*}
\]
meaning that $\delta H = 0$. This invariance of $H$ will later lead to invariance of the whole integrable model to be defined below.

Also, below we will see that $H$ will agree with the Sobolev norm appearing in the coadjoint treatment of the model.

In terms of $m$ and $v$ and for $\kappa = \epsilon/\beta$, the second bracket \((26)\) takes the following form:

\[
\begin{align*}
\{m(x), m(y)\}_2 &= -\beta (2m(x)\delta_x(x-y) + m_x(x)\delta(x-y)) - \frac{1}{2}\epsilon \left(1 - \beta^2 \delta_x^2\right) \delta'(x-y) \\
\{v(x), m(y)\}_2 &= -\beta (v(x)\delta_x(x-y) + v_x(x)\delta(x-y)) \\
\{v(x), v(y)\}_2 &= \frac{1}{2}\epsilon \delta_x(x-y) \\
\{v(x), \xi_1(y)\}_2 &= \frac{1}{2}\xi_1(x)\delta(x-y), \\
\{v(x), \xi_2(y)\}_2 &= -\frac{1}{2}\xi_2(x)\delta(x-y), \\
\{m(x), \xi_1(y)\}_2 &= \frac{1}{2}\xi_1(x)\delta(x-y) - \frac{\beta}{2} (3\xi_1(x)\delta'(x-y) + \xi_1'(x)\delta(x-y)), \\
\{m(x), \xi_2(y)\}_2 &= -\frac{1}{2}\xi_2(x)\delta(x-y) - \frac{\beta}{2} (3\xi_2(x)\delta'(x-y) + \xi_2'(x)\delta(x-y)), \\
\{\xi_1(x), \xi_2(y)\}_2 &= \frac{m(x) - v(x)}{4\beta} \delta(x-y) - \frac{1}{4}(v'(x-y) + 2v(x)\delta'(x-y)) \\
&\quad - \frac{\epsilon}{\beta} \left(\frac{1}{4}\delta'(x-y) + \frac{\beta}{4}\delta''(x-y)\right),
\end{align*}
\]

We notice that a cocycle of the second bracket provides an additional compatible bracket structure. From now on we set $\epsilon$ to zero (while maintaining $\beta \neq 0$ in \((27)\)) and describe the bihamiltonian structure consisting of brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$, which gives rise to Lenard relations:

\[
\{w_i(x), H_{-j}\}_2 = j\{w_i(x), H_{-j-1}\}_1, \quad j = 1, 2, 3, \ldots, (34)
\]

The Lenard relations \((34)\) can be used to recursively build an hierarchy of Hamiltonians. Starting with the gradient operator $H_{-2} = H$ from \((31)\) we obtain from relations \((31)\) an expression for $H_{-1} = \int w_2(x) \, dx$, which happens to be the Casimir of the first Poisson bracket. Similarly, we also find an expression for Hamiltonian $H_{-3}$:

\[
H_{-3} = \frac{1}{2} \int [w_1\phi_1\phi_2 - \beta\phi_1\phi_2^2 - 4\beta\phi_2\eta_2\eta_1 - 4\phi_1 (2\eta_2\eta_1 + \beta\eta_2\eta_1')] \, dx,
\]

as an application of Lenard relations \((34)\) (for $\epsilon = 0$).

We will study the flow of the bihamiltonian hierarchy defined by:

\[
\begin{align*}
\frac{\partial w_i}{\partial t} &= \{w_i(x), H_{-3}\}_1 = \frac{1}{2}\{w_i(x), H_{-2}\}_2 \\
\frac{\partial \xi_i}{\partial t} &= \{\xi_i(x), H_{-3}\}_1 = \frac{1}{2}\{\xi_i(x), H_{-2}\}_2, \quad i = 1, 2.
\end{align*}
\]
In this way we obtain:

\[ w_{1t} = \frac{1}{2} (w_1 \phi_1 - 4 \beta \eta \eta_1)_x \]
\[ w_{2t} = \frac{1}{2} (2\phi_1 \phi_2 + \beta \phi_2^2 + \beta \phi_1 \phi_{2x} - 4(2\eta \eta_1 + \beta \eta \eta_1 x))_x \]
\[ \xi_{1t} = \frac{1}{2} (2\eta_1 \phi_1 + \beta \eta_1 x \phi_1 + \beta \eta_1 \phi_2)_x \]
\[ \xi_{2t} = \frac{1}{2} (\beta \eta_2 \phi_2 + 2\eta_2 \phi_1 - \beta (\eta_2 \phi_1)_x)_x \]

Equations of motion (36) become in terms of \( m, u \) and \( v \):

\[ m_t = 3uu_x - \beta u u_{xxx} - 2\beta^2 u_x u_{xx} - vv_x + (\beta \eta_2 \eta_1 + \beta \xi_2 \eta_1 + \beta \eta_2 \xi_1)_x \]
\[ v_t = (vu + \beta \eta_1 \eta_2)_x \]
\[ \xi_{1t} = \frac{1}{2} (3\eta_1 u + 2\beta \eta_1 x u + \eta_1 (v + \beta u_x))_x \]
\[ \xi_{2t} = \frac{1}{2} (3\eta_2 u - 2\beta \eta_2 x u + \eta_2 (v - \beta u_x))_x \]

Let

\[ \psi_1 = (\xi_1 - \xi_2) \sqrt{2}, \quad \psi_2 = (\xi_1 + \xi_2) \sqrt{2} \]

Then the fermionic part of (33) takes in terms of \( \psi_i, i = 1, 2 \), the following form:

\[ \{v(x), \psi_1(y)\}_2 = \frac{1}{2} \psi_2(x) \delta(x - y), \]
\[ \{v(x), \psi_2(y)\}_2 = \frac{1}{2} \psi_1(x) \delta(x - y), \]
\[ \{m(x), \psi_1(y)\}_2 = \frac{1}{2} \psi_2(x) \delta(x - y) - \frac{\beta}{2} (\psi'_1(x) \delta(x - y) + 3 \psi_1(x) \delta'(x - y)) \]
\[ \{m(x), \psi_2(y)\}_2 = \frac{1}{2} \psi_1(x) \delta(x - y) - \frac{\beta}{2} (\psi'_2(x) \delta(x - y) + 3 \psi_2(x) \delta'(x - y)) \]
\[ \{\psi_1(x), \psi_1(y)\}_2 = -\{\psi_2(x), \psi_2(y)\}_2 = \frac{1}{\beta} (m(x) - v(x)) \delta(x - y) \]
\[ \{\psi_1(x), \psi_2(y)\}_2 = -2v(x) \delta'(x - y) - v'(x) \delta(x - y) \]

One can remove the deformation parameter \( \beta \) from the Poisson brackets (33) and (38) by defining

\[ M = -\frac{1}{\beta} (m - v) \]
Thus the deformation parameter $\beta$ has been effectively moved from Poisson brackets to the redefined function $M$.

Introducing
\[ \gamma_1 = \eta_1 - \eta_2, \quad \gamma_2 = \eta_1 + \eta_2 \]
we can rewrite $\psi_1, \psi_2$ as
\[ \psi_1 = (\gamma_1 + \beta \gamma_2) \sqrt{2}, \quad \psi_2 = (\gamma_2 + \beta \gamma_1) \sqrt{2}. \]

In this notation the equations of motion (37) become
\[
\begin{align*}
mt &= 3uu_x - \beta^2 uu_{xxx} - 2\beta^2 u_x u_{xx} - \nu v_x + \frac{1}{2} \left( 3\beta \gamma_2 \gamma_1 + \beta^2 \gamma_2 \gamma_2 x - \beta^2 \gamma_1 \gamma_1 x \right) x \\
v_t &= \left( vu + \frac{1}{2} \beta \gamma_1 \gamma_2 \right) x \\
(\gamma_1 + \beta \gamma_2)_t &= \frac{1}{2} \left( 3u \gamma_1 + 2\beta u \gamma_2 x + v \gamma_1 + \beta u x \gamma_2 \right) x \\
(\gamma_2 + \beta \gamma_1)_t &= \frac{1}{2} \left( 3u \gamma_2 + 2\beta u \gamma_1 x + v \gamma_2 + \beta u x \gamma_1 \right) x. 
\end{align*}
\]

These equations of motions can be cast in a manifestly supersymmetric form
\[
\left[ \beta (D_1 D_2 \Phi) - \Phi \right]_t = -2\beta \Phi (D_1 D_2 \Phi) - \Phi^2 - \frac{\beta}{2} (D_2 \Phi)(D_1 \Phi),
\]
where the covariant derivatives $D_1, D_2$ are defined in terms of Grassmannian variables $\theta_1, \theta_2$ as
\[
D_1 = \frac{\partial}{\partial \theta_1} - \theta_1 \partial_x, \quad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \partial_x
\]
and satisfy
\[
D_1^2 = -\frac{\partial}{\partial x}, \quad D_2^2 = \frac{\partial}{\partial x}.
\]

The superfield
\[
\Phi = u + \theta_1 \gamma_2 + \theta_2 \gamma_1 + \theta_2 \theta_1 \frac{u - v}{\beta}
\]
transforms under supersymmetry as
\[
\delta \Phi = (\epsilon_1 Q_1 + \epsilon_2 Q_2) \Phi,
\]
where the generators $Q_1, Q_2$ are
\[
Q_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \partial_x, \quad Q_2 = \frac{\partial}{\partial \theta_2} - \theta_2 \partial_x.
\]

The supersymmetry transformation (45) reads in components:
\[
\begin{align*}
\delta \gamma_1 &= \epsilon_1 \frac{u - v}{\beta} + \epsilon_2 u_x \\
\delta \gamma_2 &= -\epsilon_1 u_x - \epsilon_2 \frac{u - v}{\beta} \\
\delta u &= \epsilon_1 \gamma_2 + \epsilon_2 \gamma_1 \\
\delta v &= \epsilon_1 (\gamma_2 + \beta \gamma_1 x) + \epsilon_2 (\gamma_1 + \beta \gamma_2 x).
\end{align*}
\]
It is easy to explicitly verify that the equations of motion (41) are invariant under supersymmetry transformations (37).

Equation (42) agrees with supersymmetric $N = 2, \alpha = 4$ Camassa-Holm equation obtained in [18] using the hereditary recursion operator defined on a superspace.

4 Coadjoint orbit method for $N = 2$ model

Here we will develop a formalism, which derives equations of motion (37) as Euler-Arnold equations. The algebra elements of $N = 2$ superconformal algebra $\mathcal{G}$ will be denoted as $(f, a, \alpha_1, \alpha_2)$, $(g, b, \beta_1, \beta_2)$ and so on. They satisfy the commutation relations:

\[
\begin{align*}
\{ f, a, \alpha_1, \alpha_2 \}, (g, b, \beta_1, \beta_2) &= \left( f g' - f' g - \alpha_2 \beta_1 - \alpha_1 \beta_2, \\
fb' - a' g - \alpha_2 \beta_1 - \alpha_1 \beta_2 - \beta \alpha_2 \beta_1 + \beta \alpha_2 \beta_1 + \beta \alpha_1 \beta_2 - \beta \alpha_1 \beta_2, \\
&\quad \frac{1}{2} a \beta_1 + \frac{1}{2} b \alpha_1 + \frac{1}{2} f \beta_1 - \frac{1}{2} \beta g \alpha_1 + f \beta_1 - \frac{1}{2} f' \beta_1 - g \alpha_1 + \frac{1}{2} g' \alpha_1, \\
&\quad \frac{1}{2} a \beta_2 - \frac{1}{2} b \alpha_2 + \frac{1}{2} f \beta_2 + \frac{1}{2} \beta g \alpha_2 + f \beta_2 - \frac{1}{2} f' \beta_2 - g \alpha_2 + \frac{1}{2} g' \alpha_2 \}
\end{align*}
\]

(48)

Its dual space will be denoted as $\mathcal{G}^*$. The typical element of $\mathcal{G}^*$ is denoted by $(u, v, \eta_1, \eta_2)$. The pairing between $\mathcal{G}$ and $\mathcal{G}^*$ will be provided by $H^{(1)}$ inner-product defined as:

\[
\langle (u, v, \eta_1, \eta_2), (f, a, \alpha_1, \alpha_2) \rangle = \int \left[ \frac{va}{\beta} - \frac{1}{\beta} (uf + \beta^2 u' f') + 2 (\eta_1 + \beta \eta_1') \alpha_2 - 2 (\eta_2 - \beta \eta_2') \alpha_1 \right] dx
\]

\[
= \int \left[ \frac{va}{\beta} - \frac{1}{\beta} fm + 2\xi_1 \alpha_2 - 2\xi_2 \alpha_1 \right] dx
\]

(49)

The coadjoint action is obtained from

\[
\langle \text{ad}^*_{(f, a, \alpha_1, \alpha_2)} (u, v, \eta_1, \eta_2), (g, b, \beta_1, \beta_2) \rangle = \langle (u, v, \eta_1, \eta_2), [(f, a, \alpha_1, \alpha_2), (g, b, \beta_1, \beta_2)] \rangle
\]

(50)

Denote $\text{ad}^*_{(f, a, \alpha_1, \alpha_2)} (u, v, \eta_1, \eta_2)$ by $(B, B_0, B_1, B_2)$. Then equation (50) yields

\[
(1 - \beta^2 \partial_v^2) B = -(fm)' - f' m + va' - (\xi_1 \alpha_2 + \xi_2 \alpha_1) + 2\beta (\xi_1 \alpha_2 - \xi_2 \alpha_1) + \beta (\xi_1 \alpha_2 - \xi_2 \alpha_1)'
\]

\[
B_0 = - (fv)' + \xi_1 \alpha_2 + \xi_2 \alpha_1
\]

\[
(1 + \beta \partial_v) B_1 = \frac{1}{2} \left( \frac{1}{\beta} \alpha_1 m + \alpha_1' v - (\alpha_1 v)' - \frac{1}{\beta} \alpha_1 v + \frac{1}{\beta} \xi_1 a - \frac{1}{\beta} \xi_1 f - \xi_1 f' - 2 (\xi_1 f)'ight)
\]

\[
(1 - \beta \partial_v) B_2 = \frac{1}{2} \left( \frac{1}{\beta} \alpha_2 m + \alpha_2' v + (\alpha_2 v)' - \frac{1}{\beta} \alpha_2 v + \frac{1}{\beta} \xi_2 a - \frac{1}{\beta} \xi_2 f + \xi_2 f' + 2 (\xi_2 f)'ight)
\]

Substituting $(u, v, \eta_1, \eta_2)$ for $(f, a, \alpha_1, \alpha_2)$ in the above equations we see that equations of motion (37) are reproduced as the Euler-Arnold equations:

\[
\frac{d}{dt} (u, v, \eta_1, \eta_2) = -\text{ad}^*_{(u, v, \eta_1, \eta_2)} (u, v, \eta_1, \eta_2)
\]

(51)
The above equation can also be derived using the Poisson bracket structure on a coadjoint orbit $G^*$ induced by algebra commutation relations via relation (4). For

$$F = \langle (u, v, \eta_1, \eta_2) | (f, a, \alpha_1, \alpha_2) \rangle, \quad G = \langle (u, v, \eta_1, \eta_2) | (g, b, \beta_1, \beta_2) \rangle, \quad \mu = (u, v, \eta_1, \eta_2)$$

we find from relation (4) the bracket structure which agrees with $\{\cdot, \cdot\}_2$ from (33).

After multiplying the right hand side of equation (51) by $\langle (f, a, \alpha_1, \alpha_2) \rangle$ we obtain

$$-\langle \text{ad}^*_{(u,v,\eta_1,\eta_2)}(u,v,\eta_1,\eta_2) \rangle (u,v,\eta_1,\eta_2) = \langle (u,v,\eta_1,\eta_2) | \text{ad}_{(f,a,\alpha_1,\alpha_2)}(u,v,\eta_1,\eta_2) \rangle$$

In view of definition (4) and due to the above relation the Euler-Arnold equation can be cast in a form:

$$\frac{d}{dt} F = \{F, H\}$$

for $F$ given by definition (52) and for $H$ being a half of the $H^1$-like norm of $(u,v,\eta_1,\eta_2)$ given explicitly by

$$H = \frac{1}{2} \langle (u,v,\eta_1,\eta_2) | (u,v,\eta_1,\eta_2) \rangle = \frac{1}{2} \int \left[ \frac{u^2}{\beta} - \frac{1}{\beta} \left( u^2 + \beta^2 (u')^2 \right) + 2 (\eta_1 + \beta \eta_1') \eta_2 - 2 (\eta_2 - \beta \eta_2') \eta_1 \right] dx$$

with $H_{-2}$ as defined in (31).

A central extension of algebra (48) is given by the following commutation relations:

$$\left[ (f,a,\alpha_1,\alpha_2,c,c_0,c_{12}) , (g,b,\beta_1,\beta_2,c,c_0,c_{12}) \right]_c = \frac{\epsilon}{2\beta^2} \int f (1 - \beta^2 \partial_x^2) g', \int ab', 2\beta \int (1 - \beta \partial_x) \alpha'_1 \alpha_1 - (1 + \beta \partial_x) \alpha'_2 \beta_2$$

Via extension of relation (4) the above central extension leads to the Poisson bracket structure

$$\{m(x), m(y)\}_c = -\frac{1}{2} \epsilon \left( 1 - \beta^2 \partial_x^2 \right) \delta(x - y)$$

$$\{v(x), m(y)\}_c = 0$$

$$\{v(x), v(y)\}_c = \frac{1}{2} \epsilon \delta(x - y)$$

$$\{\xi_1(x), \xi_2(y)\}_c = \frac{\epsilon}{\beta} \left( \frac{1}{4} \delta'(x - y) - \frac{\beta}{4} \delta''(x - y) \right)$$

which agrees both with the first bracket structure in (30) and the central extension of the second bracket structure in (26).

Note, that the quantity

$$H_{-1} = \frac{1}{\beta} \int (v - m) dx$$
commutes with all variables with respect to the first bracket structures (55) or (30). With respect to the second bracket structure (26) \( H \) acts as a gradient operator:

\[
\{ X(x), H_{-1} \}_2 = X'(x), \quad \text{for } X = m, v, \xi_1, \xi_2
\]  

(56)

We find from the above that

\[
\{ F, H_{-1} \}_2 = \left\langle (u', v', \eta_1', \eta_2') \mid (f, a, \alpha_1, \alpha_2) \right\rangle = \int \left[ \frac{v' a}{\beta} - \frac{1}{\beta} f \left( 1 - \beta^2 \partial_x^2 \right) u' + 2 \left( 1 + \beta \partial_x \right) \eta_1^2 \alpha_2 - 2 \left( 1 - \beta \partial_x \right) \eta_2^2 \alpha_1 \right] dx
\]

\[
= \frac{\epsilon}{2\beta} \left\langle (u, v, \eta_1, \eta_2) \left| \left( f, a, \alpha_1, \alpha_2, c, c_0, c_12 \right) , \left( u, v, \eta_1, \eta_2, c, c_0, c_12 \right) \right| c \right\rangle
\]

After setting \( \epsilon = \beta \) we obtain that the above equation equals

\[
\{ F, H_{-1} \}_2 = \frac{1}{2} \left\langle (u, v', \eta_1', \eta_2') \left| \left( f, a, \alpha_1, \alpha_2, c, c_0, c_12 \right) , \frac{\delta}{\delta \mu} |(u, v, \eta_1, \eta_2)|^2 \right| c \right\rangle = \{ F, H \}
\]

for \( H \) from (53), \( F \) from (52) and with \( \mu = (u, v, \eta_1, \eta_2) \).

Consider the first bracket structure given by (55) with \( \epsilon = \beta \) or (30), then the Hamiltonian \( H \) from (53) serves as a momentum with respect to this bracket structure according to:

\[
\{ X(x), H \}_1 = \{ X(x), H_{-1} \}_2 = X'(x), \quad \text{for } X = m, v, \xi_1, \xi_2
\]  

(57)

as follows from (56) and the Lenard relation proved above. This shows that the \( H^1 \)-like norm is fully fixed by the requirement that the norm \( H \) serves as a momentum with respect to the first bracket structure defined by a central extension. Furthermore applying the Lenard argument we find

\[
\{ X(x), H_{-1} \}_1 = \{ X(x), H_0 \}_2
\]

with \( H_0 \) being a constant by a power of fields counting argument. Thus, we have proved that \( H_{-1} \) is a Casimir of the first bracket structure.

5 Hodographic transformation and Supersymmetry

It is customary to study Camassa-Holm equation in terms of hodographic variables. The basic question which we will address in this section is how the change of variables to the hodographic variables affects supersymmetry described above. The difficulty one encounters here is the fact that the hodographic variables do not commute with the regular supersymmetry transformations. This obstacle makes it necessary to define a new set of extended supersymmetry transformations obeying \( N = 2 \) supersymmetric algebra and commuting with hodographic variables. In the next step we rewrite all equations of motion (41) as continuity equations in terms of the hodographic variables which leads to the proof of their invariance under the \( N = 2 \) supersymmetry transformations.
Here, we will study a hodographic change of variables based on a continuity equation, \( v_t = (vu + \frac{1}{2} \beta \gamma_1 \gamma_2)_x \), the second of equations in (41). It follows from this continuity equation that a one-form:

\[
\omega = v \, dx + \left( vu + \frac{1}{2} \beta \gamma_1 \gamma_2 \right) \, dt
\]

is closed, meaning that \( d\omega = 0 \) with \( d = dx \frac{\partial}{\partial x} + dt \frac{\partial}{\partial t} \). Thus, it is possible to rewrite \( \omega \) as

\[
\omega = dy
\]

for some scalar function \( y \). Let \( s \) be such that \( ds = dt \) and choose now a new coordinate system with coordinates \( y \) and \( s \). Then in the new coordinate system an expression for the exterior derivative \( d \) becomes

\[
d = dy \frac{\partial}{\partial y} + ds \frac{\partial}{\partial s} = \left( v \, dx + \left( vu + \frac{1}{2} \beta \gamma_1 \gamma_2 \right) \, dt \right) \frac{\partial}{\partial y} + dt \frac{\partial}{\partial s}.
\]

Comparing with expression \( d = dx \frac{\partial}{\partial x} + dt \frac{\partial}{\partial t} \) it follows that

\[
\frac{\partial}{\partial x} = v \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \left( vu + \frac{1}{2} \beta \gamma_1 \gamma_2 \right) \frac{\partial}{\partial y} \tag{58}
\]

The fact that \( d^2 = 0 \) ensures commutativity

\[
\left[ \frac{\partial}{\partial y}, \frac{\partial}{\partial s} \right] = 0
\]

of derivatives with respect to the new variables.

From the above construction we get

\[
dx = - \left( u + \frac{1}{2} \beta \gamma_1 \gamma_2 / v \right) \, ds + \frac{1}{v} \, dy.
\]

Thus

\[
\left( \frac{1}{v} \right)_s = - \left( u + \frac{1}{2} \beta \gamma_1 \gamma_2 / v \right)_y. \tag{59}
\]

From (58) we find that

\[
\frac{\partial x}{\partial y} = \frac{1}{v}, \quad \frac{\partial x}{\partial s} = - \left( u + \frac{1}{2} \beta \gamma_1 \gamma_2 / v \right).
\]

The first of equations (41) can also be rewritten as a continuity equation. Define namely

\[
P = \frac{m}{v^2} + \beta \frac{\psi_1 \psi_2}{2v^3} \tag{60}
\]

and

\[
\rho = - v - \beta \, P \gamma_1 \gamma_2 + \frac{1}{2v} \left( 2 \beta \gamma_2 \gamma_1 + \beta^2 \gamma_2 \gamma_2 x - \beta^2 \gamma_1 \gamma_1 x \right). \tag{61}
\]
Then the first of equations (41) amounts to:

$$P_s = \rho_y$$ \hspace{1cm} (62)

Thus, there exists a potential $f$ such that

$$P = fy, \quad \rho = fs$$

The last two equations of (41) represent fermionic continuity equations and can be used to define fermionic coordinates $\tau_i, i = 1, 2$ through e.g.

$$d\tau_1 = \frac{1}{\sqrt{2}} \psi_1 dx + \frac{1}{2} (3u\gamma_1 + 2\beta u\gamma_2 + v\gamma_1 + \beta u_x \gamma_2) dt.$$

The coordinate $\tau_1$ satisfies

$$\frac{\partial \tau_1}{\partial x} = \frac{1}{\sqrt{2}} \psi_1, \quad \frac{\partial \tau_1}{\partial t} = \frac{1}{2} (3u\gamma_1 + 2\beta u\gamma_2 + v\gamma_1 + \beta u_x \gamma_2).$$

In hodographic variables the fermionic conservation laws become:

$$\left(\frac{\psi_1}{\sqrt{2}v}\right)_s = \frac{1}{2} \left( u\gamma_1 + v\gamma_1 + \beta u_x \gamma_2 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_1}{\sqrt{2}} \right)_y$$

$$\left(\frac{\psi_2}{\sqrt{2}v}\right)_s = \frac{1}{2} \left( u\gamma_2 + v\gamma_2 + \beta u_x \gamma_1 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_2}{\sqrt{2}} \right)_y. \hspace{1cm} (63)$$

Thus, after the hodographic transformation all equations of motion of the model are described by continuity equations (59), (62), (63), which all have a common form:

$$G_s = Hy.$$}

Now, we will investigate invariance of such continuity equations under the supersymmetry transformations (47). Consider the $\epsilon_1$ part of the supersymmetry transformations (47):

$$\frac{1}{\sqrt{2}} \delta_1 \psi_1 = \epsilon_1 \frac{m - v}{\beta}, \quad \frac{1}{\sqrt{2}} \delta_1 \psi_2 = -\epsilon_1 v_x$$

$$\delta_1 v = \epsilon_1 \frac{1}{\sqrt{2}} \psi_2, \quad \delta_1 m = \epsilon_1 \left( \frac{1}{\sqrt{2}} \psi_2 - \frac{\beta}{\sqrt{2}} \psi_1 \right).$$

For an arbitrary function $F(y,s)$ it holds on basis of (58) that

$$\delta_1 \frac{\partial}{\partial y} F = \frac{\partial}{\partial y} \delta_1 F - \epsilon_1 \frac{\gamma_2 + \beta \gamma_1 x}{v} \frac{\partial}{\partial y} F$$

$$\delta_1 \frac{\partial}{\partial s} F = \frac{\partial}{\partial s} \delta_1 F - \frac{\epsilon_1}{2} \left( u\gamma_2 + v\gamma_2 + \beta u_x \gamma_1 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_2}{\sqrt{2}} \right) \frac{\partial}{\partial y} F \hspace{1cm} (64)$$

Let $h_1$ be a function such that

$$h_{1y} = \frac{\psi_2}{\sqrt{2}v}, \quad h_{1s} = \frac{1}{2} \left( u\gamma_2 + v\gamma_2 + \beta u_x \gamma_1 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_2}{\sqrt{2}} \right).$$
in agreement with (63). Applying $\delta_1$ on a general continuity equation $G_s = H_y$ yields
\[
(\delta_1 G - \epsilon_1 h_1 G_y)_s = (\delta_1 H - \epsilon_1 h_1 G_s)_y = (\delta_1 H - \epsilon_1 h_1 H_y)_y,
\]
which can be rewritten as a new transformed continuity equation:
\[
\left(\tilde{\delta}_1 G\right)_s = \left(\tilde{\delta}_1 H\right)_y,
\]
where we introduced an extended supersymmetry transformation
\[
\tilde{\delta}_1 = \delta_1 - \epsilon_1 h_1 \frac{\partial}{\partial y}.
\]
The action of $\tilde{\delta}_1$ is explicitly given by
\[
\begin{align*}
\tilde{\delta}_1 \gamma_1 &= \epsilon_1 \frac{u - v}{\beta} - \epsilon_1 h_1 \gamma_1 y, \\
\tilde{\delta}_1 \gamma_2 &= -\epsilon_1 u_x - \epsilon_1 h_1 \gamma_2 y, \\
\tilde{\delta}_1 u &= \epsilon_1 \gamma_2 - \epsilon_1 h_1 u_y, \\
\tilde{\delta}_1 v &= \epsilon_1 (\gamma_2 + \beta \gamma_1 x) - \epsilon_1 h_1 v_y.
\end{align*}
\]
(65)

Consider now the $\epsilon_2$ part of the supersymmetry transformations (47) and define $h_2$ such that
\[
h_{2y} = \frac{\psi_1}{\sqrt{2v}}, \quad h_{2s} = \frac{1}{2} \left( u \gamma_1 + v \gamma_1 + \beta u_x \gamma_2 - \frac{\beta}{v} \gamma_1 \gamma_2 \psi_1 \sqrt{2} \right).
\]
In this case the extended supersymmetry transformation is defined as
\[
\tilde{\delta}_2 = \delta_2 - \epsilon_2 h_2 \frac{\partial}{\partial y}
\]
and explicitly given by
\[
\begin{align*}
\tilde{\delta}_2 \gamma_1 &= \epsilon_2 u_x - \epsilon_2 h_2 \gamma_1 y, \\
\tilde{\delta}_2 \gamma_2 &= -\epsilon_2 \frac{u - v}{\beta} - \epsilon_2 h_2 \gamma_2 y, \\
\tilde{\delta}_2 u &= \epsilon_2 \gamma_1 - \epsilon_2 h_2 u_y, \\
\tilde{\delta}_2 v &= \epsilon_2 (\gamma_1 + \beta \gamma_2 x) - \epsilon_2 h_2 v_y.
\end{align*}
\]
(66)
The supersymmetry transformation $\tilde{\delta}_2$ maps the continuity equation $G_s = H_y$ into
\[
\left(\tilde{\delta}_2 G\right)_s = \left(\tilde{\delta}_2 H\right)_y.
\]
The main point is that the continuity equations $\left(\tilde{\delta}_i G\right)_s = \left(\tilde{\delta}_i H\right)_y$ for $i = 1, 2$ are linear combinations of original continuity equations (59), (62), (63). Thus, the extended supersymmetry transformations $\tilde{\delta}_i$, $i = 1, 2$ keep the model invariant and we conclude that
\[
[\tilde{\delta}_i, \frac{\partial}{\partial y}] = 0, \quad [\tilde{\delta}_i, \frac{\partial}{\partial s}] = 0.
\]
(67)
To find the action of the extended supersymmetry transformations \( \tilde{\delta}_i, i = 1, 2 \) on \( h_i, i = 1, 2 \) we first calculate:

\[
\tilde{\delta}_1 h_1 y = -\epsilon_1 (v + h_1 h_1 y)_y, \quad \tilde{\delta}_2 h_2 y = \epsilon_2 (v - h_1 h_1 y)_y
\]

and

\[
\tilde{\delta}_2 h_1 y = -\epsilon_2 \left( \frac{m - v}{\beta v} + (h_2 h_1 y)_y \right), \quad \tilde{\delta}_1 h_2 y = \epsilon_1 \left( \frac{m - v}{\beta v} - (h_1 h_2 y)_y \right)
\]

Thus,

\[
\tilde{\delta}_1 h_1 = -\epsilon_1 (v + h_1 h_1 y + c_{11}), \quad \tilde{\delta}_2 h_2 = \epsilon_2 (v - h_1 h_1 y + c_{22})
\]

and

\[
\tilde{\delta}_2 h_1 = -\epsilon_2 \left( \int dy \frac{m - v}{\beta v} + h_2 h_1 y + c_{21} \right), \quad \tilde{\delta}_1 h_2 y = \epsilon_1 \left( \int dy \frac{m - v}{\beta v} - h_1 h_2 y + c_{12} \right)
\]

where \( c_{ij}, i, j = 1, 2 \) are integration constants.

Despite a presence of non-local terms in the above extended supersymmetry transformations one finds that their repeated application yields

\[
\tilde{\delta}_1^i \tilde{\delta}_1 X = \pm \epsilon_1 \epsilon_1' c_{11} X_y, \quad \tilde{\delta}_2^i \tilde{\delta}_2 X = \pm \epsilon_2 \epsilon_2' c_{22} X_y
\]

where \( X \) stands for \( u, v, \gamma_1, \gamma_2 \). Similarly,

\[
\left( \tilde{\delta}_2 \tilde{\delta}_1 - \tilde{\delta}_1 \tilde{\delta}_2 \right) X = \epsilon_1 \epsilon_2 (c_{21} - c_{12}) X_y
\]

Choosing the integration constants as \( c_{11} = c_{22} = 1 \) and \( c_{21} = c_{12} \) we reproduce the \( N = 2 \) algebra of the supersymmetry transformations given in eq. (47).

## 6 Outlook

In this paper we have described a general formalism which associates a bihamiltonian structure to the \( N = 2 \) superconformal Poisson algebra by taking advantage of existence of central extension of the original algebra. We have shown how this deformation uniquely determines a deformed Sobolev-type inner product for the coadjoint orbit formalism through construction of the momentum operator for the first bracket. The momentum operator is invariant under the supersymmetry transformation and induces via Lenard relations a chain of conserved hamiltonians determining the \( N = 2 \) supersymmetric Camassa-Holm hierarchy.

All the steps of the above construction are general in nature and the formalism extends easily to other similar formalisms. In a forthcoming publication [21] we will apply the formalism to the \( N = 1 \) superconformal algebra and also provide formal proofs for the observations made here.

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References


