Using ordinary multiplication to do relativistic velocity addition

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I. INTRODUCTION

Relativistic velocity addition in one dimension is a fixture of introductory relativity. It is usually treated as a pedagogical cul-de-sac: presented, examined, forgotten.

By a very simple transformation, often by inspection, velocities can be converted into velocity factors, to be defined below. Relativistic addition of velocities corresponds to ordinary multiplication of velocity factors. Using the standard formula, addition of more than two relativistic velocities quickly becomes unwieldy. Using velocity factors, such problems can often be solved by inspection. Moreover many problems beyond the practical reach of the usual formula are so easily formulated using velocity factors that they, too, can be solved by inspection.

Physically, the velocity factor is just a two-way doppler factor. They are therefore closely related to the $k$ (a one-way doppler factor) in Bondi’s $k$-calculus and also to rapidity (the inverse hyperbolic tangent of the velocity, or logarithm of $k$), but they are superior to either approach for deriving closed form answers by inspection. Moreover, one can prove their properties from the usual velocity addition formula using only elementary algebra.

II. RELATIVISTIC VELOCITY ADDITION

The one-dimensional relativistic velocity addition formula is

$$V_{ac} = V_{ab} \boxplus V_{bc} = \frac{V_{ab} + V_{bc}}{1 + V_{ab} \cdot V_{bc}},$$

where we have taken $c = 1$ and where we use “$\boxplus$” here and subsequently to denote relativistic velocity addition.

A considerable advantage of this representation of the composition of velocities is that it is clear how, for velocities small relative to $c$, relativistic velocity addition reduces to simple addition. We see at once that velocity addition is commutative. Its associativity, however, is not obvious in this representation.

III. A TRULY ADDITIVE REPRESENTATION: RAPIDITIES

Another representation is that in terms of rapidities or velocity parameters, given by

$$\alpha = b \arctanh\left(\frac{V}{c}\right),$$

where $b$ is a constant. Here, it is convenient to take $b$ as unity, but other choices can also be useful. We also continue to take $c = 1$.

A velocity between $-1$ and $1$ becomes a rapidity between $-\infty$ and $\infty$. This representation is monotonic increasing and invertible. The rapidity is zero when the velocity is.

Since

$$\tanh(\alpha_{ab} + \alpha_{bc}) = \frac{\tanh(\alpha_{ab}) + \tanh(\alpha_{bc})}{1 + \tanh(\alpha_{ab}) \cdot \tanh(\alpha_{bc})},$$

the relativistic sum $V_{ac} = V_{ab} \boxplus V_{bc}$ yields the ordinary sum $\alpha_{ac} = \alpha_{ab} + \alpha_{bc}$.

The rapidity representation is manifestly commutative and manifestly associative. Again for small velocities, the rapidity reduces to the velocity giving correspondence with Galilean velocity addition.

Rapidity is particularly useful for integrating proper acceleration. Indeed it can be interpreted as the integral of the proper acceleration: in a relativistic rocket, it is the velocity one would calculate by multiplying the rocket’s average accelerometer reading by the elapsed time on the rocket’s clock. This is the velocity that would be imputed by an ideal Newtonian inertial guidance system.

In a companion paper, we show that rapidity can be interpreted as the change in pitch of radiation fore and aft of the direction of motion.

The omission of rapidities from introductory treatments of relativistic velocity addition is puzzling. Hyperbolic tangents and their inverses have long been available on even modest scientific calculators, so that the result

$$V_1 \boxplus V_2 = \tanh(\arctanh V_1 + \arctanh V_2)$$
is easy to remember and quick to compute. Nor is it a serious objection that the calculator gives only an inexact numerical result, because in practical situations the inexactness of computation will be dwarfed by the inexactness of the measured values.

Nor can the reason for the omission of rapidities lie in the underlying theory. Using velocities \( V = \tanh \alpha \) but not rapidities \( \alpha \) in the analytic geometry of the \( x\)-\( t \) plane is strongly analogous to using slope \( s = \tan \theta \) but not angle \( \theta \) in the analytic geometry of the \( x\)-\( y \) plane. One can treat the usual addition formula for tangents as a “slope addition formula”

\[
s_1 \oplus s_2 = \frac{s_1 + s_2}{1 - s_1 s_2}.
\]

But while one can indeed formulate the analytic geometry of the Euclidean plane using slopes and never angles, it is artificial to do so. It is similarly artificial in relativity to use velocities and never rapidities.

If neither theory nor practice account for this omission, perhaps a particular kind of pedagogical convenience does. In teaching velocity addition, it is customary to use examples and problems in which each of the velocities to be added is a simple fraction of \( c \); their relativistic sum is then also a fraction of \( c \). In this case, computation using the usual velocity addition formula uses exact rational arithmetic, which makes the examples easier to follow and the problems easier to grade and to troubleshoot.

As we shall see below, the method of velocity factors shares this pedagogical virtue, while nonetheless bringing us most of the theoretical and practical virtues of rapidity.

### IV. JUSTIFICATION OF THE METHOD OF VELOCITY FACTORS

Define the velocity factor \( f \) corresponding to \( V \) by

\[
f = g(V) = \frac{1 + V}{1 - V}.
\]  

We note in passing that \( g \) is a Möbius function that rotates the Riemann sphere by a quarter turn, with fixed points \( \pm i \); since all the coefficients are real, the real axis maps to itself. In particular, if one stereographically projects the real axis onto a unit circle centred at 0, then \( g \) corresponds to a quarter turn of this circle, taking -1 to 0, 0 to 1, 1 to \( \pm \infty \), and \( \pm \infty \) to -1. Composing \( g \) twice yields the negative reciprocal function, composing it three times yields its inverse, and composing it four times yields the identity.

The connection between Möbius functions and relativity proves remarkably deep; this particular transformation has other computational uses. We shall not require these properties, though, to prove what we need.

Solving for \( V \), we get

\[
V = \frac{f - 1}{f + 1}.
\]  

The correspondence between \( V \) and \( f \) is monotonic increasing, with the velocity range \([-1, 1]\) corresponding to the velocity factor range \([0, \infty]\). \( V = 0 \) corresponds to \( f = 1 \).

Clearly, if \( V = -V \) then \( f = f^{-1} \); negation of velocities corresponds to reciprocation of velocity factors. Now

\[
f_{ab} \times f_{bc} = \frac{1 + V_{ab} \cdot 1 + V_{bc}}{1 - V_{ab} \cdot 1 - V_{bc}} = \frac{1 + V_{ab} \cdot V_{bc} + V_{ab} + V_{bc}}{1 + V_{ab} \cdot V_{bc}} = \frac{1 + V_{ab} \cdot V_{bc} + (V_{ab} + V_{bc})}{1 + V_{ab} \cdot V_{bc}} = \frac{1 + V_{ab} \cdot V_{bc} - (V_{ab} + V_{bc})}{1 + V_{ab} \cdot V_{bc}} = \frac{1}{1 - V_{ac}} = f_{ac}.
\]  

So relativistic addition of velocities corresponds to ordinary multiplication of velocity factors.

This result might have been had more quickly from the connection with rapidities,

\[
\alpha = \arctanh V = \ln \sqrt{\frac{1 + V}{1 - V}} = \frac{1}{2} \cdot \ln f = \log_{(e^2)} f.
\]  

but the derivation in Eq. (6) does not require any acquaintance with either rapidities or hyperbolic functions, or indeed logarithms, exponentials or calculus.

In our examples, we have also made use of the observation that if, for any \( N \) and \( D \),

\[
V = \frac{N}{D}
\]

then

\[
f = \frac{D + N}{D - N},
\]

and that conversely if

\[
f = \frac{\nu}{\delta}
\]
then
\[ V = \frac{\nu - \delta}{\nu + \delta}. \]  
\[ \text{(11)} \]

In particular, if either of \( V \) or the corresponding velocity factor \( f \) is rational, or more generally algebraic, then both are. Converting either way requires taking a sum and a difference and forming a ratio; this determines the target value up to a possible sign change and a possible reciprocation, both of which can easily be put in by hand if monotonicity and the following correspondences are remembered:

\[
\begin{array}{c|c}
V & f \\
-1 & 0 \\
0 & 1 \\
1 & \infty \\
\end{array}
\]

An alternative mnemonic can be derived from the trigonometric subtraction formula

\[ \tan(\psi - \phi) = \frac{\tan(\psi) - \tan(\phi)}{1 + \tan(\psi) \cdot \tan(\phi)}. \]

Let \( \psi - \phi = \pi/4 \), so that the left hand side is unity. Then we can solve for \( \tan(\psi) \) to get

\[ \tan(\psi) = \frac{1 + \tan(\phi)}{1 - \tan(\phi)}. \]  
\[ \text{(12)} \]

Comparing this with Eq. (11) we see that if a velocity \( V \) and its corresponding velocity factor \( f \) are regarded as the slopes of two lines, than the line whose slope is \( f \) is rotated by \(+45^\circ\) relative to the line whose slope is \( V \).

V. USING VELOCITY FACTORS

We now turn to the use of this multiplicative representation, in which velocities between \(-1\) and 1 become velocity factors between 0 and \( \infty \). We shall see that this representation fits somewhere between the velocity representation and the rapidity representation. This correspondence too is a monotonic increasing, invertible function of velocity, but here zero velocity corresponds to a velocity factor of 1, and negation of a velocity corresponds to reciprocation of its velocity factor.

A first example: adding three given velocities relativistically.

Suppose, e.g., that we wish to find

\[ \frac{1}{3} \oplus \frac{2}{5} \oplus \left(-\frac{1}{4}\right), \]

where we are taking \( c = 1 \).

We make a table containing the velocities we wish to sum:

\[
\begin{array}{c|c}
V & f \\
1/3 & 3+1 = 2 \\
2/5 & 5+2 = 7/3 \\
-1/4 & -1/1 = -1/1 \\
\end{array}
\]

Then we compute the corresponding velocity factors:

\[
\begin{array}{c|c}
V & f \\
1/3 & 3+1 = 2 \\
2/5 & 5+2 = 7/3 \\
-1/4 & -1/1 = -1/1 \\
\end{array}
\]

The values of the velocity factor \( f \) are computed by forming ratios of the sum and the difference of the numerator and denominator of the values of \( V \).

Whether the sum or the difference should be in the numerator and what sign the difference should carry are easily figured out by remembering that the velocity factor cannot be negative, and that positive velocities \( V \) correspond to velocity factors greater than one. Another simple mnemonic is derived below.

Next, we multiply the velocity factors we have found to get the overall velocity factor:

\[
\begin{array}{c|c}
V & f \\
1/3 & 3+1 = 2 \\
2/5 & 5+2 = 7/3 \\
-1/4 & -1/1 = -1/1 \\
\end{array}
\]

\[ \ldots \cdot \frac{2}{3} \times \frac{5}{2} = \frac{10}{6} \]

Finally, we form a ratio of the sum and difference of the denominator and numerator of the overall velocity factor on the right to get the velocity sum:

\[
\begin{array}{c|c}
V & f \\
1/3 & 3+1 = 2 \\
2/5 & 5+2 = 7/3 \\
-1/4 & -1/1 = -1/1 \\
\end{array}
\]

\[ \frac{14/5}{12/5} = \frac{9/19}{} \times \frac{2}{3} \times \frac{5}{2} = \frac{14}{5} \]

Again, we need not memorize which way around to write the difference, or whether to put it in the numerator or denominator. We need only remember that velocity factors larger than one correspond to positive velocities, and that the magnitude of a velocity can be no greater than one. So, finally,

\[ \frac{1}{3} \oplus \frac{2}{5} \oplus \left(-\frac{1}{4}\right) = \frac{9}{19}. \]

A second example: relativistic fractions

What, relativistically, is \( 3/7 \) of \( (5/8)c \)?

Regarding \((5/8)c\) as the overall result of a large number of locally equivalent small boosts, this question asks what the velocity is when \( 3/7 \) of these small boosts have been executed.
From another point of view, the question asks for the velocity of a boost, which when repeated 7 times (in successive comoving frames), gives the same result as boosting 3 times (again, in successive comoving frames) by \((5/8)c\). In other words, we want to find \(U\) such that

\[
U \oplus U \oplus U \oplus U \oplus U \oplus U \oplus U = \frac{5}{8} \oplus \frac{5}{8} \oplus \frac{5}{8}. \tag{13}
\]

Before resorting to velocity factors, let us try to solve this by repeated use of the usual velocity addition formula on each side. Then Eq. \((13)\) can be written

\[
\frac{U^7 + 21 \cdot U^5 + 35 \cdot U^3 + 7 \cdot U}{7 \cdot U^6 + 35 \cdot U^4 + 21 \cdot U^2 + 1} = \frac{1085}{1112}, \tag{14}
\]

or

\[
1112 \cdot U^7 - 7595 \cdot U^6 + 23352 \cdot U^5 - 37975 \cdot U^4 + 38920 \cdot U^3 - 22785 \cdot U^2 + 7784 \cdot U - 1085 = 0. \tag{15}
\]

Even exploiting the palindromic symmetry between the coefficients in numerator and denominator of the left hand side of Eq. \((14)\), finding a solution in closed form is non-trivial.

Using rapidities and a scientific calculator, a numerical answer is easily obtained by evaluating

\[
tanh \left( \frac{3}{7} \times \operatorname{arctanh} \left( \frac{5}{8} \right) \right) \approx 0.3043. \tag{16}
\]

Using velocity factors, we can produce a closed-form answer practically by inspection.

We first take \(5/8\) and find its velocity factor, which is \((5+8)/(8-5) = 13/3\).

We raise this to the \(3/7\) power to get \(13^{3/7}/3^{3/7}\), the velocity factor of the desired answer, and convert back to a velocity in closed form,

\[
U = \frac{13^{3/7} - 3^{3/7}}{13^{3/7} + 3^{3/7}}. \tag{17}
\]

This result can be confirmed by evaluating the left hand side of Eq. \((14)\) algebraically, as shown below in section \(\text{VIII}\).

It can also be confirmed from Eq. \((13)\) as shown below in section \(\text{IX}\). Using the velocity addition formula to confirm a correct value already supplied takes considerably more effort than finding that value using velocity factors.

VI. THE VELOCITY FACTOR IS A TWO-WAY DOPPLER FACTOR

The velocity factor also has a simple physical interpretation as the two-way doppler factor corresponding to a given separation speed. Thus if \(O\) at rest at \(x = 0\) in a vacuum sends a light pulse of duration \(T\) to an mirror \(M\) travelling with velocity \(V\) along the positive \(x\) axis, then \(O\) will observe a reflected pulse to have a duration \(f \cdot T\). Moreover, this suggests an obvious physical explanation of why velocity factors compose by multiplication. Indeed, Bondi used multiplicative composition as one of the postulates of his \(k\)-calculus\(^1\), an elegant and accessible formulation of special relativity. His \(k\), a one-way doppler factor, is not \(f^{1/2}\). Of course, any fixed nonzero power of either of these would also be a faithful multiplicative representation.

A one-way doppler factor is simpler than a two-way doppler factor, so Bondi’s \(k\) is simpler physically. Nonetheless, like rapidity, Bondi’s beautiful \(k\)-calculus has not become a part of standard part of the pedagogy of introductory physics.

VII. SUMMARY

Velocity addition using the usual formula is unwieldy and of limited usefulness. Rapidities are a more powerful, and easily applied to a broader range of questions. The internal workings of that tool are, like those of the calculators required to use them, usually left inaccessible. This leads to difficulties in error checking and interpretation.

Velocity factors are more or less the internal workings of rapidity. The correspondence between velocity factors and velocities is simple. The use of velocity factors places interesting questions in easy reach, and so encourages tinkering; relativistic velocity addition can now become a more rewarding part of the standard curriculum than at present. Velocity factors make physical sense, provide closed form answers, are at least as memorable as the usual velocity addition formula and doppler formulae to which they are equivalent—and one can use them without having to reach for a calculator.

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VIII. SOLUTION OF RELATIVISTIC FRACTION OF VELOCITY USING RAPIDITIES

Evaluating expression (10) we find

\[ \tanh \left( \frac{3}{7} \times \arctanh \left( \frac{5}{8} \right) \right) \]

\[ = \tanh \left( \frac{3}{7} \times \ln \sqrt{\frac{1 + (5/8)}{1 - (5/8)}} \right) \]

\[ = \tanh \left( \frac{3}{14} \times \ln \frac{8 + 5}{8 - 5} \right) \]

\[ = \tanh \left( \ln \left( \frac{13}{3} \right)^{3/14} \right) \]

\[ = \exp \left( \ln \left( (13/3)^{3/14} \right) \right) - \exp \left( - \ln \left( (13/3)^{3/14} \right) \right) \]

\[ = \exp \left( \ln \left( (13/3)^{3/14} \right) \right) + \exp \left( - \ln \left( (13/3)^{3/14} \right) \right) \]

\[ = \frac{(13/3)^{3/7} - 1}{(13/3)^{3/7} + 1} \]

\[ = \frac{13^{3/7} - 3^{3/7}}{13^{3/7} + 3^{3/7}} \]

as claimed in Eq. (17).

IX. SOLUTION OF RELATIVISTIC FRACTION OF VELOCITY USING THE USUAL VELOCITY ADDITION FORMULA

Evaluating the first relativistic addition on the right hand side of Eq. (13) we find

\[ \frac{5}{8} \oplus \frac{5}{8} = \frac{2(5/8)}{1 + (5/8)^2} = \frac{80}{89} \]

so the whole right hand side of Eq. (13) becomes

\[ \frac{5}{8} \oplus \frac{5}{8} \oplus \frac{5}{8} = \frac{80}{89} \]

\[ = \frac{(80/89) + (5/8)}{1 + (80/89)(5/8)} \]

\[ = \frac{1085}{1112} \]

It is in principle straightforward to evaluate the left hand side of Eq. (13).

\[ U \oplus U \oplus U \oplus U \oplus U \oplus U \]

with

\[ U = \frac{13^{3/7} - 3^{3/7}}{13^{3/7} + 3^{3/7}} \]

Evaluating expression (19) gives

\[ U^7 + 21 \cdot U^5 + 35 \cdot U^3 + 7 \cdot U \]

\[ \frac{7 \cdot U^6 + 35 \cdot U^4 + 21 \cdot U^2 + 1}{7} \]

Substituting the right hand side of Eq. (20) into this is tedious, even when one exploits the symmetries in both expressions.

We can instead reduce the labor by considering the expression

\[ \frac{p^m - q^m}{p^m + q^m} \oplus \frac{p^n - q^n}{p^n + q^n} \]

with arbitrary positive \( p \) and \( q \) and arbitrary real \( m \) and \( n \).

Expanding the relativistic sum, we get

\[ \frac{p^m - q^m}{p^m + q^m} + \frac{p^n - q^n}{p^n + q^n} \]

Multiplying numerator and denominator by \( (p^m + q^m)(p^n + q^n) \), this becomes

\[ \frac{(p^m - q^m)(p^n + q^n) + (p^n - q^n)(p^m + q^m)}{(p^m + q^m)(p^n + q^n) + (p^m - q^m)(p^n - q^n)} \]

which expands to
\[
\frac{p^{m+n} + p^n q^n - p^n q^m - q^{m+n} + p^m q^n - p^m q^m - q^{m+n}}{p^{m+n} + p^n q^n + p^n q^m + q^{m+n} + p^m q^n - p^m q^m + q^{m+n}}
\]  

(25)

or

\[
\frac{2p^{m+n} - 2q^{m+n}}{2p^{m+n} + 2q^{m+n}}.
\]

(26)

Thus we have

\[
\frac{p^m - q^m}{p^m + q^m} \oplus \frac{p^n - q^n}{p^n + q^n} = \frac{p^{m+n} - q^{m+n}}{p^{m+n} + q^{m+n}}.
\]

(27)

(This result could have been had at once from

\[
\left(\frac{p}{q}\right)^m \times \left(\frac{p}{q}\right)^n = \left(\frac{p}{q}\right)^{m+n},
\]

in which the two factors on the left and the product on the right are each taken to be velocity factors.)

We now apply this general result (27) to our problem. Taking

\[
U = \frac{p - q}{p + q} = \frac{p^1 - q^1}{p^1 + q^1},
\]

(28)

and applying Eq. (27) repeatedly, it should be clear that

\[
U \oplus U \oplus U \oplus U \oplus U \oplus U \oplus U = \frac{p^7 - q^7}{p^7 + q^7}.
\]

(29)

Now setting \( p = 13^{3/7} \) and \( q = 3^{3/7} \), the left hand side of Eq. (13) becomes

\[
U \oplus U \oplus U \oplus U \oplus U \oplus U \oplus U = \frac{(13^{3/7})^7 - (3^{3/7})^7}{(13^{3/7})^7 + (3^{3/7})^7} = \frac{13^3 - 3^3}{13^3 + 3^3} = \frac{2170}{2224} = \frac{1085}{1112},
\]

(30)

which is what Eq. (18) gave us for the right hand side of Eq. (13), so that they are equal as claimed.