We study spherically symmetric solutions in $f(R)$ theories and its compatibility with local tests of gravity. We start by clarifying the range of validity of the weak field expansion and show that for many models proposed to address the Dark Energy problem this expansion breaks down in realistic situations. This invalidates the conclusions of several papers that make inappropriate use of this expansion. For the stable models that modify gravity only at small curvatures we find that when the asymptotic background curvature is large we approximately recover the solutions of Einstein gravity through the so-called Chameleon mechanism, as a result of the non-linear dynamics of the extra scalar degree of freedom contained in the metric. In these models one would observe a transition from Einstein to scalar-tensor gravity as the Universe expands and the background curvature diminishes. Assuming an adiabatic evolution we estimate the redshift at which this transition would take place for a source with given mass and radius. We also show that models of dynamical Dark Energy claimed to be compatible with tests of gravity because the mass of the scalar is large in vacuum (e.g. those that also include $R^2$ corrections in the action), are not viable.
1 Introduction

Theories of gravity whose action is some function of the Ricci scalar have received much attention recently as possible models of dynamical Dark Energy. It is well known that when we take a gravitational Lagrangian as \( \mathcal{L}_g = \frac{M_p^2}{2} f(R) \) (where \( M_p \) is the Planck mass and \( R \) the scalar curvature), as long as one can invert the relation
\[
e^{\sqrt{\frac{2}{3} \phi M_p}} = \frac{df(R)}{dR}
\]
to find \( R(\phi) \) for a range of \( R \), a solution where \( R \) lies in this range can also be found from an equivalent action that consists of Einstein gravity minimally coupled to a scalar field, namely \( \phi \), with certain potential \([1,2]\). So these models, when applied to the Dark Energy problem, are equivalent to quintessence models where the scalar field is conformally coupled to matter\(^1\). It has been shown that for many choices of the function \( f \) one obtains acceptable cosmological solutions that could be used to describe the acceleration of the universe: in fact one can reproduce an arbitrary expansion history by choosing \( f \) suitably \([4,5]\). What has been however subject of controversy is whether one can, at the same time that one describes the Dark Energy in this way, satisfy the bounds on the existence of extra scalar fields with gravitational couplings coming from Solar System or laboratory experiments (see e.g. \([2,6,7,8]\)).

Let us explain here the reasons why this controversy has arisen. In order to check the experimental implications of a gravitational theory for weak fields one usually starts by considering a background solution and making an expansion of the equations of motion (EOM) in powers of the fluctuations over this background solution. If the second (and higher) order terms in this expansion are negligible with respect to the first order ones in a considered solution, perturbation theory is applicable. When this is the case, the EOM can be approximated by a set of linear second order differential equations, and we can usually find analytic solutions for static and symmetric sources and check the compatibility with experiments. The situation is however completely different when this expansion is not applicable. In this case we have to deal with non-linear differential equations that are usually difficult to solve, and addressing the issue of compatibility with experiments becomes challenging. We will see that this is indeed the situation for many of the considered \( f(R) \) gravities: perturbation theory breaks

\(^1\)In this paper we restrict ourselves to the conventional metric formulation of \( f(R) \) gravity and do not consider the Palatini formulation \([3]\).
down in some cases and is not applicable because non-linearities in the equations are not negligible.

In [2,6,7] this linearized expansion was considered for some \( f(R) \) theories. As we said this theory generically consists of Einstein gravity plus an extra scalar field conformally coupled to matter, so at the linear level the only terms that appear in the EOM for the scalar are the kinetic and mass terms with a source given by the trace of the energy-momentum tensor (EMT). It was therefore argued in [2,6,7] that simple models in which the mass of the scalar is very small in vacuum, such as \( f = R \pm \mu^{2n+2}/R^n \), are ruled out because it would mediate an extra long range force contradicting experiments. But this analysis was challenged in [8] where solutions with \( R \gg \mu^2 \) were considered. Intuitively, we can expect by looking at the action above that when \( R \gg \mu^2 \) the effect of the modification should be negligible. The extra terms in the equations will be suppressed by powers of the ratio \( \mu^2/R \) and the solution for any given source should be very close to that of General Relativity (GR) up to these small corrections. But while this is true, we will see in this paper that in fact there is no contradiction with the previous analysis. This is so because, in the models with inverse powers of \( R \), whenever the scalar curvature deviates from its vacuum value significantly in some region (\( \Delta R/R_0 \geq 1 \) with \( R_0 \sim \mu^2 \)), the linearized expansion breaks down, and the analysis of [2,6,7] can not be applied to those solutions. It is then conceivable that in these models an effect similar to the Chameleon mechanism of [9] could take place, where the effects of a scalar field that is very light in vacuum are hidden because non-linearities are important when considering sources. As we will see, the asymptotic boundary conditions for the scalar curvature are the crucial element that will allow us to decide whether there is a Chameleon effect or not for the extra scalar degree of freedom. In particular, for the models with inverse powers of \( R \), if the background curvature goes to a small value asymptotically (\( R_0 \sim \mu^2 \)), the Chameleon effect does not take place: astrophysical bodies do not have a “thin shell” for static solutions. In this case the perturbative solutions of [2,6,7] are valid and the scalar curvature is “locked” to the background value everywhere, \( R \sim R_0 \sim \mu^2 \) (a fact also noticed in [7]). However, when the asymptotic value of the curvature is \( R_0 \gg \mu^2 \), the Chameleon effect does take place and we recover the solutions of Einstein gravity up to small corrections.

\footnote{In these cases the mass of the scalar in vacuum would be \( \sim \mu \), and the application of these theories to the Dark energy problem demands \( \mu \sim H_0 \), where \( H_0 \) is the value of the Hubble constant today.}
We get then the following picture of gravitational dynamics in these theories when applied to the Universe: one would observe a transition from Einstein gravity at early times (when $H^2 \gg \mu^2$) to scalar-tensor gravity at late times (when $H^2 \sim \mu^2$). If we assume that this transition is adiabatic, i.e. the solution is always taken to be the equilibrium one, these types of modification are ruled out, since we would be in the scalar-tensor regime at present. However, in this paper we do not rule out the possibility that there are some models for which this transition is non-adiabatic and slow enough so we would still remain in the “GR regime” locally in the Solar System. If this was the case those models would not be in conflict with local tests of gravity. The Chameleon effect will also allow us to resolve the apparent discontinuity in the General Relativistic $\mu \to 0$ limit of these theories where we seem to end up with a massless scalar field coupled to gravity instead of GR. In the real Universe (i.e. with a non-zero background energy density) we would recover the usual gravitational dynamics in this limit only at a non-perturbative level.

On the other hand, it has been shown that for some specific forms of the action the scalar field can be made arbitrarily massive in vacuum [11,12]. If this mass is large enough it is claimed that the scalar would have passed undetected so the models do not conflict with local tests of gravity. But as we will see this conclusion is wrong and this possibility can not be realized. The reason for this is that we can not trust a weak field expansion in this case: as we raise the mass of the scalar field, we lower the energy scale where non-linearities become important. We will find that for most functions $f$ that attempt to describe Dark Energy the energy scale at which non-linearities become relevant is $\Lambda_s \simeq M_p \left( \frac{H_0}{m_s} \right)^4$, where $m_s$ is the mass of the scalar field [13]. If this mass is large compared with $H_0$, as it should be in order to avoid conflict with experiment, this scale is very small which means that in all realistic situations where we want to apply the linearization over vacuum it is not possible to do so because non-linearities in the equations are not negligible. So in this case the fact that the mass term appearing in the linearized EOM for the scalar in vacuum is large does not imply compatibility with experiments. And we will argue that, for observationally relevant distances, the true

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3We are assuming here that the mass squared of the scalar is positive in this regime, so the solutions with $R \gg \mu^2$ are stable. For instance, when $f = R \pm \mu^{2n+2}/R^n$ this would be the case only for the positive sign, see [10]. We will come back to this issue in section four.

4And in all models that satisfy some “minimal requirements” to be defined precisely later, it is at least of order $\Lambda_s \simeq M_p \left( \frac{H_0}{m_s} \right)^2$. 

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solution on vacuum is again of the scalar-tensor type with an effectively massless scalar, as in the generic models. Furthermore, we will show that models that include positive powers of $R$ to raise the scalar mass in vacuum will also conflict with experimental results on large curvature backgrounds, where $R \gg \mu^2$. In those backgrounds the mass of the scalar diminishes to a small value and the strong coupling scale $\Lambda_s$ raises so one can use the linearized equations and it is easy to conclude that one should have observed the effects of the scalar on laboratory searches of fifth forces, for instance.

The paper is organized as follows. In the next section we will review the solutions of the linearized EOM for a spherically symmetric mass in $f(R)$ gravity and we will discuss the range of validity of the weak field expansion for a generic action $f(R)$. In the third section we will focus on theories designed to explain the cosmic acceleration and we will consider the expansion over vacuum where $R_0 \sim \mu^2 \sim H_0^2$. We will distinguish two cases: the “generic” one (in which the scalar mass is $\sim \mu$) and the “fine-tuned” one (in which the mass is $\gg \mu$). In the first case we will comment on the curious property that the scalar curvature remains small in the perturbative solutions even in those regions where there is a large energy-momentum density. In this case the strong-coupling scale of the linearization is the Planck mass, so once the curvature gets locked into this phase it remains there: one would require Planck scale energies or very strong gravitational fields in order to excite the scalar curvature out of its vacuum value. In the second case we will explain why when we raise the mass of the scalar in a certain background in a model of dynamical Dark Energy, the linearized expansion over such background is actually of no use. And we will argue that the true solution exhibits the same behavior as for the generic case. Furthermore we will explain why the addition of positive powers of $R$ to the action to yield this effect in vacuum would imply a conflict with experiment whenever $R \gg \mu^2$. In the fourth section we will briefly discuss the Chameleon mechanism and the recovery of Einstein gravity in models that modify gravity only at small curvatures $\sim \mu^2$, as long as the asymptotic background curvature is $R_0 \gg \mu^2$ and the scalar mass squared is positive. We will also discuss the transition from GR to scalar-tensor gravity (or “locked curvature” phase) that would take place in these theories when applied to the Universe as the background curvature diminishes to a value $\sim \mu^2$ and the Chameleon effect gradually disappears. Finally we end with the conclusions in section five.
2 The short distance weak field expansion, general case

It is well known that the EOM derived from the action

\[ S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} f(R) + S_m \quad \text{(with } G_N \equiv 1/(8\pi M_p^2)) \, , \]

where \( S_m \) is the matter action minimally coupled to the metric, are equivalent to those of Einstein gravity minimally coupled to a scalar field that is conformally coupled to matter. This equivalence is most easily demonstrated directly at the level of the action [1,2], but it is also instructive to take a slight detour and show how the equivalence arises at the level of the EOM. The EOM derived from (2) read:

\[ E_{\mu\nu}(g_{\mu\nu}) = f'(R)R_{\mu\nu} - f(R)\frac{g_{\mu\nu}}{2} + g_{\mu\nu}\Box f'(R) - \nabla_{\mu} \nabla_{\nu} f'(R) = 8\pi G_N T_{\mu\nu} \, . \]

A striking difference with Einstein gravity is seen from the trace:

\[ E_{\mu\mu}(g_{\mu\nu}) = -2f(R) + f'(R)R + 3\Box f'(R) = 8\pi G_N T \, . \]

In contrast with GR, where the Ricci scalar is completely determined by the trace of the energy-momentum tensor \( (R = -8\pi G_N T) \) we now see that \( R \), or rather \( f'(R) \), is an independent propagating degree of freedom. It will therefore be convenient to introduce a new variable \( \psi \), that we will identify with \( f'(R) \), and cast the original set of fourth order differential equations in \( g_{\mu\nu} \) into a set of second order equations in \( g_{\mu\nu} \) and \( \psi \):

\[ E_{\mu\nu}(g_{\mu\nu}, \psi) \equiv \psi R_{\mu\nu} - f'(\mathcal{R}(\psi))\frac{g_{\mu\nu}}{2} + g_{\mu\nu}\Box \psi - \nabla_{\mu} \nabla_{\nu} \psi = 8\pi G_N T_{\mu\nu} \, , \]

\[ \mathcal{R}(\psi) = R \, , \]

where \( \mathcal{R} \) is defined as the inverse of \( f' \):

\[ f'(\mathcal{R}(\psi)) \equiv \psi \, . \]

It is clear that a solution of eqs.(5,6) also solves the original EOM (3) as long as \( \mathcal{R}(\psi) \) is well defined on the solution. Notice that the invertibility of \( f' \) is equivalent to the more familiar condition that \( f''(R) \) should be different from zero. In general \( f''(R) \) can become zero for some values of \( R \) so \( \mathcal{R} \) will be a multivalued function, and one has to
chose a certain branch. The solutions of (3) are found then by considering eqs. (5,6) with all possible branches.

Finally, it turns out to be convenient to define a new metric variable

\[ \hat{g}_{\mu\nu} \equiv f'(R)g_{\mu\nu} = \psi g_{\mu\nu}, \] (8)

since in terms of this metric the eqs. (5,6) let themselves resuffle as those of Einstein gravity, minimally coupled to the scalar \( \psi \) and with a metric \( g_{\mu\nu} = \hat{g}_{\mu\nu}/\psi \) in the matter sector:

\[
\hat{E}_{\mu\nu} \equiv \psi^{-1}E_{\mu\nu}(g_{\mu\nu}, \psi) + \frac{g_{\mu\nu}}{2}(R(\psi) - R) \\
= \hat{G}_{\mu\nu} - \frac{3}{2\psi^2}\partial_\mu\psi\partial_\nu\psi + \hat{g}_{\mu\nu} \left( \frac{3}{4\psi^2}\hat{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi + \frac{1}{2\psi} \left( R(\psi) - \frac{f(R(\psi))}{\psi} \right) \right), \\
= 8\pi\frac{G_N}{\psi} T_{\mu\nu},
\] (9)

\[
E_{\psi} \equiv E^\mu_\mu(g_{\mu\nu}, \psi) + \psi(R(\psi) - R) = 3\Box\psi + V_p(\psi) = 8\pi G_N T,
\] (10)

where \( \hat{G}_{\mu\nu} \) is the Einstein tensor corresponding to \( \hat{g}_{\mu\nu} \) and we have defined

\[
V_p(\psi) \equiv \psi R(\psi) - 2f(R(\psi)).
\] (11)

One can now verify that these equations indeed arise from the variation with respect to \( \hat{g}_{\mu\nu} \) and \( \psi \) of the action

\[
S_{eq} \equiv \frac{1}{16\pi G_N} \int d^4x \sqrt{-\hat{g}} \left( \hat{R} - \frac{3}{2\psi^2}\partial_\mu\psi\partial_\nu\psi\hat{g}^{\mu\nu} - V(\psi) \right) + S_m(\hat{g}_{\mu\nu}/\psi), \\
= \int d^4x \sqrt{-\hat{g}} \left( \frac{M_p^2}{2} \hat{R} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\hat{g}^{\mu\nu} - V(\phi) \right) + S_m(\hat{g}_{\mu\nu}e^{-\kappa\phi}), \] (12)

where

\[
V(\psi) = \frac{1}{\psi}(\mathcal{R}(\psi) - \frac{f(\mathcal{R}(\psi))}{\psi}),
\] (13)

and in the last line we have reparameterized the scalar:

\[
\phi \equiv \sqrt{\frac{3}{2}} M_p \ln \psi \equiv \kappa^{-1} \ln \psi,
\] (14)

to get a Lagrangian with the usual kinetic term and a potential \( V(\phi) \equiv M_p^2 V(e^{\kappa\phi})/2. \)

We are interested in the situation where there is some local fluctuation on a background characterized by some curvature \( R_0 \) and a density \( T_0 \). By local we mean that
the distance scale of the fluctuation is short with respect to the characteristic distance scale associated with the curvature of the background, so that we can effectively take the flat space limit. However, as we will see shortly, this does not mean at all that we can forget about the background curvature $R_0$. In the end, the reason for this is that the local fluctuations have to match asymptotic boundary conditions that obviously do depend on the background.

The weak field expansion now consists of Taylor expanding the eqs. (9,10) in powers of the fluctuations and solving them order by order. So we write:

\[ \psi \equiv \psi_0(1 + \tilde{\psi}), \]
\[ \tilde{g}_{\mu\nu} \equiv \psi_0(g^0_{\mu\nu} + h_{\mu\nu}) = \psi_0(\eta_{\mu\nu} + h_{\mu\nu}), \quad (15) \]
\[ T_{\mu\nu} \equiv T^0_{\mu\nu} + \tilde{T}_{\mu\nu}, \]

where the 0 sub/superscript stands for the background value, and as we said, we take the flat space limit $g^0_{\mu\nu} \approx \eta_{\mu\nu}$. At first order, the scalar equation (10) now only depends on the scalar fluctuation $\tilde{\psi}$ and the trace of the fluctuation of the EMT $\tilde{T} \equiv \eta^{\mu\nu}\tilde{T}_{\mu\nu}$:

\[ (\partial^2_0 - \nabla^2)\tilde{\psi} + m^2_s \tilde{\psi} = -\frac{8}{3} \pi G^\text{eff}_N \tilde{T}, \quad (16) \]

where

\[ m^2_s \equiv -\frac{1}{3} V'_p(\psi_0) = \frac{1}{3} \left( \frac{f'(R_0)}{f''(R_0)} - R_0 \right), \quad (17) \]

and we have defined the effective Newton’s constant $G^\text{eff}_N \equiv G_N/\psi_0$. Later we will also use the effective Planck mass defined as: $M_p^\text{eff} \equiv \psi_0 M_p^2$.

Given the matter source $\tilde{T}$ one can now unambiguously determine $\tilde{\psi}$ at linear order. So let us review here the case of a spherically symmetric matter distribution with constant density $\tilde{T} = -\tilde{T}_{00} = -\tilde{\rho}$, extending over a radius $r_\odot$. The static and spherically symmetric general solution of (16) outside and inside the distribution reads:

\[ \tilde{\psi}_{\text{out}} = C_1 \frac{e^{-m_s r}}{r} + C_2 \frac{e^{m_s r}}{r}, \quad (18) \]
\[ \tilde{\psi}_{\text{in}} = C_3 \frac{e^{-m_s r}}{r} + C_4 \frac{e^{m_s r}}{r} + \frac{8\pi G^\text{eff}_N \tilde{\rho}}{3m^2_s}. \quad (19) \]

One then determines the constants $C_i$ by imposing the appropriate boundary conditions. First of all we have the condition that $\psi$ takes the background value at infinity:

\footnote{Since we are only considering local fluctuations, we can drop terms like $T^0_{\mu\nu}h^{\mu\nu}$ or $\partial_\mu h^{\mu\nu}\partial_\nu \psi_0$.}
$\tilde{\psi}(r) \to 0$ as $r \to \infty$. This sets $C_2$ to zero. One then also has the condition that $\tilde{\psi}$ is regular in the origin, this sets $C_3 = -C_4$. Finally one can determine the other two constants by matching the solution outside the distribution to the solution inside: $\psi_{\text{out}}(r) = \psi_{\text{in}}(r)$ and $\psi'_{\text{out}}(r) = \psi'_{\text{in}}(r)$. In the case that $m_s \ll r^{-1}$, one finds in this way that:

$$
\tilde{\psi}_{\text{out}} \approx \frac{2G^\text{eff}_N M}{3r} e^{-m_s r},
$$

$$
\tilde{\psi}_{\text{in}} \approx \frac{4\pi G^\text{eff}_N \tilde{\rho}}{3} \left( r^2 - \frac{r^2}{3} \right),
$$

where $M = 4\pi \tilde{\rho} r^3/3$ is the total mass of the distribution. Notice that when we can neglect the scalar mass term in (16), even for more general static spherically symmetric density distributions one can use Gauss’s Law to find the same solution outside the source with $M$ being indeed the total mass of the distribution. When we can not neglect the scalar mass the solution outside will depend on the actual density distribution inside. For instance, in the constant density case, the matching of the solution outside to the one inside gives for $m_s \gg r^{-1}$:

$$
\tilde{\psi}_{\text{out}} \approx \frac{4\pi G^\text{eff}_N \tilde{\rho}}{3m_s^2} \left( 2 - \frac{r^2}{r^2} e^{m_s(r-r_\odot)} \right).
$$

We are assuming here that the background is stable so that $m_s^2 \geq 0$. (In fact $m_s^2 \sim -R_0$ would also be fine since for our purpose, the study of local fluctuations, this would mean $m_s \approx 0$ effectively.)

We have now found the first order solution for the extra scalar degree of freedom $\psi$ contained in the metric. The first order solution for the fluctuations $h_{\mu\nu}$ of $\tilde{g}_{\mu\nu}$ follows from the linearization of (9) which gives the same equations as those of Einstein gravity (with a rescaled Newton’s constant):

$$
\dot{G}^{(1)}_{\mu\nu} = 8\pi G^\text{eff}_N \tilde{T}_{\mu\nu}.
$$

So for a static spherically symmetric mass distribution we find the usual result:

$$
h_{00} \approx \frac{2G^\text{eff}_N M}{r},
$$

$$
h_{ij} \approx \delta_{ij} \frac{2G^\text{eff}_N M}{r},
$$

$$
h_{0i} = 0.
$$
We can now finally write down the linearized result for the actual metric
\[ g_{\mu\nu} = \frac{\tilde{g}_{\mu\nu}}{\psi} \simeq \eta_{\mu\nu} + h_{\mu\nu} - \tilde{\psi} \eta_{\mu\nu}. \] (28)

And in the small scalar mass case, \( m_s \ll r_0^{-1} \), we find for the metric outside the distribution the familiar result of a Brans-Dicke scalar-tensor theory with vanishing Brans-Dicke parameter \( \omega \):
\[
ds^2 = dx^\mu dx^\nu g_{\mu\nu} \]
\[
\simeq - \left(1 - \frac{2G_{\text{eff}}^M R}{r} - \frac{2G_{\text{eff}}^M}{3r}e^{-m_s r} \right) dt^2 + \left(1 + \frac{2G_{\text{eff}}^M R}{r} - \frac{2G_{\text{eff}}^M}{3r}e^{-m_s r} \right) dx^2.
\]

However, this first order result will only be a good approximation if the weak field expansion makes sense. That is, if the neglected higher order terms in the equations are subdominant to the individual first order terms that we have solved for. The weak field expansion of the kinetic terms in the equations (9,10) is the same as for GR, so we already know that this expansion will only break down as \( r \) approaches the Schwarzschild radius \( r_S \sim G_{\text{eff}}^M M \), where \( h_{\mu\nu}, \tilde{\psi} \sim 1 \). The additional ingredient now is the expansion of the potential term \( V_p(\psi) = -\psi^3 V'(\psi) \) in the scalar equation (10). Comparing individually the first order terms that we are keeping \( \psi_0 \tilde{\psi}(r), \psi_0 \tilde{\psi}'(r)/r, \psi_0 m_s^2 \tilde{\psi}(r) \) with the higher order terms coming from the Taylor expansion of \( V_p(\psi) \) that we are neglecting, we see that the weak field expansion will be valid as long as
\[
\frac{\psi_0 \tilde{\psi}(r)}{r^2} (1 + m_s^2 r^2) \gg (\psi_0 \tilde{\psi}(r))^n \left| V_p^{(n)}(\psi = \psi_0) \right| \text{ for every } n > 1.
\] (30)

For all the situations that we consider in the next sections, the terms coming from the second and third derivative of \( V_p \) will pose no real limitation on the applicability of the weak field expansion, and the breakdown of the linearization will be due to the higher order terms, that typically become important at short distances or high energies. It is then customary to describe the breakdown of the weak field expansion in terms of a strong coupling scale \( \Lambda_s \). One therefore considers fluctuations characterized by a single energy scale \( E \), such that \( h_{\mu\nu}, \tilde{\psi} \sim E/M_{\text{eff}}^p \) and \( \partial^2 h_{\mu\nu}, \partial^2 \tilde{\psi} \sim E^3/M_{\text{eff}}^p \). The strong coupling scale is then defined as that energy scale where the weak field expansion

\footnote{Throughout this paper, when writing expressions like \( A \gg B \), we will always mean \( |A| \gg |B| \).}
breaks down. From GR we know that the expansion of the kinetic terms will be valid as long as $E \ll M_{\text{eff}}$. While from the expansion of the potential term in (10) we now find the strong coupling scale

$$\Lambda_s \equiv \min_{n>3} \left\{ M_{\text{eff}}^p, \left| \frac{1}{V_p^{(n)}(\psi_0)} \left( \frac{M_{\text{eff}}^p}{\psi_0} \right)^{n-1} \right| \right\}.$$  \hspace{1cm} (31)

Notice that this strong coupling scale would be the same as we would find by simply Taylor expanding the potential of the canonically normalized field defined in (12) over the background value $\phi_0(= \kappa^{-1} \ln \psi_0)$ as

$$V(\phi) \approx m_s^2 \tilde{\phi}^2 + X \tilde{\phi}^3 + \lambda \tilde{\phi}^4 + \sum_{n=0}^{4+n} \frac{\tilde{\phi}^{4+n}}{(\Lambda_n)^n},$$  \hspace{1cm} (32)

where $\tilde{\phi} = \phi - \phi_0$. The strong coupling scale is then $\Lambda_s = \psi_0^{1/2} \times \min\{M_p, |\Lambda_n|\}$, where the factor $\psi_0^{1/2}$ converts the scales in the so called Einstein frame, described by the metric $\hat{g}_{\mu\nu} \approx \psi_0 \eta_{\mu\nu}$, to the corresponding scales in the matter frame, described by the actual metric $g_{\mu\nu} \approx \eta_{\mu\nu}$.

3 The spherically symmetric solutions on low curvature backgrounds

In this section we will focus our attention on $f(R)$ models that intend to explain the cosmic acceleration through a modification of gravity at low curvatures. In particular we will apply the results of the previous section to check under which conditions we can use the linearization of the action over a background of small curvature ($R_0 \sim H_0^2$) to compute the solution that corresponds to a spherically symmetric mass distribution. As we said we will impose some conditions on $f(R)$ so that one can talk of the acceleration as a “curvature effect”, as opposed to e.g. the $\Lambda$CDM model that could be described by $f = R - \Lambda$. In particular we will assume that $f(R) = R + F(R)$ where there is a single mass scale appearing in $F$ that we will denote by $\mu$ and is of order $\mu \sim H_0$. Then we will consider separately two cases: the generic and the fine tuned one. In the generic case we will assume that $F$ is such that

$$F^{(n)}|_{R=\mu^2} \sim \mu^{2-2n} \sim H_0^{2-2n}.$$  \hspace{1cm} (33)

\footnote{This scale is also relevant when computing for instance quantum corrections, since for momenta higher than this the loop expansion will also break down.}
for all \( n \geq 1 \). For \( n = 1 \) we mean by this that \( F'\big|_{R=\mu^2} \sim f'\big|_{R=\mu^2} \sim 1 \). For instance, this is the case of modifications to GR where \( F = \mu^{2n+2}/R^n \) or \( F = \mu^2 \log(R) \). We will see that in this case we can use the linearization in vacuum to study the metric in all the conventional weak field situations of GR, and the solutions conflict with observations.

In the second case, the fine-tuned one, we will consider the particular situation in which

\[
F''\big|_{R=12H_0^2} \ll \mu^{-2} \sim H_0^{-2},
\]

but still \( F(R) \) satisfies (33) for some \( n > 2 \). This case includes models like [11]

\[
F(R) = -\frac{\mu^4}{R} + \alpha \frac{\mu^6}{R^2},
\]

or like [12]

\[
F(R) = \mu^2 \log(R) + \alpha \frac{R^2}{\mu^2} \quad \text{and} \quad F(R) = \frac{\mu^{2n+2}}{R^n} + \alpha \frac{R^2}{\mu^2},
\]

with \( \alpha \) an order one parameter, that have been claimed to be compatible with local tests of gravity. In this case the linearized expansion over the background with \( R = 12H_0^2 \) breaks down as we approach any source and therefore can not be used to find the solution. We will nevertheless see that the true solutions for these models are essentially the same as in the generic case, with an extra massless scalar in conflict with observations.

### 3.1 Generic case

In the previous section we learned that the mass of the scalar and the range of validity of the weak field expansion follows from the derivatives of \( V_p(\psi) \) on the background. If the condition (33) holds one can easily check that \( \psi_0 \) will be an order one number and that \( V_p(\psi_0) \) and all its derivatives are of order \( \mu^2 \sim H_0^2 \). We then find that the scalar is essentially massless \( m_s \sim H_0 \) and the linearized solution (29) corresponding to a static spherical mass distribution is the one of a massless Brans-Dicke scalar field with vanishing \( \omega \) parameter, clearly ruled out by Solar System tests (\( \omega > 40.000 \) according to the latest measurements from the Cassini mission [13]). However, as we said, one should still check if the linearization is applicable, since the weak field conditions are not necessarily the same as those of GR. But in this case we find that they actually are: from (30) we find that the linearization is valid as long as the usual condition

\[
\frac{G_NM}{r_\odot} \ll 1
\]

(37)
is satisfied. So we can use the weak field expansion for the same situations as in GR and in those cases the solution (29) does indeed describe to a very high accuracy the static solution that corresponds to a spherical mass distribution and matches the low curvature background at infinity. In fact, we see that when translated into an energy scale, the effects of non-linearities would not be important until we reach the Planck scale since $\Lambda_s \sim M_p$, as happens in conventional GR. There are however significant differences with respect to GR. Perhaps one of the most striking features of these solutions is that the scalar curvature remains of order $R \sim \mu^2$ even inside sources where $G_N T \gg \mu^2$, as also noted in [7]. But this is easily understood by going to the definition of the field $\psi$. For instance, in the case when $F = \mu^{2n+2}/R^n$ we have that

$$\psi = 1 - n \frac{\mu^{2n+2}}{R^{n+1}}. \quad (38)$$

so for $R \sim \mu^2$ a shift in the scalar curvature of order $\mu^2$ implies a shift in $\psi$ of order one which corresponds to an energy scale of order $M_p$ for the canonically normalized field $\phi$. In these models, once the background scalar curvature reaches its vacuum value it stays locked into this $R \sim \mu^2$ regime, and we would need extremely energetic processes involving strong fields ($\tilde{\phi} \geq M_p$) to get it out of this “locked phase” locally.

3.2 Fine-tuned case

In [11,12] several models are proposed for which the second derivative of $F$ is zero on a particular low curvature (de Sitter) background. It is argued that since the mass of the scalar $m_s^2 \sim 1/F''(R)$ now goes to infinity, its action range goes to zero and the model becomes compatible with the Solar System experiments. In fact, to be compatible with the current limits from fifth force search experiments, $m_s \geq 10^{-3} eV$ would already be fine [14]. For instance in the case (35) one has

$$m_s^2 = - \frac{R_0^2}{2} \frac{(1 - \frac{8\mu^2}{3\alpha \mu^2})}{R_0 - 3\alpha \mu^2} \quad (39)$$

and one finds that $m_s \geq 10^{-3} eV$ for

$$R_0 = 3\alpha \mu^2 (1 \pm 10^{-60}). \quad (40)$$

This already shows the problem with this proposal. If the background curvature deviates only minutely from this fine-tuned value, the scalar would become light again, in
conflict with experiment, as we already noticed in [15]. However, one might still think that for these particular fine tuned backgrounds the static solution corresponding to a spherically symmetric mass distribution would be very close to the GR solution, corresponding to (29) with large $m_s$. But, just as in the previous subsection, we should check the weak field conditions to make sure that we can actually use the linearized solution (29). Let us therefore consider the form of $\mathcal{V}_p(\psi)$ in the neighbourhood of the background for which $F'' = 0$ (and where $m_s^2 \to \infty$). We will denote by $R_t$ and $\psi_t$ the values of $R$ and $\psi$ in such background. As we said, even on this fine-tuned background the condition (33) will still be satisfied for some $n > 2$. For instance in the cases (35,36) this condition is satisfied for all $n > 2$. In particular we have $F''(R_t) \sim 1/\mu^4$ and one can show from (6,7) and (11) that:

$$\psi - \psi_t \simeq \frac{F''(R_t)}{2} (R - R_t)^2,$$

where the approximations are valid for $(R - R_t) \ll \mu^2$ or equivalently $(\psi - \psi_t) \ll 1$ and the sign in (42) depends on the chosen branch for $\mathcal{R}$. Looking at the approximate form of $\mathcal{V}_p$ for $\psi$ close to $\psi_t$ it is obvious why the Taylor expansion over a background with $\psi_0 = \psi_t$ breaks down, since $\mathcal{V}_p$ is non-analytic at this point. This should not come as a surprise, because we know that when $F''$ becomes zero one can not invert the relation $f'(R) = \psi$ and $\mathcal{R}$ gets a branch point. In any case, one can see that the weak field conditions (30) for the expansion on a background with curvature $R_0$ close to $R_t$ now simply reduce to the condition for the validity of the expansion of $\mathcal{V}_p(\psi_0(1 + \tilde{\psi}))$ in powers of $\tilde{\psi}$, which are

$$\tilde{\psi} \ll \psi_0 - \psi_t \sim \frac{1}{F''(R_t)\mathcal{V}_p(\psi_0)^2} \sim \frac{(\mu / m_s)^4}{(H_0 / m_s)^4}.$$

Here $m_s$ corresponds to the mass of the scalar in the background with $\psi = \psi_0$. Plugging here the expressions for the linearized solutions for $\tilde{\psi}$ we find that this condition becomes

$$\frac{G_N M}{r_\odot} \ll \frac{H_0^4}{m_s^4},$$

$$\psi - \psi_t \simeq \frac{F''(R_t)}{2} (R - R_t)^2,$$
for objects with size $r_\odot \ll m_s^{-1}$, and

$$G_N \tilde{\rho} \ll \frac{H_0^4}{m_s^2}, \quad (45)$$

for objects with size $r_\odot \gg m_s^{-1}$. And it is apparent that if we take $m_s \geq 10^{-3} eV$ we can not use the linearized solution in any situation. This failure of the weak field expansion is also manifested in the very low strong coupling scale that we now find:

$$\Lambda_s \sim M_p \left( \frac{H_0}{m_s} \right)^4, \quad (46)$$

which is even smaller than the Hubble scale ($\leq H_0^2/M_p$) for $m_s \geq 10^{-3} eV$.

To find the true static solution corresponding to a spherically symmetric mass distribution with density $\tilde{\rho}$ we should look at the full equation (10). Let’s take for instance the specific background with curvature $R_0 = R_t$. If this static background is a solution of the EOM we have that $\mathcal{V}_p(\psi_t) = 0$, so using the approximate expression for $\mathcal{V}_p$, eq. (42), the scalar equation (10) now becomes:

$$\nabla^2 \tilde{\psi} \pm \left( \frac{2 \psi_t \tilde{\psi}}{9 F''(R_t)} \right)^{1/2} = -\frac{8}{3} \pi G_N^{\text{eff}} \tilde{\rho}. \quad (47)$$

Notice that this is valid as long as $\tilde{\psi}, h_{\mu \nu} \ll 1$. The correct approximate form of the equation does not correspond to a Klein-Gordon equation with a large mass term, but an unusual power $\tilde{\psi}^{1/2}$ of the fluctuation appears in the potential term. Had we inappropriately used a Taylor expansion of the potential we would have erroneously concluded that the mass term appearing in the equation diverges. Imagine now that we have a solution of this equation that matches the asymptotic background at infinity, so $\tilde{\psi} \to 0$ for $r \to \infty$ and corresponds to the spherical source with mass density $\tilde{\rho}$. Keeping the correct approximate form of the equation we can see that the Laplacian will dominate over the potential term at short distances as usual and the solution close enough to the source will be the one of a massless scalar to a good approximation:

$$\tilde{\psi} = \frac{2G_N^{\text{eff}} M}{3r} + C. \quad (48)$$

We have not fixed the constant $C$ because its precise value would depend on the non-linear interactions of $\psi$ that will be important at long distances. By comparing the terms in the Laplacian that we have solved for, with the potential term that we
have ignored, we see that the solution corresponding to a massless scalar will be a good approximation as long as $r \ll (G_N M/\mu^4)^{1/5}$ if $C < (\mu G_N M)^{4/5}$ or as long as $r \ll (G_N M/(\sqrt{C} \mu^2))^{1/3}$ if $C > (\mu G_N M)^{4/5}$. So we see that regardless of the precise value of $C$ (that, as we said, should be determined by matching with the asymptotic background), the solution to the full equation (47) in this fine-tuned case is the same as the one we found in the generic case for small distances.

Another specific problem with models of the type (36), that include terms like $\sim R^2$ in the action, is that one does not even recover Einstein gravity for high curvature backgrounds. One can see this immediately from the Lagrangian because it can be approximated by $f \simeq R + \alpha R^2/\mu^2$ when $R \gg \mu^2$. In terms of the extra scalar fluctuation this means $m_s^2 \sim R_0$ for large background curvatures $R_0 \gg \mu^2$. Notice that now $\Lambda_s \sim M_p^{eff}$, so that we can indeed trust the linearization, and we would find an extra long range force when performing local experiments. In the next section we will show how the extra scalar fluctuation disappears for large background curvatures, if the action does approximate the Einstein-Hilbert action, $f \simeq R$ for $R \gg \mu^2$.

Finally, we have only explicitly considered the case with $F'''(R_0)$ equal to its 'natural' value. One could imagine an even more fine-tuned case where $F^{(n)}(R_0) \ll H_0^{2-2n}$ for all $n < m$ and (33) is satisfied only for $n \geq m$. In that case one finds for $\psi - \psi_t \ll 1$:

$$V_p(\psi) - V_p(\psi_t) \sim \mu^2 (\psi - \psi_t)^{1/(m-1)},$$

resulting in a strong coupling scale

$$\Lambda_s \sim M_p \left( \frac{H_0}{m_s} \right)^{\frac{2m-2}{m-2}},$$

so we see that in any case $\Lambda_s \leq M_p \left( \frac{H_0}{m_s} \right)^2$ ($\sim H_0$ for $m_s \sim 10^{-3} eV$). The correct approximate equation for the scalar, eq.(47), would now generalize to

$$\nabla^2 \tilde{\psi} + \frac{1}{3} \left( \frac{(m-1)! \psi_t \psi}{F^{(n)}(R_t)} \right)^{1/(m-1)} = -\frac{8}{3} \pi G_N^{eff} \tilde{\rho},$$

and the same argument can be used to show that the solutions will approach those of the scalar-tensor theory with a massless scalar at short distances.
4 The spherically symmetric solutions on high-curvature backgrounds

4.1 The Chameleon effect

Let us begin this section by briefly reviewing the Chameleon mechanism \cite{9} for hiding the effects of a field that is otherwise very light in vacuum. Usually one assigns a range to the force mediated by a given field according to its mass, because when \( r > m^{-1} \) the potential produced by the source gets an exponential Yukawa suppression. However, when non-linear interactions are important there are more possibilities. Imagine that we have a scalar field with an arbitrary potential in vacuum, \( V(\phi) \), and a coupling to the trace of the matter EMT like\footnote{\( T \leq 0 \) for all realistic cases, so it is convenient to define \( \rho \equiv -T \geq 0 \).} \( \Delta \mathcal{L} = \alpha(\phi)T = -\alpha(\phi)\rho \). For finding static solutions we should solve the equation

\[
\nabla^2 \phi = \alpha'(\phi)\rho + V'(\phi).
\]  

(52)

Notice that the solution of this equation is unique for a given source and asymptotic boundary conditions. To understand the Chameleon effect we will consider two different spherically symmetric situations: in one \( \rho = 0 \) everywhere except in a small region \( r \leq r_\odot \) where it is constant, while in the other \( \rho \) is a constant everywhere.

Let’s deal now with the first case. If we can use a weak field expansion the solution will be completely analogous to the solution of the linearized EOM of \( f(R) \) gravity presented in the second section but let us briefly repeat it here. The asymptotic value of the field, \( \phi_0 \), is such that it minimizes its vacuum potential, \( V'(\phi_0) = 0 \). Then in the region where \( \rho \neq 0 \) the field finds itself in a non-equilibrium position, and it will acquire a non-trivial profile. In a weak field expansion in powers of the fluctuation \( \tilde{\phi} = \phi - \phi_0 \), the linearized equation becomes

\[
\nabla^2 \tilde{\phi} = \frac{\beta \rho}{M_p} + m_s^2(\phi - \phi_0),
\]  

(53)

where \( m_s^2 \equiv V''(\phi_0) \) and \( \beta \equiv M_p\alpha'(\phi_0) \). The solution outside the source \( (r \geq r_\odot) \) is

\[
\phi(r) = \phi_0 + C_1 \frac{e^{-m_s r}}{r},
\]  

(54)

where we have taken into account the asymptotic boundary conditions and \( C_1 \) is a
constant to be determined. The solution inside the source is

$$\phi(r) = \phi_0 - \frac{\beta \rho}{M_p m_s^2} + C_2 \frac{e^{-m_s r}}{r} + C_3 \frac{e^{m_s r}}{r}. \quad (55)$$

Now we can determine the constants $C_i$ by imposing $\phi'(0) = 0$ and continuity of $\phi$ and its first derivative at $r = r_\odot$. Doing this, in the limit $r_\odot \ll m_s^{-1}$ we get the usual result

$$C_1 = -\frac{\beta \rho r_\odot^3}{3 M_p} = -\frac{\beta M}{4 \pi M_p}, \quad (56)$$

where $M \equiv \frac{4}{3} \pi r_\odot^3 \rho$. We see how the localized energy-momentum density sources the field and makes it acquire a non-trivial profile outside the source.

However, in the second situation where $\rho$ is a constant everywhere, this term can be seen as another contribution to the potential of $\phi$. So the field will not acquire a non-trivial profile but will simply set to the minimum of its $\rho$-dependent effective potential $V_{\text{eff}} = V(\phi) + \alpha(\phi) \rho$. So the solution will be $\phi = \phi_s$ where $\phi_s$ satisfies $V'(\phi_s) + \alpha'(\phi_s) \rho = 0$.

The Chameleon effect will take place whenever the second situation is a good approximation for the solution inside a localized spherically symmetric source. When this is the case, the region inside the source where the field is settled to the minimum of its effective potential will not source the field outside. The “thin shell” will be the region near the surface of the massive body where the field has a profile interpolating between its equilibrium positions, $\phi_0$ and $\phi_s$. If this region has a thickness given by $\Delta r$, the effects of the force mediated by this field will be suppressed as long as $\Delta r/r_\odot \ll 1$. For the thickness of this “thin shell” one finds the approximate expression [9]:

$$\frac{\Delta r}{r_\odot} \approx \frac{(\phi_0 - \phi_s)}{6 \beta \Phi_N M_p}, \quad (57)$$

where $\Phi_N \equiv G_N M/r_\odot$ is the Newtonian potential at the surface of the body. So the condition for the Chameleon mechanism to hold will be that the difference of the values of the field that minimize the effective potential inside and outside the source should be much smaller than $\beta \Phi_N M_p$. When non-linearities are negligible we can approximate the potential and the coupling function by $V = m_s^2 \phi^2$ and $\alpha = \beta_0 \phi/M_p$. It is easy then to compute $\phi_s$ and applying the previous formula we see that there will be a thin shell only when $r_\odot \gg m_s^{-1}$, and the effects of the field will be hidden only for distances larger than the inverse mass, as expected. But when non-linearities are relevant we can
have a Chameleon effect even if \( r_{\odot} \ll m_{s}^{-1} \). Notice that this necessarily implies the breakdown of the weak field expansion. In this case only the mass contained within the thin shell will contribute to the field outside the source so we can estimate the solution for the scalar field outside the source as \[ \phi \approx \phi_{0} - \frac{3\Delta r}{4\pi M_{p}} \frac{e^{-m_{s}r}}{r_{\odot}}. \] where \( M_{ts} \) is the mass contained within the thin shell. Through this mechanism a field that is very light in vacuum could have passed experimentally undetected.

### 4.2 The Chameleon effect in \( f(R) \) gravity

We are now in a position to assess what are the necessary properties for an \( f(R) \) action in order to get a Chameleon effect for the extra scalar degree of freedom. To parallel the Chameleon literature \[9\] we will work with the canonically normalized field \( \phi \) defined in eq. (1). For \( f(R) \) theories one can see from the action (12) that the coupling of the scalar field to matter is given by \( \Delta \mathcal{L} = e^{-2\kappa \phi} T/4 \equiv -e^{-2\kappa \phi} \rho/4 \) where \( \kappa^{-1} = \sqrt{3/2} M_{p} \) and \( T \) is again the trace of the EMT. Qualitatively, what we need to get a Chameleon effect is that as \( \rho \) becomes large, the value of \( \phi \) that minimizes the effective potential, \( V_{eff} = V(\phi) + e^{-2\kappa \phi} \rho/4 \), depends very weakly on \( \rho \). If this is the case the difference \( \phi_{0} - \phi_{s} \) in the estimation of the thin shell thickness of the previous section will be small when the asymptotic background energy density is large and massive bodies will indeed develop a thin shell, so that the extra force becomes negligible. When this is the case the scalar curvature will follow roughly the Einstein equations, \( R \approx \rho/M_{p}^{2} \). The scalar curvature is however given in terms of \( \phi \) through its definition eq. (1), so the requirement that as \( \rho \) becomes large the variation of the equilibrium value for \( \phi \) becomes small can be rephrased as the requirement that as \( R \) becomes large, \( f'(R) \) becomes roughly constant. So, for modifications of the GR action characterized by a curvature scale \( \mu^{2} \), we will have a Chameleon effect at high values of the background energy density, \( \rho_{0} \gg \mu^{2} M_{p}^{2} \), (or curvature \( R_{0} \gg \mu^{2} \)) if

\[ f'(R) \approx 1 \quad \text{when} \quad R \gg \mu^{2}. \] (59)

This qualitative discussion agrees with the naive expectations that one could have, since we are simply saying that the effects of the extra degree of freedom will be
hidden when the background curvature is large if the action *looks like* the Einstein-Hilbert action when the curvature is large. However, in order to consider these large curvature backgrounds we need them to be relatively stable. This will give a condition on the departure from the Einstein-Hilbert action for large curvatures. From (59) we see that there are essentially two possibilities. Either \( f''(R) \) will be close to zero and positive; or \( f''(R) \) will be close to zero but negative. In the latter case we find a large tachyonic instability for the scalar fluctuation \( m_s^2 \sim 1/f''(R) \), resulting in a decay of the large curvature background. For instance, the models \( f(R) = 1 - \mu^{2+2n}/R^n \) that were proposed originally [16], have \( m_s^2 \sim -R(R/\mu^2)^{n+1} \) for large curvatures, so for such models those backgrounds are unstable. For this reason they fail to give a realistic expansion of the Universe at early times, as noticed in [5]. This also agrees with the results of [17], where it was shown that FRW solutions for these models never attain large curvatures, clearly in conflict with a conventional matter dominated expansion.

The story is completely different for the models with positive values of \( f''(R) \) at large curvatures, that we will consider from now on. The scalar fluctuation now has a large positive mass squared and the backgrounds are stable. So we can safely assume that the FRW solutions will only start to differ from those of Einstein gravity at the current epoch, when \( H \sim H_0 \sim \mu \). One might think that the recovery of Einstein gravity for large background curvatures now simply happens because the mass becomes large, reducing the range of the extra force. However, this assumes that the weak field expansion remains valid. But, as we preempted in the beginning of this subsection, the weak field expansion will in fact break down for realistic situations, and the recovery of Einstein gravity happens through the non-linear Chameleon mechanism. For a massive body of mass \( M \), with radius \( r_\odot \) and density \( \rho = 3M/(4\pi r_\odot^3) \) we then get the following picture. For low values of the asymptotic background energy density \( \rho_0 \) the gravitational field of the body will be of the scalar-tensor type, with an extra force with long range \( 1/m_s \). If we now increase the background density and curvature, the range of the extra force will decrease. But then at some point, when \( 1/m_s \) is still larger than \( r_\odot \), the linearized solution breaks down and the body develops a thin shell, suppressing the extra force. Let us now illustrate this for some particular form of the function \( f \). We will consider functions such that when \( R \gg \mu^2 \) can be approximated
by $f(R) \simeq R + \mu^{2n+2}/R^n$. The relation of the scalar field with the curvature will then be given by

$$e^{\kappa \phi} \simeq 1 - n \frac{\mu^{2n+2}}{R^{n+1}}.$$  \hfill (60)

Since we are assuming that $R$ is positive and much bigger than $\mu$ we will be interested in the behavior of the potential for negative values of $\phi$ close to zero. Our assumptions imply then that in this region the effective potential for the field can be approximated by

$$V_{eff} \simeq e^{-2\kappa \phi} \left( \frac{\rho_0}{4} - \frac{n+1}{2} M_p^2 \mu^2 \left( \frac{1 - e^{\kappa \phi}}{n} \right)^{\frac{1}{n+1}} \right).$$ \hfill (61)

$$\simeq \frac{\rho_0}{4} e^{-2\kappa \phi} - \frac{n+1}{2} M_p^2 \mu^2 \left( - \frac{\kappa \phi}{n} \right)^{\frac{1}{n+1}}.$$ \hfill (62)

For $\rho_0 \gg M_p^2 \mu^2$ this potential is minimized for

$$\kappa \phi_0 \simeq -n \left( \frac{M_p^2 \mu^2}{\rho_0} \right)^{n+1}.$$ \hfill (63)

We see how the equilibrium value of the field simply gets closer to zero as we increase the background energy density. As we just said, this will give a large mass for the scalar fluctuations on these large curvature backgrounds:

$$m_s^2 = V_{eff}''(\phi_0) \sim \mu^2 (-\kappa \phi_0)^{-\frac{n+2}{n+1}} \sim \frac{\rho_0}{M_p^2} \left( \frac{\rho_0}{M_p^2 \mu^2} \right)^{n+1} = R_0 \left( \frac{R_0}{\mu^2} \right)^{n+1}.$$ \hfill (64)

However, at the same time this gives a low strong coupling scale for the weak field expansion. Indeed, the expansion of the potential \[(62)\] in powers of the fluctuation $\tilde{\phi}$ breaks down for $|\tilde{\phi}| \geq |\phi_0|$, so we get a strong coupling scale

$$\Lambda_s \sim |\phi_0| \sim M_p \left( \frac{M_p^2 \mu^2}{\rho_0} \right)^{n+1},$$ \hfill (65)

and the linearized solution for $\tilde{\phi}$, that we get from eq.\[(55)\] with $\beta = -\kappa M_p/2$, is only valid when

$$\kappa \tilde{\phi}(r_\odot) \simeq \frac{2G_N M}{3r_\odot} \ll |\kappa \phi_0| \sim \left( \frac{M_p^2 \mu^2}{\rho_0} \right)^{n+1}.$$ \hfill (66)

\[9\text{Notice that we are not making any assumption here about the properties of the vacuum solutions (where } R \sim \mu^2\text{) since we are just assuming a functional form for } f \text{ that holds in the high curvature limit } R \gg \mu^2.\]
For a given source, this will be the case for background densities $\rho_0$ smaller than a certain critical density $\rho_c$ given by

$$\rho_c \equiv \frac{M_p^2 \mu^2}{\Phi_N^{1/(n+1)}},$$

(67)

where $\Phi_N$ stands again for the Newtonian potential at the surface of the body. If the background density is much larger than $\rho_c$ the linearization breaks down and the full non-linear equation is approximately solved by the Chameleon thin shell solution (58). Indeed, from eq.(57) we find that the body develops a thin shell if:

$$\frac{\Delta r}{r_\odot} \sim \left( \frac{M_p^2 \mu^2}{\rho_0} \right)^{n+1} \Phi_N^{-1} = \left( \frac{\rho_c}{\rho_0} \right)^{n+1} \ll 1.$$  

(68)

Notice that massive bodies with strong gravitational fields in their surface (i.e. large $\Phi_N$) will develop a thin shell more easily than smaller ones for a given asymptotic background curvature. We also see how, when we take the limit $\mu \to 0$ leaving $\rho_0$ fixed, all sources will develop an (infinitely) thin shell and we recover the solutions of GR, in which the field (curvature) is a constant outside the source and jumps discontinuously ($\Delta r \to 0$) to a different value inside the source. So we see that the true limit to GR in these models would be $\mu^2 M_p^2 \rho_0 \to 0$, where $\rho_0$ is a background energy density.

In these estimations, if we are dealing with the cosmological background, we can approximate $\frac{\rho_0}{M_p^2 \mu^2} \sim (1+z)^3$. Assuming then an adiabatic evolution (i.e the solution is always taken to be the equilibrium, static one for the prescribed asymptotic value of the curvature,) we can estimate using eq.(68) at what cosmological time a given isolated source would change its gravitational field from GR to scalar-tensor. For instance a star has typically $\Phi_N \sim 10^{-6}$, while a galaxy or galaxy cluster can have $\Phi_N$ in the range $\sim 10^{-4} - 10^{-7}$. It is clear thus from (68) that we can expect that the gravitational field of these astrophysical sources would have changed to the scalar-tensor type already at high redshifts, when $(1+z) \sim 10^{4/(3n+3)} - 10^{7/(3n+3)}$. Notice that if we had used the invalid linearized result (54,56), we would simply assign a range to the force mediated by the field given by its inverse mass, $r_s \sim m_s^{-1}$, and we could estimate the redshift dependence of this distance as $r_s(z) \sim m_s^{-1}(z) \sim \mu^{-1}(1+z)^{-3(n+2)/2}$. So we would conclude that the field would have very long range even at high redshifts. However,

\[\begin{align*}
\text{We are assuming here that the density of the body, } \rho, \text{ is } \rho \gg \rho_c; \text{ one can show from (64) that this indeed implies that } 1/m_s(\rho_c) \gg r_\odot.
\end{align*}\]
here we have shown that non-linear interactions provide a further suppression of the effects of the scalar field through the Chameleon effect.

We should point out here that in this section we have just presented order-of-magnitude estimations for the necessary conditions to recover Einstein gravity. But since these solutions are non-perturbative, in order to study its behavior at a more quantitative level when for instance $\Delta r/r_\odot$ is not very small, a numerical integration of the equations would be mandatory. Also, we have assumed an adiabatic evolution in the estimation of the cosmological time at which the gravitational field of a given source would change from GR to scalar-tensor. Under this assumption the models would be ruled out, since we would be in the scalar-tensor regime at present. But to study the time-dependence of these solutions, thereby checking explicitly if their evolution is really adiabatic, one might have to resort again to numerical methods. On the basis of our analysis we can not exclude that models exist for which the GR to scalar-tensor transition is slow and non-adiabatic. In this case such $f(R)$ models could perhaps be brought into accord with observations.

5 Conclusions

In this paper we have studied the solutions corresponding to spherically symmetric sources in $f(R)$ theories of gravity. We have started by clarifying the range of validity of the linearized expansion, giving the conditions that have to be satisfied for this expansion to be valid for an arbitrary function $f$. Then we have shown that for the models that represent a modification of gravitational dynamics only at low curvatures, the linearized expansion on vacuum breaks down as long as the scalar curvature departs significantly from its vacuum value. These models are characterized by a function that can be approximated by the Einstein-Hilbert action ($f \approx R$) when $R \gg \mu^2$, but is non-trivial when $R \sim \mu^2$. However, when we fix our asymptotic boundary conditions for the curvature to the vacuum value, in most cases the linearized solutions (that are in clear conflict with Solar system experiments) are valid as long as $G_N M/r \ll 1$, the same condition that one finds in GR. Indeed, the strong coupling scale of this linearization is the Planck mass, as in GR. This implies that, in these models, once the scalar curvature diminishes to a value $\sim \mu^2$, it gets “locked” into this value everywhere, and one would require very strong gravitational fields or Planck scale energies to be
able to get out of this phase locally. It is worth to mention here that the situation is completely different for models that include inverse powers or logarithms of other curvature invariants beyond the scalar curvature. In particular if we include invariants that do not vanish in the Schwarzschild solution, even in the case when the linearized solutions are the same, the strong coupling scale for the linearization on vacuum is significantly smaller, $\Lambda_s \sim (\mu^3 M_p)^{1/4}$, as we showed in [18]. For those models one can never use the linearized solutions, and one does recover the GR solutions at short distances as required by Solar System or laboratory experiments [18].

Also, we have seen why raising the mass of the scalar field in vacuum by adding, e.g. an $R^2$ term to the action does not imply compatibility with local tests of gravity, as claimed in [12]. In fact, it is easy to see that those models are ruled out because when adding positive powers of $R$ to the action we are modifying gravity also at high curvatures. For instance in the $R^2$ case one can approximate the action by $f = R + R^2/m^2$ with $m \sim \mu$ in the high curvature regime ($R \gg \mu^2$). In those situations the scalar field becomes effectively massless and its effects would have been observed. Furthermore, we have argued that even the solutions in vacuum are of scalar-tensor type for observationally relevant distances in these cases.

On the other hand, for the models that modify gravity only at small curvatures, when the asymptotic value of the curvature is large (i.e. $\gg \mu^2$ because there is a large, constant background energy-momentum density), the linearization breaks down and the behavior of the extra scalar field, that we can associate to the extra degree of freedom contained in the metric, is governed by non-linear dynamics. This observation allows one to recover quantitatively what intuitively could seem obvious: the solutions of the theory approach those of GR when $R \gg \mu^2$. This is achieved through the so-called Chameleon mechanism [9]: whenever there is a localized massive body in a high curvature background, the scalar field quickly adopts a new equilibrium position inside it as a consequence of its non-linear interactions even if its mass is small on the background. As a consequence only a “thin shell” of matter in the surface of the body acts as a source for the field outside. This “thin shell” is the region near the surface where the scalar field interpolates between its equilibrium positions. We have estimated this thickness for some forms of the function $f$, giving a quantitative estimation of the necessary value of the background curvature in order for this effect to take place for a body of given mass and radius. Using this estimation, and under the assumption
of an adiabatic evolution, we have argued that in the application of these theories to the Universe one should observe a transition from GR to scalar-tensor gravity as the Universe expands, taking place already at high redshifts for most astrophysical sources. However we have not ruled out the possibility that this gravitational decay could be slow and non-adiabatic for some models. If the Earth and its environment would still remain in the GR regime in some cases, those models would be compatible with local tests of gravity. But a quantitative determination of the time scale associated to the decay of the scalar curvature to its equilibrium value $\sim \mu^2$ inside sources as the asymptotic background curvature approaches its vacuum value would require the study of time-dependent non-perturbative solutions which lie beyond the scope of the present paper.

Finally, our analysis also allows us to make some more general statements on the applicability of the linearization for these theories. For low curvatures of the background $R_0 \sim \mu^2$ one can safely use the weak field expansion in the same situations as for GR. However, for large curvatures, one has to be careful when using this expansion. For instance, the linearized equations for the cosmological perturbations that were used in [5] will become invalid at early times, at least if one considers realistic (i.e. high curvature) FRW backgrounds. The recovery of Einstein gravity is essentially non-perturbative as we illustrated in the previous section. And in such models we can expect that in general there will be a range of background curvatures for which the linear solutions are those of scalar-tensor gravity, whereas the true non-linear solutions are in agreement with GR.

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**References**


