Exact spectrum of scalar field perturbations in a radiation deformed closed de Sitter universe

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ABSTRACT: We observe that the equation of motion for a free scalar field in a closed universe with radiation and a positive cosmological constant is given by Lamé’s equation. Computing the exact power spectrum of scalar field perturbations, the presence of both curvature and radiation produces a red tilt weakly dependent on the amount of radiation.

KEYWORDS: Inflation, QFT on curved spacetime.
1. Introduction

There is by now strong and growing evidence that the very early universe went through a phase of inflation [1]. What led to this phase of accelerating expansion, and what the underlying microscopic theory is, are questions that are still under intense investigation. Our current understanding from the nearly scale invariant spectrum of the cosmic microwave background (CMB) anisotropies is that such an inflationary phase (at least the observable e-folds) can be well described by an approximate de Sitter universe.

An interesting modification of a (quasi-) de Sitter spacetime that would still be consistent with an inflationary history of the universe is the introduction of some primordial radiation. At late times, this radiation would get inflated away and the naive expectation is that this radiation will therefore not leave any observable effects on the CMB temperature anisotropy power spectrum. Various cosmological models have been proposed in which it is natural to include some amount of radiation in the early (quasi-) de Sitter phase [2–4] and some general results on radiation-filled FRW universes can be found in [5]. Moreover, going beyond the mini-superspace approximation in selecting the wave function of the universe [6–8] naturally leads to an effective description in terms of a radiation deformed de Sitter geometry, where the radiation density is related to the fundamental (string) length scale in the problem [9, 10].

\[^{1}\text{A similar conclusion was reached in [11] and [12] based on string thermodynamic arguments and quantum gravitational loop effects, respectively.}\]
All these considerations warrant a more detailed study of a radiation deformed de Sitter geometry. In particular, it would be interesting to see whether primordial radiation could leave some observational signatures. Here we report that in a de Sitter plus radiation universe, remarkably, the power spectrum of scalar field fluctuations is determined by a known differential equation: Lamé’s equation. This allows for a precise analysis of the spectrum and the modifications with respect to a pure de Sitter geometry.

Scalar fields in pure and deformed de Sitter spacetime have of course been well studied. [14,15] studied the mode functions of a scalar field on a pure de Sitter background. The open radiation deformed de Sitter geometry is considered in [17]. We study a closed radiation deformed de Sitter geometry. There is a crucial difference between the open and the closed geometries. In an open radiation deformed geometry, the universe begins in a singularity, followed by a phase of radiation domination, after which the universe becomes dominated by the cosmological constant. In contrast, the radiation deformed global de Sitter spacetime interpolates between a pure closed de Sitter and an Einstein static universe as the total amount of radiation is varied from zero to the maximally allowed value, and there is no singularity in the geometry. The presence of any more radiation (than the maximum allowed amount) leads to a spacelike singularity. In Euclidean signature, the geometry corresponds to a deformation of an $S^4$ to a “barrel” shaped geometry [9,10]. So below the maximally allowed value for the radiation density the initial singularity is absent and the geometry can safely be analytically continued to a bouncing Lorentzian geometry.

Our study is motivated in part because the Euclidean geometry allows an interpretation as the instanton process involved in selecting the wave function of the universe [6–8]. What might be a natural choice for the wave function of the universe is a subject of long debate [13], which we will not address in this paper (see [9,10] for discussion). In the original Hartle-Hawking context, it allows for an unambiguous determination of the vacuum state of fluctuations [23, 24]. The cosmological vacuum ambiguity has recently received renewed interest as future CMB experiments are sensitive enough to probe it [30]. We will point out that in a Hartle-Hawking cosmological scenario with radiation the Bunch-Davies initial state is no longer unambiguously selected by analytic continuation to the Euclidean geometry.

To briefly summarize our main result: the effect of the radiation is to lead to a moderation of the closed dS red tilt of the scalar field power spectrum. The obvious explanation for this is that the low $k$-modes are sensitive to the spatial curvature, i.e. the compactness, of the closed de Sitter geometry. The radiation deformation effectively shrinks the volume of the spatial $S^3$'s, which changes the red tilt in the fluctuations. Consistency with the observed flatness of our universe, however, make the red-tilt due to curvature barely observable. Similarly, the additional red-tilt due a radiation deformation is unlikely to show up in future experiments.

2. A closed universe with a positive cosmological constant and radiation

The scale factor for a closed universe with a positive cosmological constant $\Lambda$ plus a radi-
ation term with energy density $\rho*/a^4$ obeys the equation

$$2 \frac{1}{a(t)} \frac{d^2 a(t)}{dt^2} + \frac{1}{a(t)^2} \left( \frac{da(t)}{dt} \right)^2 + \frac{1}{a(t)^2} = \Lambda - \frac{\rho*}{a(t)^4},$$

(2.1)

We use $t$ for time in Lorentzian signature metric, $\tau$ for time in Euclidean signature metric, natural units in which $8\pi G = c = \hbar = 1$. The solution to Eqn. (2.1) is given by

$$a(t) = \frac{1}{\sqrt{2L}} \sqrt{1 + \sqrt{1 - \beta \cosh(2Lt)}}.$$  

(2.2)

Here $\beta = 4L^2 \rho*$ and $L = \sqrt{\frac{1}{4} \Lambda}$ is the late-time horizon (see appendix B). The Euclidean version is a deformation of $S^4$. We will refer to it as the “barrel” geometry [10]

$$a(\tau) = \frac{1}{\sqrt{2L}} \sqrt{1 + \sqrt{1 - \beta \cos(2L\tau)}}.$$  

(2.3)

As $\beta \to 0$ (no radiation) one recovers the pure de Sitter geometry. In the opposite limit $\beta \to 1$ one obtains the Einstein Static universe. The Euclidean geometry corresponds to an $S^4$ for $\beta = 0$, a cylinder ($R^1 \times S^3$) for $\beta = 1$, and for general $\beta$ interpolates between these two, resembling a barrel (Fig.(1)).

Perhaps a geometrically more intuitive way to write the scale factor is

$$a(\tau) = \frac{1}{L \sqrt{1 + 2\Delta^2}} \sqrt{\Delta^2 + \cos^2(L\tau)}.$$  

(2.4)

where $\Delta^2 = \frac{1 - \sqrt{1 - \beta}}{2\sqrt{1 - \beta}}$. In the limit of vanishing radiation, $\Delta \to 0$, one manifestly recovers pure de Sitter.

Compared to Lorentzian de Sitter (Eq.(2.2)), the size of the spatial $S^3$’s shrink as a consequence of the radiation, which is most pronounced around the waist at $t = 0$. At this bounce, the waist is reduced by a factor $\sqrt{\frac{1 + \Delta^2}{1 + 2\Delta^2}}$. This effect reaches a minimum in the (critical) limit $\beta \to 1$, or equivalently $\Delta^2 \to \infty$. The geometry reduces to a pure de Sitter spacetime as $t \to \pm \infty$ due to the redshifting of the radiation. The Penrose diagram of the radiation deformed solution corresponds to a slightly elongated version of the perfect square of pure de Sitter [18]. This shows that the compact nature of the spatial slices can in principle be ascertained in a finite amount of proper time by fiducial observers, which is impossible in pure de Sitter.

3. Scalar field in a closed de Sitter universe with radiation

To show that the fluctuation spectrum is determined by the Lamé equation, consider a minimally coupled scalar field

$$\mathcal{L}_\Phi = -\frac{1}{2} g^{ab} (\partial_a \Phi)(\partial_b \Phi) - \frac{M^2}{2} \Phi^2.$$  

(3.1)
The wave equation (in Euclidean signature) is given by
\[
\left( -\frac{\partial^2}{\partial \tau^2} - \frac{3}{a(\tau)} \frac{\partial}{\partial \tau} - \frac{\Delta_3}{a(\tau)^2} + M^2 \right) \Phi = 0 ,
\]
where \( \dot{a} \equiv da(\tau)/d\tau \), and \( \Delta_3 \) is the Laplacian on the unit three-sphere. As is standard, we decompose the field into \( S^3 \) spherical harmonics
\[
\Phi = f_k(\tau) Y_{k\ell m}(\chi, \theta, \phi)
\]
with eigenvalues
\[
\Delta_3 Y_{k\ell m} = -(k - 1)(k + 1) Y_{k\ell m} \quad (k - 1) \geq \ell \geq |m| \geq 0.
\]
The mode function \( f_k(\tau) \) must then obey the following equation

\[
-\ddot{f}_k - 3 \frac{\dot{a}}{a} \dot{f}_k + \left[ \frac{(k^2 - 1)}{a^2} + M^2 \right] f_k = 0.
\] (3.5)

We are following the common convention in a closed cosmology to let the minimum (co-moving) momentum variable \( k \) equal 1, corresponding to the homogeneous mode of Eq.(3.4). Note that in our notation the scale factor \( a(\tau) \) has dimensions of length, implying that the co-moving momentum \( k \equiv a(\tau) p \) is a dimensionless number. An important thing to realize is that in these conventions the same co-moving scale in initially different closed universes (for example one with or without primordial radiation), will not correspond to the same physical scale at late cosmological time.

In what follows, we shall simplify the above equation and show that the mode functions \( f_k(\tau) \) are given in terms of solutions to Lamé’s equation. First make the substitution

\[
F_k(\tau) = a(\tau) f_k(\tau) .
\] (3.6)

then \( F_k \) must satisfy

\[
\ddot{F}_k + \frac{\dot{a}}{a} \dot{F}_k - \left( \frac{\dot{a}}{a} + \frac{a^2}{a^2} \right) F_k - \left( \frac{(k^2 - 1)}{a^2} + M^2 \right) F_k = 0.
\] (3.7)

From the trace of the Einstein equations it is easy to show that for a closed FRW universe (in Euclidean signature)

\[
\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -2L^2 + \frac{1}{a^2} .
\] (3.8)

This leads to the equation

\[
\ddot{F}_k + \frac{\dot{a}}{a} \dot{F}_k + L^2 \left( \mu(\mu + 1) - \frac{k^2}{L^2 a^2} \right) F_k = 0 ,
\] (3.9)

where \( \mu(\mu + 1) = 2 - M^2/L^2 \). Substituting the explicit form of the scale factor, we obtain the following equation

\[
\left( 1 + \frac{\Delta^2}{\sin^2(L \tilde{\tau})} \right) \ddot{F}_k + L \cot(L \tilde{\tau}) \dot{F}_k + L^2 \left( \mu(\mu + 1) + \frac{\Delta^2(\mu(\mu + 1)) - k^2(1 + 2\Delta^2)}{\sin^2(L \tilde{\tau})} \right) F_k = 0
\] (3.10)

in terms of \( \tilde{\tau} = \tau + \pi/2L \). In the limit \( \Delta \to 0 \), one recovers the associated Legendre equation for pure de Sitter space using global coordinates (closed \( S^3 \) slicing) [19, 20].

Let us now express everything in terms of a variable \( y = \cos^2(L \tilde{\tau}) \). This yields

\[
F_k''(y) + \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - z} \right) F_k'(y) + \frac{(\mu(\mu + 1) - k^2) + \Delta^2(\mu(\mu + 1) - 2k^2) - \mu(\mu + 1)y}{4y(y - 1)(y - z)} F_k(y) = 0 ,
\] (3.11)
where $z = 1 + \Delta^2$. Eq. (3.11) is an ordinary differential equation with four singular points (at $0, 1, z, \infty$). This equation is well studied and is known as Heun’s equation [21]. In fact, Eq. (3.11) is a special type of Heun’s equation called Lamé’s equation. That scalar field fluctuations in closed de Sitter universes plus radiation reduce to Lamé’s equation was also noted in [5]. Here we use this to compute an exact power-spectrum of scalar field fluctuations. In this context it is interesting to note that according to [21], Lamé’s equation is the most general equation of Heun type which can arise from a Laplacian after separation of variables. For the sake of completeness the general features of Heun’s and Lamé’s equation are discussed in appendix A. Solutions to Lamé’s equation exist both as a power series and as a series in hypergeometric functions.

### 3.1 Initial conditions and vacuum states

To determine the complete solution to the mode equation (3.5) we must supply initial conditions or equivalently define the vacuum state. This is where physics enters. To construct the vacuum states we follow [15, 16]. In the previous sections we wrote the scalar field equation of motion in a metric using Euclidean time. In this section we will work in Lorentzian signature, $a(t) = \frac{1}{L\sqrt{1+\Delta^2}}\sqrt{\Delta^2 + \cosh^2(Lt)}$.

In order to follow the notation of [16], we start from the Lorentzian counterpart of Eq. (3.9), and define a new time variable $\eta$ given by

$$\eta = 2 \arctan (e^{Lt}) . \tag{3.12}$$

As $-\infty < t < \infty$, $0 < \eta < \pi$. For pure de Sitter this variable simply corresponds to conformal time, but this is not true in general. In terms of this variable the equation of motion becomes

$$\frac{d^2 F_k}{d\eta^2} + \frac{\Delta^2 \sin \eta \cos \eta}{1 + \Delta^2 \sin^2 \eta} \frac{dF_k}{d\eta} + \left( \frac{(1 + 2\Delta^2)k^2}{1 + \Delta^2 \sin^2 \eta} - \frac{(2 - M^2/L^2)}{\sin^2 \eta} \right) F_k = 0 . \tag{3.13}$$

It will be useful to recast this equation one more time into a deformation of the defining associated Legendre equation by introducing a new function $G_k$ related to $F_k$ as $F_k = \left( \sqrt{\sin \eta}/(1 + \Delta^2 \sin^2 \eta)^{1/4} \right) G_k$. Then in terms of the variable $x = -\cos(\eta)$ one obtains

$$\frac{(1 - x^2) d^2 G_k(x)}{dx^2} - 2x \frac{dG_k(x)}{dx} + \frac{1}{1 + \Delta^2 (1 - x^2)} \left( \frac{(1 + 2\Delta^2)(k^2 - 1/4) - v^2(1 + \Delta^2)(1 - x^2)}{1 - x^2} + \frac{3\Delta^2 (1 + \Delta^2)}{4(1 + \Delta^2(1 - x^2))} \right) G_k(x) = 0 , \tag{3.14}$$

where $v^2 = 9/4 - M^2/L^2$.

It is worth briefly recapitulating the construction of vacua for de Sitter spacetime before we complicate matters by taking into account the radiation term. When $\Delta^2 = 0$, the above equation is by construction an associated Legendre equation, and the time variable
\( \eta \) corresponds to conformal time. The most general solution is a linear combination of the associated Legendre polynomials
\[
F_k(\eta) = \sqrt{\sin \eta} \left[ A_k P_{k-1/2}^v(-\cos \eta) + B_k Q_{k-1/2}^v(-\cos \eta) \right].
\] (3.15)

The coefficients \( A_k \) and \( B_k \) are not independent. The orthonormality of the physical mode functions requires that
\[
F_k \frac{dF^*_k}{d\eta} - F^*_k \frac{dF_k}{d\eta} = i,
\] (3.16)
and this leads to the constraint
\[
A_k B_k^* - B_k A_k^* = \frac{i \Gamma(k + 1/2 - v)}{\Gamma(k + 1/2 + v)}.
\] (3.17)

A priori there is no preferred vacuum state and all solutions to equation (3.17) are allowed. On physical grounds however, one state, the Bunch-Davies or Euclidean vacuum is usually preferred. The Euclidean vacuum is the unique set of initial conditions for which the analytically continued Euclidean Green’s function has no singularities at the antipodal point [22]. For our purposes this can simply be interpreted to mean that the state should reduce to the usual flat space vacuum at physical momentum scales much larger than the curvature scale, i.e. \( k/a > H \). This requirement singles out the following values for the coefficients [15, 16]
\[
A_k = \left( \frac{\pi \Gamma(k + 1/2 - v)}{4 \Gamma(k + 1/2 + v)} \right)^{1/2} e^{i\pi v/2},
\]
\[
B_k = -\frac{2i}{\pi} A_k.
\] (3.18)

In the Hartle-Hawking “tunneling from nothing” scenario for the wavefunction of the universe, analyticity has been argued to uniquely resolve the cosmological vacuum ambiguity in favor of this Euclidean Bunch-Davies vacuum [23, 24]. Clearly this analyticity argument no longer holds with even a small amount of radiation present; there is no regularity condition at the south pole of the “barrel”.

### 3.2 Vacua for the radiation deformed geometry

We would like to see how this de Sitter result - the one complex parameter family of vacua given by Eq.(3.17) - gets modified by the presence of radiation. We do so by a power series analysis of solutions to Eq.(3.14). Let us denote the two independent solutions by \( G^v_{k-1/2} \) and \( H^v_{k-1/2} \). We will require that in the limit of vanishing radiation \( G^v_{k-1/2} \) reduces to \( P^v_{k-1/2}(x) \) and \( H^v_{k-1/2} \) reduces to \( Q^v_{k-1/2} \). Using the power series ansatz \( G(x) = \sum_{r=0}^{\infty} c_r (1 - x)^{r+\alpha} \) it is straightforward to verify that the two independent solutions to Eq.(3.14) are given by
\[
\alpha = v/2, \quad \text{or} \quad \alpha = -v/2,
\]
\[
c_1/c_0 = \frac{2\alpha(2\alpha + 1) - 2(k - 1/2)(k + 1/2)}{(2\alpha + 2)^2 - v^2} + 4\Delta^2 \frac{v^2 - 4\alpha^2 - (k - 1/2)(k + 1/2)}{(2\alpha + 2)^2 - v^2}.
\] (3.19)
Here we have listed only the ratio of the first two coefficients. This suffices to study the late time behavior \((x \to 1)\). The coefficient \(c_0\) is fixed by requiring the solutions to reduce to the associated Legendre functions for zero radiation. This gives
\[
c_0 = \frac{((-1)^v\Gamma(k + v + 1/2))}{(2^{v/2}v!\Gamma(k - v + 1/2))}.
\]
The general \(n\)-th term recursion relation is easy to find.

The above two values of \(\alpha\) (viz. \(\alpha = +v/2, -v/2\)) give two independent solutions \(G_{k-1/2}^v\) and \(G_{k-1/2}^v\). These are the analogs of \(P_{k-1/2}^v\) and \(P_{k-1/2}^{-v}\), respectively. To find \(H_{k-1/2}^v\) - the analog of \(Q_{k-1/2}^v\) - we remind the reader of the relationship for associated Legendre functions [25]
\[
Q_{k-1/2}^v(x) = \frac{\pi}{2\sin(v\pi)} \left( P_{k-1/2}^v(x)\cos(v\pi) - \frac{\Gamma(k + v + 1/2)}{\Gamma(k - v + 1/2)} P_{k-1/2}^{-v}(x) \right). \tag{3.20}
\]
We can find the corresponding relation for \(H_{k-1/2}^v\) as follows. In Eq.(3.20), the coefficients of the associated Legendre functions are independent of the variable \(x\), representing time. We expect the same to hold true for the modified relationship. If we can fix these coefficients at one instant of time, they will have those values at all times. Now note that at late times (as \(x \to 1\)), Eq.(3.14) takes the form
\[
(1 - x^2) \frac{d^2G_k(x)}{dx^2} - 2x \frac{dG_k(x)}{dx} + \left( (1 + 2\Delta^2)(k^2 - 1/4) - \frac{v^2}{1 - x^2} \right) G_k(x) = 0.
\]
(3.21)

This is almost the equation for de Sitter spacetime, except for the factor of \((1 + 2\Delta^2)\). This might seem puzzling. At late times one expects the radiation to have no effect. However, recall that even at (fixed) late times, the overall size of a compact spatial slice is scaled down due to the presence of the radiation. Now we can construct a relationship analogous to Eq.(3.21). At late times, \(G_{k-1/2}^v\) approaches \(P_{\chi-1/2}^v\), with \((\chi^2 - 1/4) = (1 + 2\Delta^2)(k^2 - 1/4)\). This suggests to simply replace \(k\) by \(\chi\), giving
\[
H_{k-1/2}^v(x) = \frac{\pi}{2\sin(v\pi)} \left( G_{k-1/2}^v(x)\cos(v\pi) - \frac{\Gamma(\chi + v + 1/2)}{\Gamma(\chi - v + 1/2)} G_{k-1/2}^{-v}(x) \right). \tag{3.22}
\]

In terms of these two independent solutions \(G_{k-1/2}^v\) and \(H_{k-1/2}^v\), the general solution to the mode functions reads
\[
F_k = \frac{\sqrt{\Gamma(\sin\eta)}}{(1 + \Delta^2\sin^2\eta)^{1/4}} G_k \tag{3.23}
\]
\[
= \frac{\sqrt{\sin\eta}}{(1 + \Delta^2\sin^2\eta)^{1/4}} [\tilde{A}_k G_{k-1/2}^v(-\cos\eta) + \tilde{B}_k H_{k-1/2}^v(-\cos\eta)].
\]

A careful analysis of the orthonormality constraints shows that the two complex parameters \(\tilde{A}_k\) and \(\tilde{B}_k\) must be related as
\[
\tilde{A}_k \tilde{B}^*_k - \tilde{B}_k \tilde{A}^*_k = i \sqrt{1 + 2\Delta^2} \frac{\Gamma(\chi + 1/2 - v)}{\Gamma(\chi + 1/2 + v)} \tag{3.24}
\]

The appearance of another \(\sqrt{1 + 2\Delta^2}\) factor in this condition is important. It appears because in the radiation deformed geometry the \(\eta\) variable does not correspond to conformal
time. From Eq. (2.4) one sees that in the late time limit, the scale factor reduces to \( a(t) \approx \frac{\cosh Lt}{L\sqrt{1+2\Delta^2}} \). As compared to pure de Sitter space, this simply amounts to a rescaling of the conformal time variable with \( \sqrt{1+2\Delta^2} \). This explains the factor in Eq. (3.24). In addition we simply replaced \( k \) by \( \chi \) on the right-hand-side of this equation, following our earlier argument that at late times the Eq. (3.14) reduces to the associated Legendre equation and that the normalization condition is independent of time.

To find the analog of the Bunch-Davies vacuum in the (late time) radiation deformed geometry, we proceed in exactly the same way as before. We demand that our solution reduces to the usual flat space vacuum at high momentum \( p = k/a \). In addition it should reduce to the usual de Sitter Bunch-Davies vacuum in the limit \( \Delta^2 \rightarrow 0 \). Keeping in mind that \((\chi^2 - 1/4) = (1 + 2\Delta^2)(k^2 - 1/4)\), this singles out the following solution

\[
\tilde{A}^{BD}_k = (1 + 2\Delta^2)^{1/4} \left( \frac{\pi}{4\Gamma(\chi + 1/2 - v)} e^{i\pi v/2} \right),
\]

\[
\tilde{B}^{BD}_k = -\frac{2i}{\pi} \tilde{A}^{BD}_k.
\]

So we see that the only difference as compared to the pure de Sitter case is the replacement of \( k \) with \( \chi \) and the additional factor of \((1 + 2\Delta^2)^{1/4}\) due to the change in the normalization condition Eq. (3.24).

Fundamentally there is deeper difference between the Bunch-Davies in the radiation deformed case and that of the pure de Sitter. Were one to insist on analytic behavior of the wave function in the Euclidean past, as e.g. in the Hartle-Hawking scenario, then one must choose the Euclidean or Bunch Davies vacuum in pure de Sitter [23, 24] to avoid a singularity at the south pole of the sphere. Because even a tiny amount of radiation opens up the South pole to a barrel, there appears to be no reason to insist on well-defined behavior at the edge. The Bunch-Davies state is now only preferred for the usual reasons.

Finally note that one should no longer consider the limit \( \Delta^2 \rightarrow \infty \) in the late-time de Sitter solution; the late-time de Sitter limit is implicitly defined as \( t \geq t_f \), with \( \cosh^2(Lt_f) \gg \Delta^2 \). Instead we know that the homogeneous solution reduces to the Einstein Static Universe in the limit \( \Delta^2 \rightarrow \infty \). It is not difficult to see that in that case the mode-functions will reduce to those of ordinary flat space, i.e. plane waves. As we will be interested in a (late-time) inflationary stage, the late-time de Sitter solution is the appropriate one to study the leading effects due to the radiation deformation.

4. The power spectrum of scalar field perturbations

The large scale density perturbations present in the CMBR temperature anisotropy we see today crossed the horizon about 55 e-folds before the end of the inflation.\(^2\) Subsequently they remained frozen till horizon re-entry. Assuming we can construct a suitable closed inflationary model based on the radiation deformed de Sitter geometry, the question whether there will be any imprint on these perturbations probably depends on whether the radiation

\(^2\)The precise e-folds at which the observable modes crossed the horizon depend on the reheating temperature. In some extreme cases, the horizon crossing can be as late as 25 e-folds before the end of inflation.
was a sizeable component at horizon exit. So we expect that the large wavelength/small $k$ perturbations are more likely to be affected by the radiation (as they cross the horizon and get frozen in early on during inflation).

Let us remind the reader that we will calculate the power spectrum of massless scalar field perturbations, which is not the same as the primordial power spectrum of scalar density perturbations. Nevertheless, as is usual, we will implicitly assume that parts of our results continue to hold for the primordial spectrum of scalar density perturbations in a slow-roll inflationary generalization of the background geometry, which is the relevant quantity responsible for the observable CMBR anisotropy spectrum.

The advantage of having an exact (late-time) solution to the equations of motion is that the power spectrum of fluctuations is precisely given by

$$P_\phi = \lim_{t \to \infty} \frac{k^3}{2\pi^2} |f_k(t)|^2,$$

where the correctly normalized mode function for a massless field ($v = 3/2$) equals

$$f_k = \sqrt{\frac{L \sqrt{1 + 2 \Delta^2} (\sin \eta)^{3/2}}{(1 + \Delta^2 \sin^2 \eta)^{3/4}}} \left[ \tilde{A}_k \frac{G_{k-1/2}^{3/2}}{\Delta_k} + \tilde{B}_k H_{k-1/2}^{3/2} \right].$$

In the late time limit, just like in the pure de Sitter case, the second term will have vanishing contributions, $H_{k-1/2}^{3/2} \to 0$. Using that in the late time limit $G_{k-1/2}^{3/2}$ approaches the associated Legendre polynomial $P_{\chi-1/2}^{3/2}(x)$, with asymptotic behavior

$$\lim_{x \to 1} P_m^v(x) = (1 - x)^{-v/2} \frac{2^{v/2}}{\Gamma(1 - v)},$$

the late time mode function is

$$|f_k|^2 = (1 + 2 \Delta^2)^{3/2} \frac{L^2}{2(\chi^2 - 1)} + \ldots.$$  

To evaluate $\chi$, recall that $(\chi^2 - 1/4) = (1 + 2 \Delta^2)(k^2 - 1/4)$. Solving for $\chi$ we get

$$\chi = k \sqrt{1 + 2 \Delta^2 \frac{(k^2 - 1/4)}{k^2}}.$$  

Thus the exact scalar field power spectrum is given by

$$P_\phi = \frac{k^3}{2\pi^2} |f_k|^2 = (1 + 2 \Delta^2)^{3/2} \frac{L^2}{4\pi^2} \frac{k^3}{(\chi^2 - 1)\chi}.$$  

This is our main result. For large wavenumber $k$ it reduces to

$$P_\phi = \frac{L^2}{4\pi^2} \left( 1 + \frac{1}{k^2} \left[ 1 + \frac{3 \Delta^2}{1 + 2 \Delta^2} \right] + \ldots \right).$$

The overall $\frac{L^2}{4\pi^2}$ term corresponds to the well-known de Sitter result. Notice the cancelation of the radiation dependent factors between the norm of the mode-function and the explicit
dependence. The term in between brackets corresponds to a small red-tilt (more power at small $k$) of the spectrum of scalar field perturbations. At low values of $k$, the departure from scale invariance is pronounced and the radiation component is of the same order (in $1/k$) as the effect due to curvature alone [19]. The red tilt is of course easy to understand. Low $k$-values correspond to those modes which exited the horizon early on and were, therefore, more sensitive to the radiation present and the compact nature of the spatial $S^3$ slices. At large values of $k$, the radiation as well as the spatial curvature has inflated away and the corresponding modes would fail to notice any sizeable effect due to radiation or curvature.

With the exact power spectrum in hand, the next question is naturally whether the deviations from pure de Sitter are observable. Obviously in short inflation scenarios there is a better chance of detecting such an effect. Recall that in closed universes there is a minimal wavenumber corresponding to the size of the spatial $S^3$. The longer inflation lasts, the higher the wavenumbers are that are responsible for the observed CMB, and thus the smaller the effect. Inflation must have lasted long enough to solve the horizon and flatness problems, however, and observability thus hinges on the minimal length allowed by these constraints. Flatness in particular constrains the present scale factor $a_p$ in terms of the observed relative density

$$\Omega_{\text{tot}}(t) = 1 + \frac{1}{a^2 H^2} \rightarrow a_p^2 H_p^2 \geq \frac{1}{\Omega_{\text{obs}} - 1}. \quad (4.8)$$

The largest scale visible in the CMB, $k = a_p H_p$, must therefore be equal to or larger than

$$k^2 \geq a_p^2 H_p^2 = \frac{1}{(\Omega_{\text{obs}} - 1)} \quad (4.9)$$

Using the observed value $\Omega_{\text{obs}} \simeq 1.02$, the maximal correction to the power spectrum is therefore of order $10^{-2}$. A red tilted correction of this size is in practical terms barely distinguishable due to cosmic variance.

5. Discussion and Conclusion

To summarize, a radiation deformed inflationary phase is an interesting extension to the current minimal paradigm. A small amount of radiation is in fact quite generic. Surprisingly, the radiation deformed geometry allows for an exact analysis of probe scalar field perturbations. Calculating the power spectrum of scalar field fluctuations we found that the relative power at low multipoles is enhanced. Both the curvature and the radiation component contribute at the same order and together give rise to a red power spectrum. The observed near critical density of the universe, however, implies that the magnitude of this effect is barely detectable. Finally, in the context of a Hartle-Hawking “tunneling-from-nothing” scenario, the small amount of radiation expected to be present [9, 10, 26] implies that the Euclideanized geometry - a barrel rather than a sphere - does no longer single out a unique vacuum state. For the barrel there is no analyticity constraint that selects the the Bunch-Davies initial state.

We conclude with a (partial) list containing some suggestions on how our results might connect to other work.
• As pointed out, in (short) closed inflation scenarios, the radiation effects are most prevalent. In the context of the string theory landscape of de Sitter vacua, a Hartle-Hawking “tunneling from nothing” origin of the universe might favor short inflation scenarios over inflationary phases with a large number of e-folds because it is far easier to arrange a few e-folds in the landscape. Assuming a random form of the de Sitter potentials in the landscape, a statistical analysis of the possibility of inflation can be done as in [27]. In fact, the proposal in [28] (“living dangerously”) suggests that the physical parameters of our universe should be on the verge of violating important experimental bound (favoring, for example, short inflation).

• In [3] an initial state of the universe is constructed from closed string tachyons. This initial state corresponds to a thermal excitation above the Bunch-Davies initial state. Clearly, due to backreaction of the thermally excited state (i.e. radiation), the geometry will be modified and one might expect the power spectrum to be affected in a similar way as discussed here.

• In brane inflation scenarios, one expects a period of collision and annihilation of branes prior to the last phase of brane inflation directly preceding the reheating of our universe. In such scenarios, again assuming that short inflation is favored over long periods of inflation, we expect that there will generically be a radiation component (due to brane annihilation) during inflation.

• It is interesting to compare our result with the warm inflation scenario [2]. Warm inflation consists of a self interacting inflaton field with a dissipative mechanism that generates a sizeable radiation density during inflation. So instead of having a reheating stage following a supercooled inflationary phase, radiation is constantly generated during inflation. As discussed in [2], for certain models of the inflaton potential and dissipation mechanisms there can be a blue, instead of a red, tilt in the density perturbation for small wavenumbers due to thermal effects.

• The authors of [29] consider a scenario where the initial state of the universe is described by a positive cosmological constant and radiation (motivated by the modified wave function discussed in [9, 10, 26]). The subsequent evolution of the universe is quite interesting. The presence of a periodic potential (motivated by the string landscape) leads to a Bloch band for the wave function, and the universe cascades down this band structure (having originated somewhere near the top of the band as implied by the modified wave function) as it evolves. In the process, the energy in the positive cosmological constant gets converted into radiation until there is too little energy in the cosmological constant to support an inflationary phase. The initial state of a scalar field in this scenario will again be described by the mode functions that we have found in this paper. The subsequent evolution of the mode functions through the cascades warrants a more detailed study.
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A. Appendix: Heun’s and Lamé’s Differential Equation

Heun’s differential equation is of Fuchsian type with regular singularities at $z = 0, 1, a, \infty$. The canonical form of Heun’s differential equation is

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-a)} y = 0 \quad (A.1)$$

where $y$ and $z$ are regarded as complex variables and $\alpha, \beta, \gamma, \delta, \epsilon, q, a$ are complex and arbitrary parameters, and $a \neq 0, 1, \infty$. The first five parameters are linked by the equation

$$\gamma + \delta + \epsilon = \alpha + \beta + 1 \quad (A.2)$$

The exponents at the singular points $z = 0, 1, a, \infty$ are respectively $(0, 1 - \gamma); (0, 1 - \delta); (0, 1 - \epsilon); (\alpha, \beta)$. According to the general theory of Fuchsian equations the sum of these exponents must be equal to 2. This requirement leads to Eq. (A.2). It can be shown that any Fuchsian second-order differential equation with four singularities can be reduced to the form given by Eq. (A.1) which may, therefore, be regarded as the most general form.

Heun’s equation was originally constructed as a generalization of the hypergeometric equation. There are three ways in which Heun’s equation degenerates to hypergeometric equation.

- Setting $a = 1$ and $q = \alpha \beta$ yields the canonical hypergeometric equation for $F(\alpha, \beta, \gamma; z)$.
- Setting $\epsilon = 0, q = a\alpha \beta$ also yields the canonical hypergeometric equation for $F(\alpha, \beta, \gamma; z)$.
- Finally setting $a = q = 0$ in Eq. (A.1) yields the hypergeometric equation for $F(\alpha, \beta, \alpha + \beta - \delta + 1; z)$.

_Lamé’s equation_ is a special case of Heun’s general equation (Eq. (A.1)) for which

$$\gamma = \delta = \epsilon = \frac{1}{2} \quad (A.3)$$

and thus $\alpha + \beta = \frac{1}{2}$. It is common to write

$$\alpha = -\frac{1}{2}\nu, \quad \beta = \frac{1}{2}(\nu + 1), \quad q = -\frac{1}{4}ah \quad (A.4)$$
With these redefinitions the equation takes the canonical form of Lamé’s equation

\[
y''(z) + \frac{1}{2} \left( \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-a} \right) y'(z) + \frac{ah - \nu(\nu + 1)z}{4z(z-1)(z-a)} y(z) = 0 \tag{A.5}
\]

The parameter \( \nu \) is called the order of the equation.

Solutions can be constructed to Heun’s or Lamé’s differential equation both as a Frobenius series in \( z \) and as a series in hypergeometric functions. In this paper we have made use of the series solution. Upon constructing a series solution to Heun’s equation one typically obtains three-term recursion relations. It is, in general, impossible to write down the general solution for three-term recursion relations, unlike two-term recursion relations. One has to construct the coefficients of the different powers order by order. For further details and discussions on Heun’s differential equation see reference [21].

B. Appendix: The Horizon

When we work with global coordinates in a de Sitter spacetime, at late times we simply get the usual flat slice results. So the horizon is \( L^{-1} \). Let us find the late time horizon for the geometry at hand. The event horizon at time \( t \) is given by

\[
l(t) = a(t) \int_{t}^{\infty} \frac{dt'}{a(t')} \tag{B.1}
\]

Using Eq.(2.2) for \( a(t) \), we get

\[
l(t) = \frac{L^{-1}}{\sqrt{1+\Delta^2}} \sqrt{\Delta^2 + \cosh^2(Lt) \left( I(\frac{\pi}{2}, \delta) - I(\theta, \delta) \right)} \tag{B.2}
\]

where \( \theta = \arcsin(\tanh(Lt)) \), \( \delta = \Delta/\sqrt{1+\Delta^2} \), and \( I(\alpha, \beta) \) is the elliptic integral of the first kind defined as

\[
I(\alpha, \beta) = \int_{0}^{\alpha} \frac{d\phi}{\sqrt{1 - \beta^2 \sin^2 \phi}} \tag{B.3}
\]

At late times \( \theta \to \pi/2 \). We shall use the following identities

\[
I(\frac{\pi}{2}, \delta) = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; \delta^2 \right),
\]

\[
I(\theta, \delta) = \theta F \left( \frac{1}{2}, \frac{1}{2}; 1; \delta^2 \right) - \sin \theta \cos \theta \left( a_0 + \frac{2}{3} a_1 \sin^2 \theta + \frac{24}{35} a_2 \sin^4 \theta + \ldots \right) \tag{B.4}
\]

where \( F \left( \frac{1}{2}, \frac{1}{2}; 1; \delta^2 \right) \) is the hypergeometric series, and the coefficients are given by

\[
a_0 = F \left( \frac{1}{2}, \frac{1}{2}; 1; \delta^2 \right) - 1,
\]

\[
a_n = a_{n-1} - \left( \frac{(2n-1)!!}{2^n n!} \right)^2 \delta^{2n}. \tag{B.5}
\]
At late times, \( \sqrt{\Delta^2 + \cosh^2(Lt)} \to e^{Lt}/2 \), and \( \pi/2 - \theta \to 2e^{-Lt} \). So we get the following late time behavior for the horizon

\[
l(t) = \frac{L^{-1}}{\sqrt{1 + \Delta^2}} \left( F\left(\frac{1}{2}, \frac{1}{2}, 1; \delta^2\right) + a_0 + \frac{2}{3}a_1 + \frac{2}{3}a_2 + \ldots \right)
\]

\[
\simeq L^{-1} + \ldots
\]  

(B.6)

where we use the identity \( F(1/2, 1; \delta^2) = 1/\sqrt{1 - \delta^2} \) and the fact that \( \delta^2 = \Delta^2/(1 + \Delta^2) \). So at late times the horizon is the same as the de Sitter horizon \( L^{-1} \).

References


