Relativistic action-at-a-distance interactions: Lagrangian and Hamiltonian to terms of second order

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Abstract

Relativistic systems of particles interacting pairwise at a distance (interactions not mediated by fields) in flat spacetime are studied. It is assumed that the interactions propagate at the speed of light in vacuum and that all masses are scalars under Poincaré transformations. The action functional of the theory depends on multiple times (the proper times of the particles). In the static limit, the theory has three components: a linearly rising potential, a Coulomb-like interaction and a dynamical component to the Poincaré invariant mass. In this Letter we obtain explicitly, to terms of second order, the Lagrangian and the Hamiltonian with all the dynamical variables depending on a single time. Approximate solutions of the relativistic two-body problem are presented.

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After the discovery of the action-at-a-distance formulation of electrodynamics\cite{1} - \cite{7}, several relativistic non-instantaneous action-at-a-distance theories have been investigated \cite{8} - \cite{21}. Instantaneous action-at-a-distance formulations have been studied using a variety of approaches \cite{22} - \cite{30}.

In a recent paper \cite{31} the relativistic action-at-a-distance approach of Wheeler and Feynman \cite{1} - \cite{7} was extended in order to explore what other types of interparticle interactions are allowed in special relativity. It was assumed that the interactions propagate at the speed of light and that the theory can be described by an action that does not depend on the four-vector accelerations or on higher derivatives. A general action functional depending only on the four-vector velocities and relative positions of the particles was investigated. Assuming that the Poincaré invariant parameters that label successive events along the world lines of the particles can be identified with the particles’ proper times and that all masses are scalars, the most general form of the interaction terms in the action was determined explicitly \cite{31}.

The theory presented in \cite{31} is described by an action which depends on multiple times. Quantum mechanics, on the other hand, is based on the hamiltonian formalism (or the Dirac hamiltonian formalism for constrained systems\cite{32} - \cite{35}), which is built on the assumption that one can choose a single time variable to describe the evolution of the whole system. The lagrangian (or covariant) quantization procedure \cite{36} - \cite{38} also assumes a single time variable.

One may attempt to transform a theory with multiple time variables into an equivalent theory with just one time variable, in a given inertial reference frame, using Taylor series expansions involving the particles’ present motions \cite{39}. The problem of finding a single time variable to describe a relativistic system of interacting particles has been investigated by several authors \cite{40} - \cite{42}.

In this Letter we present a reformulation of the theory given in \cite{31} as an action-at-distance theory with a single time variable. We use series expansions up to terms of second order ($\frac{v^2}{c^2}$).

We obtain the approximate Lagrangian for this theory. Approximate expressions for the total energy, total linear momentum and total angular momentum are presented. We also obtain the Hamiltonian to terms of second order.

We present a formula for the Hamiltonian of two interacting particles in the center of momentum reference frame. Using this result, direct predictions for the energy spectrum can be made.

We also present approximate circular orbits solutions of the relativistic two-body problem.

It is well known that an isolated system of $N$ particles interacting electromagnetically may be described, to terms of second order, by the Darwin Lagrangian \cite{43},\cite{44}:

$$L_{Darwin} = -c^2 \sum_i m_i + \frac{1}{2} \sum_i m_i v_i^2 + \frac{1}{8c^2} \sum_i m_i v_i^4 - \frac{1}{2} \sum_i \sum_{j \neq i} \frac{e_i e_j}{r_{ij}} + \frac{1}{4c^2} \sum_i \sum_{j \neq i} \frac{e_i e_j}{r_{ij}} ((\vec{v}_i \vec{v}_j) + (\vec{r}_{ij} \vec{v}_i)(\vec{n}_{ij} \vec{v}_j))$$

(1)

where, $m_i$ and $e_i$ are the mass and electric charge of particle $i$ ($i = 1, 2, ..., N$), $\vec{v}_i$ its velocity, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ the relative position of particle $i$ with respect to particle $j$ and $c$ the speed of light. In \cite{11}, $\vec{n}_{ij} \equiv \frac{\vec{r}_{ij}}{r_{ij}}$.

A generalization of the Darwin Lagrangian that allows the inclusion of other types of interactions was proposed by Woodcock and Havas in the 70’s \cite{45}. Other formulations have recently been studied \cite{46}. For quantum chromodynamics, quark-antiquark interactions to
terms of second order have been studied by several authors \[47-53\]. Gravitational interac-
tions, to terms of second order, are described by the Einstein-Infeld-Hoffmann Lagrangian \[54\].

To terms of fourth order \((v^4c^4)\), ignoring the radiation effects (which for Faraday-Maxwell
electrodynamics appear at third order) the Lagrangian for an isolated system of \(N\) particles
interacting electromagnetically was obtained in \[55, 56\]. For General Relativity, gravita-
tional radiation effects occur at fifth order. To terms of fourth order, the Lagrangian for an
isolated system of \(N\) particles with gravitational interactions was derived in \[57\]. In recent
years, for the gravitational two-body problem, results to sixth order (3rd post-Newtonian
approximation) and higher have been obtained \[58\].

The Lagrangian \((1)\) depends only on the velocities and the relative positions of the
particles, which are functions of a single time variable in a given inertial reference frame \(K\). The Darwin Lagrangian can be derived either from Faraday-Maxwell’s field theory of electrodynamics \[44\] or from the relativistic action-at-a-distance theory of Wheeler and Feynman \[2\].

Wheeler-Feynman’s fully relativistic action-at-a-distance theory of electrodynamics is
described by the Fokker action \[1 - 7\]:

\[
S = -\sum_i m_i c \int d\lambda_i \dot{\zeta}_i^2 \frac{1}{2c} \sum_i \sum_{j \neq i} e_i e_j \int \int d\lambda_i d\lambda_j \xi_{ij} \delta(\rho_{ij}) \quad (2)
\]

In \((2)\), \(\lambda_i\) is a Poincaré invariant parameter that labels the events along the world line
\(z_i^\mu(\lambda_i)\) of particle \(i\) in flat Minkowski spacetime and,

\[
\zeta_i = \dot{z}_i^2
\]

\[
\xi_{ij} = (\dot{z}_i \dot{z}_j)
\]

\[
\rho_{ij} = (z_i - z_j)^2
\]

are scalars under Poincaré transformations.

The metric tensor: \(\eta_{\mu\nu} = diag(+1, -1, -1, -1)\). In \((3)\), \(\dot{z}_i^\mu = \frac{dz_i^\mu}{d\lambda_i}\).

The Dirac delta function in \((2)\) accounts for the interactions propagating at the speed of
light forward and backward in time.

In action-at-a-distance electrodynamics the interactions carry energy and momentum to
and from the particles and may simulate a field between them. However, in the absence of
electrically charged particles outside the system, this fictitious field cannot carry energy or
momentum into or away from the system. If there are no other electrically charged particles
in the universe, the total energy, linear momentum and angular momentum of a system of
point particles are conserved (assuming that only electromagnetic interactions take place).

In \[31\], a general action functional depending on the four-vector velocities and relative
positions of the particles was investigated. Assuming that the Poincaré invariant parameters
that label successive events along the world lines of the particles can be identified with the
particles’ proper times in flat spacetime and that all masses are scalars, the most general
form of the interaction terms in the action was determined explicitly.

The theory obtained in \[31\] is described by the following action functional:

\[
S = -\sum_i m_i c \int d\lambda_i \dot{\zeta}_i^2 \frac{1}{2c} \sum_i \sum_{j \neq i} g_{ij} g_{ji} \int \int d\lambda_i d\lambda_j \delta(\rho_{ij}) \left( \alpha \gamma_{ij} \gamma_{ji} + \beta \xi_{ij} + \gamma \chi_i \chi_j \right) \quad (6)
\]
In (6), $\alpha$, $\beta$ and $\gamma$ are constants, and
$$\gamma_{ij} = (\dot{z}_i(z_j - z_i)) \tag{7}$$

The Poincaré invariant parameters $\lambda_i$ are identified with the proper times ($\tau_i$) of the particles ($\lambda_i = s_i = c\tau_i$) \[31\]:
$$d\lambda_i^2 = \eta_{\mu\nu}dz_i^\mu dz_i^\nu \tag{8}$$

In the static limit the theory has three components: a linearly rising potential, a Coulomb-like interaction and a dynamical component to the Poincaré invariant mass.

There is strong experimental evidence indicating that, for large separations, the interactions between quarks can be effectively described by a linearly rising potential \[51\], \[52\].

Several relativistic generalizations of a linearly rising potential have been studied \[14\] - \[18\]. From quantum chromodynamics, it has been shown that the quark-antiquark bound states are effectively described by a static potential, which is a sum of a linearly rising potential and a Coulomb-like interaction \[59\]. It would be interesting to study the effect of a variable scalar mass for quarks, as given by the formula (10) below or in some other form.

From the action (6), using the variational principle, one can obtain the relativistic equations of motion for a system of $N$ interacting particles \[31\]:
$$\ddot{\bar{m}}_i z_i^\mu = \bar{K}_i^\mu \tag{9}$$

In (9) $\bar{m}_i$ is the dynamical mass of particle $i$ ($i = 1, 2, \ldots, N$):
$$\bar{m}_i = m_i + \gamma g_i \sum_{j \neq i} g_j \int ds_j \delta (\rho_{ij}) \tag{10}$$

and $\bar{K}_i^\mu$ is the four-vector force acting on particle $i$ ($i = 1, 2, \ldots, N$):
$$\bar{K}_i^\mu = g_i \left( F_{i}^{\mu\nu} \dot{z}_i^\nu + \Gamma_{i\alpha\beta}^\mu \dot{z}_i^\alpha \dot{z}_i^\beta \right) \tag{11}$$

where $F_{i}^{\mu\nu}$ is an antisymmetric tensor ($F_{i}^{\nu\mu} = -F_{i}^{\mu\nu}$) and $\Gamma_{i\alpha\beta}^\mu$ is a symmetric tensor ($\Gamma_{i\alpha\beta}^\mu = \Gamma_{i\beta\alpha}^\mu$) in flat spacetime. They are given by the expressions \[31\]:

$$F_{i}^{\mu\nu} = \sum_{j \neq i} \frac{g_j}{c^2} \int ds_j \frac{\delta (\rho_{ij})}{\gamma_{ji}^2} \left[ \alpha \gamma_{ji}^2 \left( \dot{z}_i^\mu - \dot{z}_j^\mu \right) \dot{z}_j^\nu - \dot{z}_i^\nu (\dot{z}_i^\mu - \dot{z}_j^\mu) \right]$$
$$+ \beta \left[ \left( \dot{z}_i^\mu - \dot{z}_j^\mu \right) \ddot{z}_i^\nu - \ddot{z}_j^\nu (\dot{z}_i^\mu - \dot{z}_j^\mu) \right] (1 - (\ddot{z}_j(z_i - z_j)))$$
$$+ \left( \dot{z}_i^\mu - \dot{z}_j^\mu \right) \ddot{z}_j^\nu - \ddot{z}_j^\nu (\dot{z}_i^\mu - \dot{z}_j^\mu) \right] \gamma_{ji} \
$$

$$\Gamma_{i\alpha\beta}^\mu = \gamma \sum_{j \neq i} \frac{g_j}{c^2} \int ds_j \frac{\delta (\rho_{ij})}{\gamma_{ji}^2}$$
$$\left[ \left( \dot{z}_i^\mu - \dot{z}_j^\mu \right) \eta_{\alpha\beta} - \frac{1}{2} \left( \delta_{\alpha}^\mu (\dot{z}_i\beta - \dot{z}_j\beta) + \delta_{\beta}^\mu (\dot{z}_i\alpha - \dot{z}_j\alpha) \right) \right] (1 - (\ddot{z}_j(z_i - z_j)))$$
$$- \left( \dot{z}_j^\alpha \dot{z}_j^\beta + \delta_{\beta}^\mu \dot{z}_j^\alpha \right) \eta_{\alpha\beta} \right] \gamma_{ji} \tag{12}$$
It should be emphasized that the action functional (6) is the most general expression one can obtain from the following four assumptions [31]:

1. the action does not depend on the four-vector accelerations or on higher derivatives,
2. the interactions propagate at the speed of light in vacuum,
3. the Poincaré invariant parameters that label successive events along the world lines of the particles can be identified as their proper times in flat spacetime,
4. all masses are scalars under Poincaré transformations.

As it can be seen from (6), significant constraints on the form of the action functional result from these assumptions. The theory depends only on three undetermined constants. It reduces to the Wheeler-Feynman theory of electrodynamics when two of the constants (\(\alpha\) and \(\gamma\)) are assumed to be equal to zero.

Based on these results, it is natural to ask how the theory described by the action (6) relates to physical phenomena that may be observed experimentally. One possibility is to look at the theory as a modified theory of electrodynamics and search for effects arising from the new interaction terms (assuming that \(\alpha\) or \(\gamma\) are not exactly equal to zero). Another possibility is to consider the theory as an incomplete model for the description of strong nuclear interactions (since the theory incorporates a linearly rising potential, but does not take into account the internal degrees of freedom of the particles).

The fully relativistic equations of motion (9-13) admit exact circular solutions for any number of particles. For electrodynamics such solutions were obtained in [6].

Our purpose now is to obtain, to terms of second order, the Lagrangian, the Hamiltonian and the equations of motion, with all dynamical variables depending on a single time in an inertial reference frame \(K\). To this end, it is convenient to rewrite the action functional (6) in the following form:

\[
S = -\sum_i m_i c^2 \int dt_i \left( 1 - \frac{v_i^2}{c^2} \right)^{\frac{3}{2}} - \frac{c}{2} \sum_i \sum_{j \neq i} g_i g_j \int \int dt_i dt_j \delta \left( c^2 (t_i - t_j)^2 - (\vec{r}_i - \vec{r}_j)^2 \right) F_{ij}
\]

(14)

where,

\[
F_{ij} = \alpha \left( c(t_i - t_j) - \frac{(\vec{v}_j (\vec{r}_i - \vec{r}_j))}{c} \right) \left( c(t_j - t_i) - \frac{(\vec{v}_i (\vec{r}_j - \vec{r}_i))}{c} \right) + \beta \left( 1 - \frac{(\vec{v}_i \cdot \vec{v}_j)}{c^2} \right) + \gamma \left( 1 - \frac{v_i^2}{c^2} \right)^{\frac{1}{2}} \left( 1 - \frac{v_j^2}{c^2} \right)^{\frac{1}{2}}
\]

(15)

From (14, 15) we can obtain the action for an individual particle \(i\), assuming a fixed motion of the other particles, as follows:

\[
S_i = \int dt \left[ -m_i c^2 \left( 1 - \frac{v_i^2}{c^2} \right)^{\frac{3}{2}} - c g_i \sum_{j \neq i} g_j \int dt_j \delta \left( c^2 (t - t_j)^2 - (\vec{r}_i - \vec{r}_j)^2 \right) F_{ij} \right]
\]

(16)

where in (16) \(F_{ij}\) depends on \(t \equiv t_i\) and on \(t_j\) (\(j \neq i\)).

The Dirac delta function can be expressed as follows [7]:
\[
\delta \left( c^2(t - t_j)^2 - (\vec{r}_i - \vec{r}_j)^2 \right) = \frac{1}{2c} \left( \frac{\delta(t_j - t_j^{(i,-)})}{R_{ij}^{\text{ret}} - (\vec{r}_i^{\text{ret}}e_j^{(-)})/c} + \frac{\delta(t_j - t_j^{(i,+)})}{R_{ij}^{\text{adv}} + (\vec{r}_i^{\text{adv}}e_j^{(+)})/c} \right) \quad (17)
\]

In (17), \( t_j^{(i,s)} \) (\( s = -, + \)) are the two roots of the equation:
\[
c^2(t - t_j)^2 - (\vec{r}_i(t) - \vec{r}_j(t_j))^2 = 0 \quad (18)
\]
and,
\[
R_{ij}^{\text{ret}} = c \left( t_j - t_j^{(i,-)} \right) \quad (19)
\]
\[
R_{ij}^{\text{adv}} = c \left( t_j^{(i,+)} - t \right) \quad (20)
\]
\[
\vec{R}_{ij}^{\text{ret}} = \vec{r}_i - \vec{r}_j^{(-)} \quad (21)
\]
\[
\vec{R}_{ij}^{\text{adv}} = \vec{r}_i - \vec{r}_j^{(+)} \quad (22)
\]

\( t - t_j^{(i,-)} \) is the time it takes for a signal to travel forward in time at the speed of light from particle \( j \) to particle \( i \) in \( K \).
\( t_j^{(i,+)} - t \) is the time it takes for a signal to travel backward in time at the speed of light from particle \( j \) to particle \( i \) in \( K \).

The action functional (16) for particle \( i \) (assuming a fixed motion of the other particles) can therefore be rewritten as:
\[
S_i = \int dt \left[ -m_ic^2 \left( 1 - \frac{v_i^2}{c^2} \right)^{\frac{3}{2}} - \frac{g_i}{2} \sum_{j \neq i} g_j \left( \frac{F_{ij}^{(-)}}{R_{ij}^{\text{ret}} - (\vec{r}_i^{\text{ret}}e_j^{(-)})/c} + \frac{F_{ij}^{(+)}}{R_{ij}^{\text{adv}} + (\vec{r}_i^{\text{adv}}e_j^{(+)})/c} \right) \right] \quad (23)
\]
where,
\[
F_{ij}^{(-)} = -\alpha \left( R_{ij}^{\text{ret}} - (\vec{r}_i^{\text{ret}}e_j^{(-)})/c \right) \left( R_{ij}^{\text{ret}} - (\vec{r}_i^{\text{ret}}e_j^{(-)})/c \right)
+ \beta \left( 1 - \frac{(\vec{r}_i e_j^{(-)})^2}{c^2} \right) + \gamma \left( 1 - \frac{v_i^2}{c^2} \right)^{\frac{1}{2}} \left( 1 - \frac{v_j^{(-)^2}}{c^2} \right)^{\frac{1}{2}} \quad (24)
\]
\[ F_{ij}^{(+)} = -\alpha \left( R_{ij}^{adv} + \frac{R_{ij}^{adv} v_j^2}{c} \right) \left( R_{ij}^{adv} + \frac{R_{ij}^{adv} v_j^2}{c} \right) \]
\[ + \beta \left( 1 - \frac{(\vec{v}_i \vec{v}_j^2)}{c^2} \right) + \gamma \left( 1 - \frac{v_i^2}{c^2} \right) \left( 1 - \frac{v_j^2}{c^2} \right) ^{\frac{3}{2}} \]  

(25)

To terms of second order we can write:

\[-m_i c^2 \left( 1 - \frac{v_i^2}{c^2} \right) ^{\frac{1}{2}} \frac{g_i}{2} \sum_{j \neq i} g_j \left( R_{ij}^{(+) -} \right) + \left( R_{ij}^{cont} \right) \left( R_{ij}^{(+)} + \frac{R_{ij}^{adv} v_j^2}{c} \right) \approx \]
\[-m_i c^2 + \frac{m_i v_i^2}{2} + \frac{m_i v_i^4}{8c^2} - g_i \sum_{j \neq i} g_j \left( -\alpha r_{ij} \left( 1 - \frac{(\vec{v}_i \vec{v}_j^2)}{c^2} \right) + \frac{v_j^2}{2c^2} + \frac{v_i^2}{2c^2} - \frac{r_{ij} v_j^2}{2c^2} \right) \]
\[ + \frac{\beta}{r_{ij}} \left( 1 - \frac{(\vec{v}_i \vec{v}_j^2)}{c^2} \right) - \frac{v_j^2}{2c^2} + \frac{v_j^2}{2c^2} - \frac{r_{ij} v_j^2}{2c^2} \]
\[ + \frac{\gamma}{r_{ij}} \left( 1 - \frac{v_i^2}{2c^2} - \frac{(\vec{v}_i \vec{v}_j^2)}{2c^2} \right) \]  

(26)

We find the approximate action for particle \(i\) to be (integration by parts allows to obtain a formula without explicit dependence on the accelerations):

\[ S_i \approx \int dt L_i \]  

(27)

where,

\[ L_i = -m_i c^2 + \frac{m_i v_i^2}{2} + \frac{m_i v_i^4}{8c^2} + g_i \sum_{j \neq i} g_j \left( \alpha r_{ij} - \frac{(\beta + \gamma)}{r_{ij}} \right) \]
\[ + \frac{g_i}{2c^2} \sum_{j \neq i} g_j \left[ -\alpha r_{ij} \left( \frac{(\beta - \gamma)}{r_{ij}} \right) (\vec{v}_i \vec{v}_j) + \left( \alpha r_{ij} + \frac{(\beta + \gamma)}{r_{ij}} \right) (\vec{v}_i \vec{v}_j) (\vec{n}_i \vec{v}_j) \right] \]
\[ + \frac{\gamma}{r_{ij}} \left( v_i^2 + v_j^2 \right) \]  

(28)

From (28) we can find the Lagrangian of the whole system of \(N\) interacting particles. It is given by the expression:

\[ L = -c^2 \sum_i m_i + \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} + \frac{1}{2} \sum_i \sum_{j \neq i} g_i g_j \left( \alpha r_{ij} - \frac{(\beta + \gamma)}{r_{ij}} \right) \]
\[ + \frac{1}{4c^2} \sum_i \sum_{j \neq i} g_i g_j \left[ \left( -\alpha r_{ij} + \frac{(\beta - \gamma)}{r_{ij}} \right) (\vec{u}_i \vec{v}_j) + \left( \alpha r_{ij} + \frac{(\beta + \gamma)}{r_{ij}} \right) (\vec{n}_{ij} \vec{v}_i) + \frac{\gamma}{r_{ij}} (v_i^2 + v_j^2) \right] \]

The Lagrangian (29) reduces to the Darwin Lagrangian (1) if \( \alpha = \gamma = 0 \) and \( \beta = 1 \).

From the Lagrangian (29) one can obtain the equations of motion to terms of second order:

\[
\begin{align*}
(m_i + \frac{\gamma g_i}{c^2} \sum_{j \neq i} \frac{g_j}{r_{ij}}) \left( 1 + \frac{v_i^2}{2c^2} \right) \ddot{a}_i + \frac{(\vec{v}_i \vec{a}_i)}{c^2} &= \\
g_i \sum_{j \neq i} \frac{g_j}{r_{ij}} \left[ \vec{r}_{ij} \left( \alpha + \frac{(\beta + \gamma)}{r_{ij}^2} \right) \left( 1 - \frac{(\vec{r}_{ij} \vec{a}_j)}{2c^2} \right) - \frac{\gamma}{r_{ij}} \frac{v_i^2}{2c^2} \right] \\
+ \left( \alpha + \frac{\beta}{r_{ij}^2} \right) \left( \frac{v_i^2}{2c^2} - \frac{(\vec{v}_i \vec{a}_i)}{c^2} \right) - \left( \alpha + \frac{3(\beta + \gamma)}{r_{ij}^2} \right) \frac{(\vec{n}_{ij} \vec{v}_i)^2}{2c^2} \right) \\
+ \left( \alpha + \frac{(\gamma - \beta)}{r_{ij}^2} \right) \frac{\vec{a}_j r_{ij}^2}{2c^2} \\
\end{align*}
\] (30)

The approximate equations (30) can also be derived directly from the exact equations of motion (32 - 33).

From the Lagrangian (29) we can obtain the total energy \( E \), the total linear momentum \( \vec{P} \) and the total angular momentum \( \vec{L} \), of a system of \( N \) interacting relativistic particles, to terms of second order:

\[
\begin{align*}
E &= c^2 \sum_i m_i + \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} - \frac{1}{2} \sum_i \sum_{j \neq i} g_i g_j \left( \alpha r_{ij} - \frac{(\beta + \gamma)}{r_{ij}} \right) \\
&+ \frac{1}{4c^2} \sum_i \sum_{j \neq i} g_i g_j \left[ \left( -\alpha r_{ij} + \frac{(\beta - \gamma)}{r_{ij}} \right) (\vec{u}_i \vec{v}_j) + \left( \alpha r_{ij} + \frac{(\beta + \gamma)}{r_{ij}} \right) (\vec{n}_{ij} \vec{v}_i) + \frac{\gamma}{r_{ij}} (v_i^2 + v_j^2) \right] \tag{31}
\end{align*}
\]

\[
\begin{align*}
\vec{P} &= \sum_i \left( m_i \left( 1 + \frac{v_i^2}{2c^2} \right) + \frac{\gamma g_i}{c^2} \sum_{j \neq i} \frac{g_j}{r_{ij}} \right) \vec{v}_i \\
&+ \frac{1}{2c^2} \sum_i \sum_{j \neq i} g_i g_j \left[ \left( -\alpha r_{ij} + \frac{(\beta - \gamma)}{r_{ij}} \right) \vec{v}_j + \left( \alpha r_{ij} + \frac{(\beta + \gamma)}{r_{ij}} \right) (\vec{n}_{ij} \vec{v}_i) \vec{n}_{ij} \right] \tag{32}
\end{align*}
\]

\[
\begin{align*}
\vec{L} &= \sum_i \left( m_i \left( 1 + \frac{v_i^2}{2c^2} \right) + \frac{\gamma g_i}{c^2} \sum_{j \neq i} \frac{g_j}{r_{ij}} \right) (\vec{r}_i \times \vec{v}_i) \\
\end{align*}
\]
\[ + \frac{1}{2c^2} \sum_i \sum_{j \neq i} g_i g_j \left[ \left( -\alpha r_{ij} + \frac{\beta - \gamma}{r_{ij}} \right) (\vec{r}_i \times \vec{v}_j) - \left( \alpha r_{ij} + \frac{\beta + \gamma}{r_{ij}} \right) \right] (\vec{n}_{ij} \vec{v}_j) (\vec{n}_i \times \vec{n}_j) \]

In order to develop the Hamiltonian formulation of the theory, we find the conjugate momenta:

\[ \vec{p}_i = \frac{\partial L}{\partial \vec{v}_i} = m_i \left( 1 + \frac{v_i^2}{2c^2} + \frac{\gamma g_i}{m_i c^2} \sum_{j \neq i} g_j \right) \vec{v}_i \]

+ \frac{g_i}{2c^2} \sum_{j \neq i} \frac{g_j}{m_j} \left[ \left( -\alpha r_{ij} + \frac{\beta - \gamma}{r_{ij}} \right) \vec{v}_j + \left( \alpha r_{ij} + \frac{\beta + \gamma}{r_{ij}} \right) (\vec{n}_{ij} \vec{v}_j) \right] \vec{n}_i \tag{33} \]

The Hamiltonian (to terms of second order) for \( N \) interacting particles is given by the formula:

\[ H = c^2 \sum_i m_i + \sum_i \frac{p_i^2}{2m_i} - \sum_i \frac{p_i^4}{8m_i^3 c^2} - \frac{1}{2} \sum_i \sum_{j \neq i} g_i g_j \left( \alpha r_{ij} - \frac{(\beta + \gamma)}{r_{ij}} \right) \]

- \frac{1}{4c^2} \sum_i \sum_{j \neq i} \frac{g_i g_j}{m_i m_j} \left[ \left( -\alpha r_{ij} + \frac{(\beta - \gamma)}{r_{ij}} \right) (\vec{p}_i \vec{p}_j) + \left( \alpha r_{ij} + \frac{(\beta + \gamma)}{r_{ij}} \right) (\vec{n}_{ij} \vec{p}_i) (\vec{n}_{ij} \vec{p}_j) \right] \]

- \frac{\gamma}{4c^2} \sum_i \sum_{j \neq i} \frac{g_i g_j}{r_{ij}} \left( \frac{p_i^2}{m_i^2} + \frac{p_j^2}{m_j^2} \right) \tag{34} \]

For \( N = 2 \) in the center of momentum frame (c.m.f.) (\( \vec{p}_1 = -\vec{p}_2 = \vec{p} \)) the Hamiltonian \( H \) takes the simple form:

\[ H = c^2 (m_1 + m_2) + \frac{p^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{p^4}{8c^2} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) - g_1 g_2 \left( \alpha r - \frac{(\beta + \gamma)}{r} \right) \]

- \frac{p^4}{8c^2} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) - g_1 g_2 \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \frac{\gamma p^2}{r} \]

+ \frac{g_1 g_2}{2c^2 m_1 m_2} \left( -\alpha r + \frac{(\beta - \gamma)}{r} \right) p^2 + \left( \alpha r + \frac{(\beta + \gamma)}{r} \right) (\vec{n} \vec{p})^2 \tag{35} \]

Direct predictions for the energy spectrum can be made from \( H \).

Comparing our expression for the Hamiltonian \( H \) with the general expression presented in \[49\], we can determine the Lorentz structure of the interactions.

The general formula for the semirelativistic two-particle Hamiltonian presented in \[49\] can be written in the c.m.f. as follows:

\[ H = c^2 (m_1 + m_2) + \frac{p^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{p^4}{8c^2} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) + W \tag{36} \]

\[ H = c^2 (m_1 + m_2) + \frac{p^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{p^4}{8c^2} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) + W \tag{37} \]
where $W$ is the sum of vector and scalar interactions:

$$W = W_V + W_S$$

(38)

Given a potential $V$ for vector exchange, $W_V$ in the c.m.f. is given by the formula:

$$W_V = V(r) + \frac{1}{2m_1m_2c^2} \left( (V(r)p^2 - rV'(r)(\vec{n}\vec{p})^2 \right)$$

(39)

Given a potential $S$ for scalar exchange, $W_S$ in the c.m.f. can be written as follows:

$$W_S = S(r) - S(r) \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \frac{p^2}{2c^2} - \frac{1}{2m_1m_2c^2} \left( S(r)p^2 + rS'(r)(\vec{n}\vec{p})^2 \right)$$

(40)

Comparing (36) with (37 - 40) we conclude that, for the theory presented in this Letter, the potentials corresponding to vector and scalar interactions are given by the following expressions:

$$V = g_1g_2 \left( -\alpha r + \frac{\beta}{r} \right)$$

(41)

$$S = \frac{g_1g_2\gamma}{r}$$

(42)

The interaction responsible for the variable masses is scalar in nature. The linearly rising potential is purely a vector potential, and so is the Coulomb-like potential. All this can also be seen directly from the fully relativistic action (4).

For $N = 2$, the approximate equations of motion (30) admit circular orbits solutions with the angular frequency $\omega$ and the distance between the particles $r = r_1 + r_2$ obeying the generalized Kepler relation:

$$\omega^2 = -\frac{g_1g_2}{\mu r^3} \left( \alpha r^2 + \beta + \gamma \right) \left[ 1 - \frac{g_1g_2}{\mu c^2 r} \left( 3\alpha r^2 \beta + \gamma \nu - \frac{1}{2} + \gamma \nu \right) \right]$$

(43)

where,

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

(44)

$$\nu = \frac{m_1m_2}{(m_1 + m_2)^2}$$

(45)

For two-body circular orbits the total energy of the system in the c.m.f., to terms of second order, is given by the formula:

$$E = c^2(m_1+m_2) + \frac{g_1g_2}{2r} \left[ -3\alpha r^2 + \beta + \gamma + \frac{g_1g_2}{4c^2\mu r} \left( (\alpha r^2 + \beta + \gamma)^2 (1 - \nu) - 2(\alpha r^2 + \beta + \gamma)\gamma \nu - \gamma^2 (1 + \nu) \right) \right]$$

(46)

The total angular momentum in the c.m.f, to terms of second order, is:

$$L = \left| g_1g_2(\alpha r^2 + \beta + \gamma)\mu r \right|^\frac{1}{2} \left[ 1 + \frac{g_1g_2}{4c^2\mu r} \left[ 4\alpha r^2 - \beta + 2\gamma \right] \right]$$

(47)

One can use the procedure presented in this paper to obtain results at higher orders in the series expansions.
At the quantum level one can describe a system of two spinning particles interacting electromagnetically, including second order terms, by the Breit equation [60]-[62]. An extended Breit equation may be devised in agreement with the Lagrangian (29) to account for the dynamical contributions to the masses and the effect of a linearly rising potential.

It may also be possible to obtain a fully relativistic two-body Dirac equation from the action (6), by extending the results of Crater and Van Alstine [62] to include the additional interactions explored in this Letter.

References