Asymptotic Expansions of Feynman Amplitudes in a Generic Covariant Gauge

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We show in this paper how to construct Symanzik polynomials and the Schwinger parametric representation of Feynman amplitudes for gauge theories in an unspecified covariant gauge. The complete Mellin representation of such amplitudes is then established in terms of invariants (squared sums of external momenta and squared masses). From the scaling of the invariants by a parameter we extend for the present situation a theorem on asymptotic expansions, previously proven for the case of scalar field theories, valid for both ultraviolet and infrared behaviors of Feynman amplitudes.

I. INTRODUCTION

Asymptotic expansions for amplitudes related to Feynman diagrams have been investigated for a long time and continue to be a topic of interest. This is specially true in what regards the infrared and ultraviolet behaviors of gauge theories, studies of critical phenomena and attempts to understand confinement in the field-theoretical framework of quantum chromodynamics. With respect to gauge theories, whose interaction mediator field has a propagator of the form (in Euclidean space)

\[
\frac{1}{k^2 + \mu^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + \mu^2} \right),
\]

where \(\mu\) is an infrared mass regulator and \(\xi\) is the gauge-fixing parameter, amplitude computations are usually performed by choosing, for simplicity, the \((\xi = 1)\) Feynman gauge. This has been done, for example, in a recent paper directly connected to the subject of our interest \(\text{[1]}\), where its authors were able to produce asymptotic expansions for some 2- and 3-loop vertex corrections in QED, using a method based on the Mellin–Barnes transform of Feynman amplitudes. There are, however, some circumstances in which it is desirable to work with other choices of \(\xi\), normally involving non-gauge-invariant quantities. This is the case, for instance, of recent discussions on gluon condensates and form factors for the gluon and ghost propagators in infrared QCD \(\text{[2, 3]}\), whose computations are mostly done in the Landau gauge, \(\xi = 0\). Also, in the context of noncommutative supersymmetric field theories, there are instances in which radiative corrections cannot be calculated in the Feynman gauge \(\text{[4]}\). The intrinsic complexity of such studies have led us to consider the usefulness of obtaining asymptotic expressions in a general covariant regime, which includes, among others, the Landau gauge. We shall do this by using the method of the complete Mellin representation of Feynman amplitudes.

Thus, in the present paper, we wish to extend the method when one considers the amplitudes in a generic covariant gauge. We here intend to take up the long tradition that started in the 1970’s with the papers \(\text{[5, 6, 7]}\) within the Bogoliubov–Parasiuk–Hepp–Zimmermann formalism. Their approach is based on the Schwinger representation of amplitudes. This representation has the advantage of being expressed in terms of so-called Symanzik polynomials in the Schwinger parameters, which can be deduced in a straightforward way from the topological characteristics of the diagrams one is interested in. From the topological point of view, the diagrams are considered as a set of interconnected lines, ignoring distinctions between those related to gauge fields and those to matter fields. These ‘topological formulas’ are thus determined from inspection of the 1-trees and 2-trees constructed from the diagram by omitting the adequate number of lines \(\text{[8, 9]}\). The Mellin transform technique was then used to prove theorems leading to the asymptotic expansion of convergent Feynman diagrams. External momenta are scaled by some parameter \(\lambda\) and through the Mellin transform with respect to the scaling parameter, as the limit of \(\lambda\) going to infinity is taken, one can prove the existence of an asymptotic series in powers of \(\lambda\) and powers of logarithms of \(\lambda\), and the series coefficients computed. Divergent diagrams were treated in \(\text{[10]}\).

In the subsequent years, the subject had further advancements, with the inclusion of the internal squared masses as possible invariants which may be scaled, in order to study the infrared behavior of the amplitudes as well \(\text{[12]}\). There the concept of ‘FINE’ polynomials was introduced, that is, those having the property of being factorizable in each Hepp sector \(\text{[11]}\) of the variables. It was then argued that they may be ‘desingularized’, which means that the integrand has a meromorphic structure, so that the residues of its various poles may be obtained. However, this is not the general situation, and in many diagrams the FINE property does not occur, which is required to hold by the Mellin...
transform method. A first solution to this problem was presented in Ref. [12] by introducing the so-called ‘multiple Mellin’ representation, which consists essentially in splitting the Symanzik polynomials in a certain number of pieces, each one of which having the FINE property. Then, after scaling by the parameter \( \lambda \), an asymptotic expansion can be obtained as a sum over all Hepp sectors. This is always possible to do, if one adopts, as was done in Refs. [13, 14, 15], the extreme point of view which consists in splitting the Symanzik polynomials in all its monomials. Moreover, this apparent complication is compensated by the fact that one can dispense with the use of Hepp sectors altogether. This came to be known as the ‘complete Mellin’ (CM) representation.

The CM representation provides a general proof of the existence of an asymptotic expansion in powers and powers of logarithms of the scaling parameter and no change of variables is needed: not only the Schwinger variables provide a desingularization, but the integrations over such variables can be explicitly performed, and we are left with the pure geometrical study of convex polyhedra in the Mellin variables. It has been proven [13] that the same results of [12] are obtained in a simpler way, and asymptotic expansions are computed in a more compact form, without any division of the integral into Hepp sectors. Another feature of the CM representation is that it allows a unified treatment of the asymptotic behavior of both ultraviolet convergent and divergent amplitudes. Indeed, as is shown in Refs. [13, 14], the renormalization procedure does not alter the algebraic structure of integrands in the CM representation. It only changes the set of relevant integration domains in the Mellin variable space.

Other applications of the method included a study of the infrared behavior of amplitudes relevant to critical phenomena [10] and of dimensional regularization [15].

In Sec. 2 of the present paper, we show how to write down a Schwinger representation for a Feynman amplitude, when the gauge-fixing parameter \( \xi \) is left unchosen, through the use of a ‘topological formula’. In this case, then, as seen in eq. (1), every gauge propagator in a diagram contributes with a second term with momentum dependence \((k^2 + \mu^2)^{-2}\), in addition to the \((k^2 + \mu^2)^{-1}\) dependence of the first term, the latter one being the same that occurs in scalar theories and gauge theories in the Feynman gauge. This fact entails the introduction of two Schwinger parameters to the same gauge-field internal line of the diagram. Also, one should consider the momentum dependence occurring in the numerator, coming not only from the gauge-field propagator, but also from those of fermionic particles, and, in principle, from momentum-dependent vertices and counterterms. Numerators, however, modify the topological formulas in a systematic way [10, 11, 12].

Sec. 3 reviews the complete Mellin transform method, and we demonstrate in Sec. 4 the existence of asymptotic series for gauge theories in our generalized case, which again has the same structure in comparison with the scalar one. In Sec. 5 we present our conclusions.

The Euclidean metric will be used throughout the paper.

II. PARAMETRIC REPRESENTATION AND TOPOLOGICAL FORMULAS

We start by considering an arbitrary connected Feynman diagram \( G \) in a purely scalar theory and let us denote by \( I_G \) its contribution to some \( n \)-point Green’s function in momentum space. It is a function of the relativistic invariants built from the external momenta and squared masses. Topologically, the Feynman diagram \( G \) is a set of \( I \) internal lines, \( V \) vertices and \( L \) loops and these quantities are well-known to satisfy the relation \( L = I - V + 1 \). A \( q \)-tree of the diagram \( G \) is a subdiagram of \( G \) having \( q \) connected components, without loops, and linking all the vertices of \( G \). Of particular interest for us are the cases of \( q = 1 \) (1-trees) and \( q = 2 \) (2-trees).

Let us define \( P_v \) as the sum of all external momenta that end at the vertex \( v \) and let us assume that they are such that \( \sum_{v=1}^{V} P_v = 0 \). Apart from global momentum conservation and overall constant factors coming from the vertices (for instance, such as \((−i\lambda)^V \) in \( \lambda \phi^4/4! \) theory) or from a possible symmetry factor, the arbitrary amplitude reads

\[
I_G(P_v; m_l^2) = \int \prod_{l=1}^{L} \frac{d^d k_l}{(2\pi)^d} \frac{1}{k_l^2 + m_l^2} \prod_{v=1}^{V} \int (2\pi)^d d^d \phi(\epsilon_{vl} k_l).
\]

In this formula, we integrate the Euclidean propagators (with possibly different masses \( m_l \)) over all momenta \( k_l \) associated to each internal line \( l \). The last factor ensures momentum conservation at each vertex. In it we have introduced the so-called incidence matrix, which is defined as

\[
\epsilon_{vl} = \begin{cases} 
+1 & \text{if the vertex } v \text{ is the starting point of line } l \\
-1 & \text{if the vertex } v \text{ is the ending point of line } l \\
0 & \text{otherwise}.
\end{cases}
\]

The Schwinger parametric integral \( \alpha \)-representation is introduced by expressing each propagator in the form

\[
\frac{1}{k^2 + m^2} = \int_0^{\infty} d\alpha \, e^{-\alpha(k^2 + m^2)}
\]
and the delta function is expressed as an integral as well,

$$
(2\pi)^d \delta^d \left( P_v - \sum_{l=1}^t \epsilon_{vl} k_l \right) = \int d^d y \ e^{-iy \cdot (P_v - \sum_{l=1}^t \epsilon_{vl} k_l)}. \tag{5}
$$

Upon integration over each $k_l$ and then over the $y$ variables, the amplitude is finally expressed as

$$
I_G(P_v; m_f^2) = \int_0^\infty \prod_{l=1}^t \frac{d\alpha_l}{U^{d/2}(\alpha)} \prod_{l=1}^t \frac{\mathcal{P}(k_l)}{k_l^2 + m_f^2} \prod_{l=1}^t (2\pi)^d \delta^d \left( P_v - \sum_{l=1}^t \epsilon_{vl} k_l \right), \tag{6}
$$

where $U$ and $N$ are homogeneous polynomials in the $\alpha_l$ parameters, of order $L$ and $L + 1$, respectively, constructed with topological relations defined by the 1- and 2-trees of the diagram $G$:

$$
U(\alpha) = \sum_T \prod_{l \in T} \alpha_l, \quad N(\alpha) = \sum_K s_K \left( \prod_{l \in K} \alpha_l \right). \tag{7}
$$

The symbols $\sum_T$ and $\sum_K$ denote, respectively, summation over the 1-trees $T$ and 2-trees $K$ of $G$. $s_K$ is the cut-invariant of one of the two connected pieces of the 2-tree $K$, that is, the square of the total external momentum $P_v$ entering one of pieces of $K$ (any one of them equivalently, by momentum conservation). $U$ and $N$ are known in the literature as the Symanzik polynomials.

Let us consider now a Yukawa-type scalar–fermion theory. A generic amplitude contains a certain number of fermion propagators which introduce a momentum-dependent polynomial $\mathcal{P}(k_l)$ as a numerator in its expression, $m_f$ being the fermion mass. Differently from eq. (2), the amplitude now is expressed as

$$
I_G(P_v; m_f^2) = \int \left( \prod_{l=1}^t \frac{d^d k_l}{(2\pi)^d} \right) \mathcal{P}(k_l) \prod_{l=1}^t \frac{1}{k_l^2 + m_f^2} \prod_{l=1}^t (2\pi)^d \delta^d \left( P_v - \sum_{l=1}^t \epsilon_{vl} k_l \right). \tag{8}
$$

The internal line index $l$ runs indistinguishably over scalar or fermionic lines and $m_f$ is to be taken either as the scalar field mass or the fermion mass, accordingly. If we take the integral representation \[5, 7, 14\], the amplitude now is expressed as

$$
\frac{\mathcal{P}(k_l)}{k_l^2 + m_f^2} = \int_0^\infty \frac{d\alpha_l}{U^{d/2}(\alpha)} \left( -\frac{1}{\alpha_l} \frac{\partial}{\partial \alpha_l} \right) e^{-\alpha_l (k_l^2 + m_f^2 + k_l \cdot \zeta)} \bigg|_{\zeta = 0}, \tag{9}
$$

the integrations in the amplitude may be performed in a similar way as in the purely scalar case, with the result that

$$
I_G(P_v; m_f^2) = \int_0^\infty \prod_{l=1}^t \frac{d\alpha_l}{U^{d/2}(\alpha)} \left( -\frac{1}{\alpha_l} \frac{\partial}{\partial \alpha_l} \right) e^{-\sum_i \alpha_l \bar{m}_l^2 \cdot \bar{m}_l \cdot \zeta} e^{-N(s_K; \alpha)/U(\alpha)} \bigg|_{\zeta = 0}, \tag{10}
$$

where

$$
\bar{m}_l^2 = m_f^2 - \zeta^2/4 \tag{11}
$$

and $s_K$ is the same cut-invariant of the 2-tree $K$ that occurs in \[7\], for which the external momenta $P_v$ is replaced by

$$
P_v = P_v + \sum_{l} \epsilon_{vl} \bar{G}_l/2. \tag{12}
$$

Let us now pass to our main subject, fermionic gauge theories. For a given generic diagram $G$, let us call $F$ the set of its (fermionic) matter field lines and $\Gamma$ the set of its gauge-field propagators. The set of all lines in $G$ is denoted by $E = F \cup \Gamma$. The number of lines in each of these sets are denoted by $N_F$ and $N_{\Gamma}$, respectively, and therefore the total number of propagators is $N_E = N_F + N_{\Gamma}$. In an arbitrary gauge, using the gauge-field propagator \[10\], we distinguish two terms in the amplitude $I_G$, related to the diagram $G$,

$$
I_G = I_G^{(1)} + I_G^{(2)}, \tag{13}
$$

where

$$
I_G^{(1)} = C_G \int \left( \prod_{l=1}^L \frac{d^d k_l}{(2\pi)^d} \right) \prod_{i=1}^{N_F} \mathcal{P}(q_i) \prod_{j=1}^{N_{\Gamma}} \prod_{j=2}^{N_{\Gamma}} \frac{1}{q_j^2 + m_f^2} \prod_{j=1}^{N_{\Gamma}} \frac{1}{p_j^2 + \mu^2}, \tag{14}
$$
and

\[ I_G^{(2)} = C_G (\xi - 1)^{N_T} \int \left( \prod_{i=1}^{L} \frac{d^4 k_i}{(2\pi)^d} \right) \prod_{i=1}^{N_F} \mathcal{P}(q_i) \frac{1}{q_i^2 + m_i^2} \prod_{j=1}^{N_T} \mathcal{Q}(p_j) \frac{1}{(p_j^2 + \mu^2)^2}. \]  

We have taken this separation due to the different structure of the denominators. \( I_G^{(1)} \) has a similar structure with respect to that of the Yukawa case above, and should give the expression of the amplitude in the Feynman gauge, \( \xi = 1 \). Here, in \( C_G \) we collect all global factors; the parameters \( m_i \) are the (possibly different) matter-field masses and \( \mu \) is the gauge-field infrared regulator. As before, let \( P \) denote the set of external momenta; thus, to shorten the expression, we have considered the momenta \( q_i(P, \{ k_i \}) \) and \( p_j(P, \{ k_i \}) \) as linear functions of the set of \( k_i \) obtained by momentum conservation at each vertex. \( \mathcal{P} \) and \( \mathcal{Q} \) are polynomials in the momenta, coming from the numerators of the propagators. They also contain factors such as the metric tensor and Dirac matrices, appropriate for a given diagram \( G \).

For both of the integrals above, a Schwinger parametric representation can be written down. In order to do this, we particularize eq. (6), in which a 4-vector \( \zeta_i \) is attached to each fermionic line \( i \) such that the momenta in the \( \mathcal{P} \) polynomial are substituted by a corresponding derivative with respect to \( \zeta_i \) at \( \zeta_i \) equal to zero. It then results that the integral \( I_G^{(1)} \) is written as

\[ I_G^{(1)} = C_G \int_0^\infty \frac{\prod_{i=1}^{N_F} d\alpha_i}{U^{d/2}} \prod_{i \in P} \mathcal{P} \left( -\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i} \right) e^{-\sum_{i=1}^{N_F} \bar{m}_i^2 \alpha_i - \mu^2 \sum_{i=1}^{N_T} \alpha_i} e^{-N(\bar{s}_K)/U} \bigg|_{\zeta_i=0}, \]  

with the Symanzik polynomials given by the same definitions as above, eqs. (7), modified by the presence of the auxiliary variables \( \zeta_i \), just as in the scalar–fermion case, eqs. (11) and (12).

For \( I_G^{(2)} \), we remind that the starting point for the parametric representation is eq. (9). Therefore, in eqs. (15), as we have products of \((p_j + \mu^2)^2\) in the denominator, we should associate two parameters, say, \( \beta_1 \) and \( \beta_2 \), to the same line, that is,

\[ \frac{1}{(p^2 + \mu^2)^2} = \left( \int_0^\infty d\beta_1 e^{-\beta_1(p^2 + \mu^2)} \right) \left( \int_0^\infty d\beta_2 e^{-\beta_2(p^2 + \mu^2)} \right). \]  

This means that the set of lines of the diagram, related to the momenta \( q_i \) and \( p_j \), are such that to each matter-field line \( i \) is attributed a parameter \( \alpha_i \), whereas for each gauge-field line \( r \) is associated a couple of parameters \( \beta_{r_1}, \beta_{r_2} \). Thus the parametric representation for \( I_G^{(2)} \) has the form

\[ I_G^{(2)} = C_G (\xi - 1)^{N_T} \int_0^\infty \prod_{i=1}^{N_T} d\alpha_i \prod_{r_1 \neq r_2} d\beta_{r_1} d\beta_{r_2} \times \prod_{i \in F} \mathcal{P} \left( -\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i} \right) \prod_{r_1 \in \Gamma} \mathcal{Q}_1 \left( -\frac{1}{\beta_{r_1}} \frac{\partial}{\partial \zeta_{r_1}} \right) \prod_{r_2 \in \Gamma} \mathcal{Q}_2 \left( -\frac{1}{\beta_{r_2}} \frac{\partial}{\partial \zeta_{r_2}} \right) \times \exp \left[ -\sum_{i=1}^{N_F} \bar{m}_i^2 \alpha_i - \sum_{r_1=1}^{N_T} \beta_{r_1}^2 \beta_{r_1} - \sum_{r_2=1}^{N_T} \beta_{r_2}^2 \beta_{r_2} \right] e^{-N(\bar{s}_K)/U} \bigg|_{\zeta_i=0}. \]  

where now

\[ U = \sum_T \prod_{i \notin T} \alpha_i \prod_{r_1 \notin T} \beta_{r_1} \prod_{r_2 \notin T} \beta_{r_2}, \quad N(\bar{s}_K) = \sum_K \bar{s}_K \prod_{i \notin K} \alpha_i \prod_{r_2 \notin K} \beta_{r_1} \prod_{r_2 \notin K} \beta_{r_2}. \]

### III. THE COMPLETE MELLIN REPRESENTATION

In this section we consider the asymptotic behavior of Feynman amplitudes in an arbitrary covariant gauge under scaling of any subset of invariants. These include either some invariants going to infinity or some masses going to zero. In other words, we have a unified treatment of both ultraviolet and infrared behaviors of the amplitudes.
As shown in Ref. [13], the scalar Feynman amplitude (6) has a complete Mellin representation, obtained in the following way. Let us rewrite the Symanzik polynomials as

\[ U(\alpha) = \sum_{j} \prod_{l=1}^{l} \alpha_{ij}^{m_{ij}} = \sum_{j} U_{j}, \quad N(\alpha) = \sum_{K} s_{K} \left( \prod_{l=1}^{l} \alpha_{ij}^{n_{K} \alpha_{ij}} \right) \equiv \sum_{K} N_{K}, \]  

(20)

where the label \( j \) runs over the set of the 1-trees \( T \) and \( K \) over the set of the 2-trees. Also, we have defined

\[ u_{ij} = \begin{cases} 0 & \text{if the line } i \text{ belongs to the 1-tree } j \\ 1 & \text{otherwise} \end{cases} \]  

(21)

and

\[ n_{iK} = \begin{cases} 0 & \text{if the line } i \text{ belongs to the 2-tree } K \\ 1 & \text{otherwise} \end{cases}. \]  

(22)

Moreover,

\[ \sum_{i} u_{ij} = L; \quad \sum_{i} n_{iK} = L + 1, \]  

(23)

for all \( j \) and \( K \).

Let us first consider a scalar amplitude of the type of eq. (6). The idea of the complete Mellin representation has its roots in the so-called ‘multiple Mellin’ representation introduced in [12]. In this case, the polynomial \( N \) in eq. (6) is split into pieces, \( N = \sum_{a} N_{a} \), in such a way that each piece \( N_{a} \) has the property of being a FINE polynomial, that is, in each Hepp sector the orderings of the \( \alpha \)'s induce one and only one dominant monomial of \( N \) [11]. The point is that not all polynomials have the property of being FINE. Nevertheless, there always exists the solution of splitting the \( N \) polynomial in all its monomials, which will always be FINE. In other words, if we adopt this extreme point of view, each piece \( N_{a} \) is a monomial of \( N \), which is necessarily FINE. In general, if the pieces \( N_{a} \) are not the monomials of \( N \), they will generate a ‘multiple Mellin’ representation of the amplitude [12]. In the extreme situation of splitting \( N \) in all its monomials, \( N = \sum_{K} N_{K} \), a ‘complete Mellin’ representation will be generated [13]. This is the point of view which we adopt here.

To proceed, let us remember the following theorem [17]: for a function \( f(u) \), piecewise smooth for \( u > 0 \), if the integral

\[ g(x) = \int_{0}^{\infty} du u^{-x-1} f(u) \]  

(24)

is absolutely convergent for \( \alpha < \text{Re } x < \beta \), then

\[ f(u) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dx g(x) u^{x}, \]  

(25)

with \( \alpha < \sigma < \beta \).

Then, if we take \( u = N_{K}/U \) in the above theorem, it is easy to see that

\[ e^{-N_{K}/U} = \int_{\tau_{K}} \Gamma(-y_{K}) \left( \frac{N_{K}}{U} \right)^{y_{K}}, \]  

(26)

where \( \int_{\tau_{K}} \) is a short notation for \( \int_{-\infty}^{+\infty} \frac{d(\text{Im } y_{K})}{2\pi i} \), with \( \text{Re } y_{K} \) fixed at \( \tau_{K} < 0 \). We may now recall the identity [18]

\[ \Gamma(u) (A + B)^{-u} = \int_{-\infty}^{\infty} \frac{d(\text{Im } x)}{2\pi i} \Gamma(-x) A^{x} \Gamma(x + u) B^{-x-u} \]  

(27)

and let us take \( A \equiv U_{1}(x) \) and \( B \equiv U_{2} + U_{3} + \cdots \). Using iteratively the identity above, it can be shown that, for

\[ u = \sum_{K} y_{K} + d/2, \]  

\[ \Gamma \left( \sum_{K} y_{K} + \frac{d}{2} \right) U^{-\sum_{K} y_{K} - \frac{d}{2}} = \int_{\sigma} \prod_{j} \Gamma(-x_{j}) U_{j}^{x_{j}}, \]  

(28)
with $\text{Re } x_j = \sigma_j < 0$, $\text{Re } (\sum_K y_K + \frac{d}{2}) = \sum_K \tau_K + \frac{d}{2} > 0$, and $\int_\sigma$ means $\int_{-\infty}^{+\infty} \prod_j \frac{d(\text{Im } x_j)}{2\pi i}$ with $\sum_j x_j + \sum_K y_K = -\frac{d}{2}$. Then, after replacing eqs. (26) and (28) in eq. (3), the amplitude is written as

$$I_G(s_K, m_l^2) = \int_\Delta \prod_j \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K s^{yk}_K \Gamma(-y_K) \int_0^\infty \prod_l d\alpha_l \alpha_l^{\phi_l-1} e^{-\sum_i \alpha_i m_i^2},$$

(29)

where

$$\phi_i \equiv \sum_j u_{ij} \sigma_j + \sum_K n_{iK} \tau_K + 1$$

(30)

and $\Delta$ is the nonempty convex domain ($\sigma$ and $\tau$ standing respectively for $\text{Re } x_j$ and $\text{Re } y_K$),

$$\Delta = \left\{ \sigma, \tau \mid \sigma_j < 0; \tau_K < 0; \sum_j x_j + \sum_K y_K = -\frac{d}{2}; \forall i, \text{Re } \phi_i \equiv \sum_j u_{ij} \sigma_j + \sum_K n_{iK} \tau_K + 1 > 0 \right\}$$

(31)

and the symbol $\int_\Delta$ means integration over the independent variables $\frac{\text{Im } x_j}{2\pi i}, \frac{\text{Im } y_K}{2\pi i}$.

The $\alpha$ integrations may be performed, using the well-known formula for the gamma function, so that we have

$$\int_0^\infty d\alpha_l e^{-\alpha_l m_l^2} \alpha_l^{\phi_l-1} = \Gamma (\phi_l) (m_l^2)^{-\phi_l}$$

(32)

and we finally get the complete Mellin representation of the amplitude in the scalar case:

$$I_G(s_K, m_l^2) = \int_\Delta \prod_j \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K s^{yk}_K \Gamma(-y_K) \prod_l (m_l^2)^{-\phi_l} \Gamma (\phi_l).$$

(33)

One should emphasize that the integral in eq. (29) above is assumed for simplicity to be ultraviolet convergent, but if this is not the case, it is shown in [13, 14] that the renormalization procedure does not alter the algebraic structure of integrands of the CM representation. It only changes the set of relevant integration domains in the Mellin variable space, all what follows remaining valid.

IV. THE CM REPRESENTATION AND ASYMPTOTIC BEHAVIORS FOR GAUGE THEORIES

Turning now to gauge theories, we see that in both equations for $I_G^{(1)}$ and $I_G^{(2)}$, given by eqs. (16) and (18), the integrals over the Schwinger parameters have the same form as in the scalar case in terms of the Symanzik polynomials, except for taking into account the substitutions $m_i^2 \rightarrow \tilde{m}_i^2$ and $s_K \rightarrow \tilde{s}_K$ and having to perform the derivatives on the auxiliary variables $\zeta$ which modify $m_i^2$ and $s_K$ and are put to zero at the end of the calculation. Here $m_l$ stands generically for both the fermion masses and the infrared gauge-field regulator. These derivatives are always attached to Schwinger parameters, which then alter the power $\partial_\zeta$ in the integrations corresponding to (32), a different alteration for each monomial of the $P$ polynomial in the case of $I_G^{(1)}$, or of the $P$, $Q_1$ and $Q_2$ polynomials for $I_G^{(2)}$.

For $I_G^{(1)}$ in eq. (16), let us assume that the polynomial $P$ has the form

$$P \left( -\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i} \right) = \sum_A \alpha_A \left( -\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i} \right)^d_{\alpha_i}$$

(34)

Then, according to [14], for each monomial $\frac{\partial}{\partial \zeta_i}$ of the polynomial $P$ the following complete Mellin representation holds:

$$F^{(1)}(s_K, m_l^2) = \int_{\Delta_A} \prod_i \left( -\frac{\partial}{\partial \zeta_i} \right)^d_{\alpha_i} \prod_j \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K \tilde{s}^{yk}_K \Gamma(-y_K) \prod_l (\tilde{m}_l^2)^{-\phi_l+d^A} \Gamma (\phi_l-d^A) \bigg|_{\zeta=0},$$

(35)

where $\tilde{m}_l$ stands for both the fermion masses $\tilde{m}_i$ and the infrared regulator $\mu$ of the gauge-field propagators; for the latter ones, $d^A = 0$. Also, $\Delta_A$ is the modified nonempty convex domain

$$\Delta_A = \left\{ \sigma, \tau \mid \sigma_j < 0; \tau_K < 0; \sum_j x_j + \sum_K y_K = -\frac{d}{2}; \forall i, \text{Re } (\phi_l-d^A) = \sum_j u_{ij} \sigma_j + \sum_K n_{iK} \tau_K - d^A + 1 > 0 \right\}.$$
We see that now the derivatives may be taken out of the integration, so that we define

\[ F_A^{(1)}(s_K, m_I^2) = \prod_l \left( -\frac{\partial}{\partial \zeta_l^i} \right) d_l^A F_A^{(1)}(\tilde{s}_K, \tilde{m}_I^2) \bigg|_{\zeta = 0}, \tag{37} \]

with

\[ F_A^{(1)}(\tilde{s}_K, \tilde{m}_I^2) = \int_{\Delta_A} \prod_l \Gamma(-x_j) \prod_K \tilde{s}_K^{\gamma_K} \Gamma(-y_K) \prod_l (\tilde{m}_I^2)^{-\phi_l + d_l^A} \Gamma (\phi_l - d_l^A). \tag{38} \]

A general asymptotic regime is defined by scaling the invariants \( s_K \) and \( m_I^2 \):

\[ \tilde{s}_K \to \lambda^{a_K} s_K, \]
\[ \tilde{m}_I^2 \to \lambda^{a_I} m_I^2, \tag{39} \]

where \( a_K \) and \( a_I \) may have positive, negative, or null values, and letting \( \lambda \) go to infinity. We then obtain under this scaling

\[ F_A^{(1)}(\lambda, \tilde{s}_K, \tilde{m}_I^2) = \int_{\Delta_A} \prod_l \Gamma(-x_j) \prod_K \tilde{s}_K^{\gamma_K} \Gamma(-y_K) \prod_l (\tilde{m}_I^2)^{-\phi_l + d_l^A} \Gamma (\phi_l - d_l^A) \bigg|_{\zeta = 0}, \tag{40} \]

where the exponent of \( \lambda \) is the following linear function of the Mellin variables:

\[ \psi^A = \sum_K a_K y_K - \sum_I a_I \left[ \phi_l(x_j, y_K) - d_l^A \right]. \tag{41} \]

Then the proof of the theorem given in [12] for functions of the form [33] can be straightforwardly extended for [35], and the following theorem is valid: as \( \lambda \to \infty \), we have an asymptotic expansion of the integral \( F_A^{(1)}(\lambda, \tilde{s}_K, \tilde{m}_I^2) \) of the form

\[ F_A^{(1)}(\lambda, \tilde{s}_K, \tilde{m}_I^2) = \sum_{q \geq 0} \sum_{p = p_{\text{max}}}^{-\infty} F_{pq}^{(1)}(\tilde{s}_K, \tilde{m}_I^2) \lambda^p \ln^q \lambda, \tag{42} \]

where \( p \) runs over the rational values of a decreasing arithmetic progression, with \( p_{\text{max}} \) as a ‘leading power’, and \( q \), for a given \( p \), runs over a finite number of nonnegative integer values.

A systematic way of evaluating the coefficients \( F_{pq} \) was presented in Ref. [13], by successive analytical continuations of the linear form \( \psi \) to more restrictive cells of the type \( \Delta \) in eq. [31]. Under the scaling [39], and summing over all monomials, \( I_G^{(1)} \) in eq. [16] becomes

\[ I_G^{(1)}(\lambda) = C_G \sum_A c_A \prod_l \left( -\frac{\partial}{\partial \zeta_l^i} \right) d_l^A \left( \sum_{p = p_{\text{max}}}^{-\infty} \sum_{q = 0}^{q_{\text{max}}(p)} F_{pq}^{(1)}(\tilde{s}_K, \tilde{m}_I^2) \right) \lambda^p \ln^q \lambda \bigg|_{\zeta = 0}, \]

\[ = C_G \sum_{p = p_{\text{max}}}^{-\infty} \sum_{q = 0}^{q_{\text{max}}(p)} \left[ \sum_A c_A \prod_l \left( -\frac{\partial}{\partial \zeta_l^i} \right) d_l^A F_{pq}^{(1)}(\tilde{s}_K, \tilde{m}_I^2) \right] \lambda^p \ln^q \lambda \bigg|_{\zeta = 0}, \]

\[ = C_G \sum_{p = p_{\text{max}}}^{-\infty} \sum_{q = 0}^{q_{\text{max}}(p)} G_{pq}^{(1)}(s_K; m_I^2) \lambda^p \ln^q \lambda, \tag{43} \]

where we have defined

\[ G_{pq}^{(1)}(s_K; m_I^2) = \sum_A c_A \prod_l \left( -\frac{\partial}{\partial \zeta_l^i} \right) d_l^A F_{pq}^{(1)}(\tilde{s}_K, \tilde{m}_I^2) \bigg|_{\zeta = 0}. \tag{44} \]

For \( I_G^{(2)} \), generalizing what has been done for eq. [33], we consider the product of polynomials \( P \), \( Q_1 \) and \( Q_2 \) in eq. [18] in the form

\[ P Q_1 Q_2 = \sum_B c_B \prod_i \left( -\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_l^i} \right) d_i^B \left( -\frac{1}{\beta_{r_1}} \frac{\partial}{\partial \zeta_{l_{r_1}}} \right) e_{r_1}^B \left( -\frac{1}{\beta_{r_2}} \frac{\partial}{\partial \zeta_{l_{r_2}}} \right) e_{r_2}^B. \tag{45} \]
In order to proceed, notice that eq. (20) is modified in the present case to

\[ U(\alpha, \beta) = \sum_j \prod_{i \in F} a_i \prod_{r_1 \in \Gamma} \beta_r^{v_{1r}} \prod_{r_2 \in \Gamma} \beta_r^{v_{2r}} \]  

(46)

\[ N(\alpha, \beta) = \sum_K \bar{s}_K \prod_{i \in F} \alpha_i^{n_{1i}} \prod_{r_1 \in \Gamma} \beta_r^{m_{1Kr_1}} \prod_{r_2 \in \Gamma} \beta_r^{m_{2r_2}}. \]  

(47)

We now replace the above expressions in eq. (18), together with eqs. (26) and (28). After some manipulations, we find for \( I_G^{(2)} \),

\[ I_G^{(2)} = C_G (\xi - 1)^N N_B \sum_B \exp \left( \prod_j \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K \Gamma(-y_K) \right) \]

\[ \times \left\{ \prod_i \left( \frac{-\partial}{\partial \zeta_i} \right) d^B_i \prod_{r_1} \left( \frac{-\partial}{\partial \zeta_{r_1}} \right) e^B_{r_1} \prod_{r_2} \left( \frac{-\partial}{\partial \zeta_{r_2}} \right) e^B_{r_2} \right\} \]

(48)

where \( \phi_i = \sum_j u_{ij} x_j + \sum_K n_{iK} y_K \) as before, and

\[ \phi_{r_1,2} = \sum_j v_{jr_1,2} x_j + \sum_K m_{Kr_1,2} y_K \]  

(49)

and now the domain of integration is

\[ \Delta_B = \left\{ \sigma, \tau \mid \sigma_j < 0; \tau_K < 0; \sum_j x_j + \sum_K y_K = -\frac{d_B}{2}; \sum_{i} u_{ij} \sigma_j + \sum_K n_{iK} \tau_K - d^B_i + 1 > 0; \sum_{r_1, r_2} v_{jr_1,2} \sigma_j + \sum_K m_{Kr_1,2} \tau_K - e^B_{r_1,2} > 0 \right\} \]  

(50)

The integrals over all Schwinger parameters \( \alpha \) and \( \beta \) are performed in the same fashion as in eq. (32), so the amplitude is written as

\[ I_G^{(2)} = C_G (\xi - 1)^N N_B \sum_B \exp \left( \prod_j \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K \Gamma(-y_K) \right) \]

\[ \times \int_{\Delta_B} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K \bar{s}_K \Gamma(-y_K) \]

\[ \times \Gamma(\phi_i - d^B_i) \left( \tilde{m}_i^{2B} \right)^{-\phi_i + d^B_i} \Gamma(\phi_{r_1} - e^B_{r_1}) \left( \tilde{\mu}_r^{2B} \right)^{-\phi_{r_1} + e^B_{r_1}} \Gamma(\phi_{r_2} - e^B_{r_2}) \left( \tilde{\mu}_r^{2B} \right)^{-\phi_{r_2} + e^B_{r_2}} \bigg|_{\zeta=0}, \]  

(51)

where \( j \) runs over the whole set of monomials of \( U(\alpha, \beta) \), eq. (46). Then, performing steps similar to those that led to eq. (37) for \( I_G^{(1)} \), we get for each monomial of the product of polynomials \( PBQ_1Q_2 \) in eq. (45) the expression

\[ F_B^{(2)}(s_K, m_i^2, \mu^2) = \prod_i \left( \frac{-\partial}{\partial \zeta_i} \right) d^B_i \prod_{r_1} \left( \frac{-\partial}{\partial \zeta_{r_1}} \right) e^B_{r_1} \prod_{r_2} \left( \frac{-\partial}{\partial \zeta_{r_2}} \right) e^B_{r_2} \right\} \]

(52)

\[ F_B^{(2)}(s_K, \bar{m}_i^2, \bar{\mu}_{r_1,2}) = \int_{\Delta_B} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_K \bar{s}_K \Gamma(-y_K) \prod_i \left( \tilde{m}_i^{2B} \right)^{-\phi_i + d^B_i} \Gamma(\phi_i - d^B_i) \bigg|_{\zeta=0}, \]  

(53)
We now consider an asymptotic regime defined by

\[ \begin{align*}
\tilde{s}_k & \to \lambda^{a_k} s_k, \\
\tilde{m}_i^2 & \to \lambda^{a_i} m_i^2, \\
\tilde{\mu}_{r_1,2}^2 & \to \lambda^{a_{r_1,2}} \tilde{\mu}_{r_1,2}^2
\end{align*} \]

(54)

where \(a_k, a_i\) and \(a_{r_1,2}\) may have positive, negative, or null values, and we let \(\lambda\) go to infinity. Under the above scaling, \(F_B^{(2)}\) in eq. (53) then becomes a function of \(\lambda\), given by

\[ F_B^{(2)}(\tilde{s}_k, \tilde{m}_i^2, \tilde{\mu}_{r_1,2}^2) = \int_{\Delta_B} \prod \frac{\Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod \tilde{s}_k^{y_k} \Gamma(-y_k) \prod (\tilde{m}_i^2)^{-\phi_i} d^B \Gamma(\phi_i - d_i^B) \lambda^{\psi_i}, \]

(55)

with the exponent of \(\lambda\) being

\[ \psi = \sum_K a_K y_K - \sum_i a_i(x_j, y_K) - d_i^B - \sum_{r_1} a_{r_1}(x_j, y_K) - e_{r_1} - \sum_{r_2} a_{r_2}(x_j, y_K) - e_{r_2}. \]

(56)

Since the integral in the function (55) has the same form of that of the corresponding one in the scalar field theory, the following asymptotic expansion holds:

\[ F_B^{(2)}(\lambda, \tilde{s}_k, \tilde{m}_i^2, \tilde{\mu}_{r_1,2}^2) = \sum_{p=p_{\text{max}}}^{-\infty} \sum_{q=0}^{q_{\text{max}}(p)} F^{(2)B}_{pq}(\tilde{s}_k, \tilde{m}_i^2, \tilde{\mu}_{r_1,2}^2) \lambda^p \ln^q \lambda. \]

(57)

Therefore, from (55), and by replacing eq. (57) into eq. (52), we have

\[ I_G^{(2)}(\lambda) = C_G (\xi - 1)^{N_F} \sum_{p=p_{\text{max}}}^{-\infty} \sum_{q=0}^{q_{\text{max}}(p)} G^{(2)}_{pq}(s_K; m_i^2, \mu^2) \lambda^p \ln^q \lambda, \]

(58)

where

\[ G^{(2)}_{pq}(s_K; m_i^2, \mu^2) = \sum_B c_B \prod_i \left( -\frac{\partial}{\partial \zeta_i} \right) d_i^B \prod_{r_1} \left( -\frac{\partial}{\partial \zeta_{r_1}} \right) e_{r_1} d_i^B \prod_{r_2} \left( -\frac{\partial}{\partial \zeta_{r_2}} \right) e_{r_2} d_i^B F^{(2)B}_{pq}(\tilde{s}_k, \tilde{m}_i^2, \tilde{\mu}_{r_1,2}^2) \bigg|_{\zeta=0}. \]

(59)

Combining both results for \(I_G^{(1)}(\lambda)\) and \(I_G^{(2)}(\lambda)\), we finally obtain the asymptotic expansion for the entire amplitude \(I_G(\lambda)\),

\[ I_G(\lambda) = C_G \sum_{p=p_{\text{max}}}^{-\infty} \sum_{q=0}^{q_{\text{max}}(p)} G_{pq}(s_K; m_i^2, \mu^2) \lambda^p \ln^q \lambda, \]

(60)

where

\[ G_{pq}(s_K; m_i^2, \mu^2) = G^{(1)}_{pq}(s_K; m_i^2) + (\xi - 1)^{N_F} G^{(2)}_{pq}(s_K; m_i^2, \mu^2), \]

(61)

with \(G^{(1)}_{pq}\) and \(G^{(2)}_{pq}\) given respectively by eqs. (44) and (59).

V. CONCLUDING REMARKS

We have stated a theorem which generalizes for gauge field theories in an unspecified gauge previous results on the asymptotic behaviors of scalar field theories. This has been possible by conveniently changing the invariants \(s_K\), \(m_i^2\) and \(\mu^2\) into new objects \(\tilde{s}_k\), \(\tilde{m}_i^2\) and \(\tilde{\mu}_{r_1,2}^2\) by means of dummy variables \(\zeta\) associated to each internal line of the diagram, to be taken equal to zero in the end. It results that the relevant Feynman integrals are expressed as polynomials of derivatives with respect to these dummy variables, acting on integrals over the Schwinger parameters. The integrals over which the derivatives act have the same formal structure as those of the simplest of cases, that of a scalar field theory. Thus, complete Mellin representations can be written for these integrals and a previous theorem for asymptotic behaviors of scalar amplitudes can be directly generalized for them. This implies that the asymptotic
expansions in our case have exactly the same structure as in scalar theories, but the coefficients of the expansion are modified by the polynomials of derivatives entering in the Schwinger representation.

We have adopted in this paper the point of view of a mathematical physicist: we have stated at the mathematical level the validity of a theorem for the asymptotics of gauge theories. However, we have not yet given at this stage a direct prescription of its use in applications to physical situations where the choice of gauges other than the Feynman gauge is mandatory. These situations do exist, as we have mentioned in the introduction. They are under current investigation by us and the results will be presented elsewhere.

Acknowledgments

This work received partial support from CNPq/MCT.