Local Distinguishability of Any Three Quantum States

Somshubhro Bandyopadhyay\(^1,2\) and Jonathan Walgate\(^1,3\)

\(^1\)Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada.
\(^2\)DIRO, Université de Montréal, C.P. 6128, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3J7.
\(^3\)Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, Canada N2L 2Y5.

(Dated: December 5, 2006)

We prove that any three linearly independent pure quantum states can always be locally distinguishable with nonzero probability regardless of their dimension, entanglement, or multipartite structure.

PACS numbers: 89.70.+c, 03.65.-w

Global operations on a quantum system can process information in ways that local operations on the system’s parts cannot. All uses of entanglement in quantum information theory flow from this one fact, from teleportation [1] to Shor’s factoring algorithm [2]. However a fundamental question remains unanswered: When is global information in ways that local operations cannot? This question can be formally posed as a local state discrimination task. Given one copy of a system in one of a known set of quantum states \(|\psi_i\rangle\}, how much ‘which state’ information can be gleaned by local operations and classical communication (LOCC), and how much more information is revealed by global measurements?

This problem has attracted much attention in recent years, after surprising results showed perfect local distinguishability was not directly linked to entanglement. Bennett and coworkers presented sets of orthogonal unentangled states that were not perfectly locally distinguishable [3]. JW, Short, Hardy and Vedral proved orthogonal pairs of states are always perfectly locally distinguishable, irrespective of their entanglement [4].

There are two natural approaches to quantum state discrimination. Optimal discrimination seeks the best possible guess as to the state of the system [5]. Conclusive discrimination seeks certain knowledge of the state of the system, balanced against a possibility of failure [6]. It follows directly from the results of Walgate et al. [4] and Virmani et al. [7] that local parties can always gain some amount of ‘which state’ information about the pure state of a shared system, and use it to improve their guesswork. Optimal state discrimination is always locally feasible in this sense, although the local optimum may be significantly worse than the global. Conclusive discrimination is more interesting. All pairs of pure quantum states can be conclusively discriminated equally well locally and globally [8]. But there are sets of four orthogonal pure quantum states that are not conclusively locally distinguishable at all. In this case local parties can never gain certain knowledge of which state they possess; the Bell states are the simplest example of such a set [9].

What about sets of three states? We show that provided they are linearly independent (only linearly independent states are globally distinguishable) three pure quantum states can be conclusively locally distinguished. Local protocols may not succeed as often as global measurements, but they can succeed some of the time. No triplet of pure states, no matter how entangled, conceals any fraction of its ‘which state’ information from local parties with certainty.

We present our results in the following framework. A multipartite quantum system \(Q\) is shared between \(n\) different local parties, each with access to one of \(n\) local Hilbert spaces: \(H_Q = \bigotimes_{j=1}^{n} H_j\). It has been prepared in one of a known set of possible pure states \(\mathcal{S} = \{|\psi_i\rangle\}\), each with some nonzero (but potentially unknown) probability \(p_i\). The local parties are set the task of discovering with certainty which of the states \(\mathcal{S}\) they have been given, using only LOCC. We will use the following definitions.

**Definition 1.** A state \(|\psi\rangle \in \mathcal{S}\) is **conclusively locally identifiable** if and only if there is a LOCC protocol whereby with some nonzero probability \(p > 0\) it can be determined that \(Q\) was certainly prepared in state \(|\psi\rangle\).

**Definition 2.** A set of states \(\mathcal{S}\) is **stochastically locally distinguishable** if and only if it contains at least one state that is conclusively locally identifiable.

We use the term ‘stochastic local distinguishability’ (SLOCC distinguishability) to highlight a subtlety: we do not require every member of \(\mathcal{S}\) to be potentially identifiable, merely one. SLOCC distinguishability is the most general definition of conclusive local state discrimination. Certainly if \(|\psi_i\rangle \in \mathcal{S}\) is conclusively locally identifiable, then the local parties can identify the state of \(Q\) with certainty some fraction \(0 < p \leq p_i\) of the time. Although our analysis centers on this general case, we also consider the restriction that all the possible states should be conclusively locally identifiable. In fact our proof shows that, with a single surprising class of exceptions, all sets

\*Electronic address: bandyo@iro.umontreal.ca
\*Electronic address: jwalgate@perimeterinstitute.ca
of three states are fully conclusively locally identifiable \((∀\psi_i \mid ψ_i \rangle \text{ is conclusively locally identifiable})\).

Stochastic local distinguishability has qualitative links to entanglement. It was proved by Horodecki et al. that orthonormal bases are SLOCC distinguishable if and only if they contain product states, in which case only the product states are conclusively locally identifiable \([10]\). We show below in Corollary 1 that sets of orthogonal states are SLOCC indistinguishable only if they are completely entangled, and in Corollary 2 that unentangled bases are always SLOCC distinguishable even when they are nonorthogonal.

We begin by proving a necessary and sufficient condition for a set of states to be stochastically locally distinguishable. We will then show that this condition always holds for sets of three states.

**Lemma 1.** Let a multipartite quantum system \(Q\) be prepared in one of a set of pure, linearly independent multipartite quantum states \(S = \{\mid ψ_i \rangle\}\).

If and only if there exists a member of the set \(\mid ψ_x \rangle\) and a product state \(\mid φ \rangle\) such that \(∀i \neq x \langle ψ_i \mid φ \rangle = 0\), and \(\langle ψ_x \mid φ \rangle \neq 0\), then \(\mid ψ_x \rangle\) is conclusively locally identifiable in \(S\) and \(S\) is stochastically locally distinguishable.

Proof of sufficiency: Assume that \(\mid ψ_x \rangle\) and \(\mid φ \rangle\) exist. The parties can locally project into a product basis of \(\mathcal{H}_Q\) that includes \(\mid φ \rangle\). If the state of \(Q\) is \(\mid ψ_x \rangle\) they will obtain the result projecting onto \(\mid φ \rangle\) with probability \(|\langle ψ_x \mid φ \rangle|^2\), which is greater than zero. In this case, they have conclusively locally identified \(\mid ψ_x \rangle\) since no other state \(\mid ψ_i \rangle\) ever yields this projection result. Since \(\mid ψ_x \rangle\) is conclusively locally identifiable in \(S\), \(S\) is stochastically locally distinguishable, and the condition is sufficient. □

Proof of necessity: Assume that \(S\) is SLOCC distinguishable. Then at least one state in \(S\) is conclusively locally identifiable, and we label it \(\mid ψ_x \rangle\). There is a LOCC protocol describable by a separable superoperator, which can produce at least one measurement outcome conclusively identifying \(\mid ψ_x \rangle\). This outcome corresponds to some separable POVM element \(M^i M = A^i A \otimes B^i B \otimes \ldots\), because it identifies \(\mid ψ_x \rangle\) must satisfy \(∀i \neq x \langle ψ_i \mid M^i M \mid ψ_i \rangle = 0\), and \(\langle ψ_x \mid M^i M \mid ψ_x \rangle \neq 0\). \(M^i M\) is decomposable into a set of rank one projection operators onto product states \(\{\mid P_i \rangle\}\): \(M^i M = \sum_{j_k} A_{j_k}^i A_{j_k} \otimes B_{j_k}^i B_{j_k} \otimes \ldots = \sum_i (\mid P_i \rangle \langle P_i \mid) (\mid P_i \rangle \langle P_i \mid)\). These product states must satisfy \(∀i \neq x \langle ψ_i \mid P_i^i P_i \mid ψ_i \rangle = 0\), and \(∃i \langle ψ_x \mid P_i^i P_i \mid ψ_x \rangle \neq 0\). Let the product state satisfying both conditions be \(\mid φ \rangle\). Thus there exists a member of the set \(\mid ψ_x \rangle\) and a product state \(\mid φ \rangle\) such that \(∀i \neq x \langle ψ_i \mid φ \rangle = 0\), and \(\langle ψ_x \mid φ \rangle \neq 0\), and the condition is necessary. □

**Corollary 1.** All sets of pure orthogonal states containing at least one product state are SLOCC distinguishable, and every unentangled member of such a set is conclusively locally distinguishable.

**Theorem 1.** Let a multipartite quantum system \(Q\) be prepared in one of a set of three pure, linearly independent multipartite quantum states \(S = \{\mid ψ_1 \rangle, \mid ψ_2 \rangle, \mid ψ_3 \rangle\}\).

\(S\) is SLOCC distinguishable.

We will prove this result separately for three different cases. First we will deal with systems whose three possible states cannot be composed on a chain of qubits (i.e. where \(\mathcal{H}_Q \neq \bigotimes_{i=1}^3 \mathcal{H}_i\), with each local party holding just one qubit). Then we will consider \(\mathcal{H}_2 \otimes \mathcal{H}_2\) systems. Lastly, we will prove our result for larger arrays of qubits: \(\mathcal{H}_Q = \bigotimes_{i=1}^n \mathcal{H}_2\). These three cases cover all possible multipartite situations.

**Lemma 2 (Higher-dimensional states).** Let a multipartite quantum system \(Q\) be prepared in one of a set of three pure, linearly independent multipartite quantum states \(S = \{\mid ψ_1 \rangle, \mid ψ_2 \rangle, \mid ψ_3 \rangle\}\). Let the space spanned by \(S\) be such that it cannot be expressed in the form \(\bigotimes_{i=1}^n \mathcal{H}_2\). \(S\) is SLOCC distinguishable.

**Proof:** If the space spanned by \(S\) cannot be expressed in the form \(\bigotimes_{i=1}^n \mathcal{H}_2\), then at least one of the local parties has an irreducibly three- or higher-dimensional Hilbert space \(\mathcal{H}_i\). We call this party ‘Alice’. We can write the states thus:

\[
\begin{align*}
|ψ_1\rangle &= \sum_i a_i |i\rangle_A |η_i\rangle_{BC\ldots}, \\
|ψ_2\rangle &= \sum_i b_i |i\rangle_A |ν_i\rangle_{BC\ldots}, \\
|ψ_3\rangle &= \sum_i c_i |i\rangle_A |μ_i\rangle_{BC\ldots},
\end{align*}
\]

**Corollary 2.** All sets of pure linearly independent product states spanning a multipartite Hilbert space \(\mathcal{H}_Q = \bigotimes_{i=1}^n \mathcal{H}_i\) are SLOCC distinguishable, and every member of such a set is conclusively locally distinguishable. □
where the vectors $|\eta_i\rangle_{BC...}$, $|\nu_i\rangle_{BC...}$, and $|\mu_i\rangle_{BC...}$ are normalized, and $a_i$, $b_i$, and $c_i$ are complex coefficients satisfying $\sum_i a_i^* a_i = 1$. Following the strategy of Lemma 1 we will show that there exists a product state $|\phi\rangle$ such that $\langle\psi_1|\phi\rangle = \langle\psi_2|\phi\rangle = 0$ and $\langle\psi_3|\phi\rangle \neq 0$. Let us write the product state thus:

$$|\phi\rangle = \left(\sum_i x_i |\psi_i\rangle\right) \otimes |\theta\rangle_{BC...},$$

with $\sum_i x_i^* x_i = 1$. We choose $|\theta\rangle$ such that it is a product state amongst the parties $B, C...$ and so that for all $i$, $\langle\eta_i|\theta\rangle \neq 0$, $\langle\nu_i|\theta\rangle \neq 0$, and $\langle\mu_i|\theta\rangle \neq 0$. (We can always do this, because by inspection for any finite set of pure quantum states we can always write down a product state nonorthogonal to all of them.) $|\phi\rangle$ must satisfy the following conditions:

$$\langle\psi_1|\phi\rangle = \sum_i x_i a_i^* \langle\eta_i|\theta\rangle = 0,$$
$$\langle\psi_2|\phi\rangle = \sum_i x_i b_i^* \langle\nu_i|\theta\rangle = 0,$$
$$\langle\psi_3|\phi\rangle = \sum_i x_i c_i^* \langle\mu_i|\theta\rangle \neq 0. \quad (2)$$

The quantities $a_i^* \langle\eta_i|\theta\rangle$, $b_i^* \langle\nu_i|\theta\rangle$, and $c_i^* \langle\mu_i|\theta\rangle$ are all fixed by our arbitrary choice of basis $\{|\psi_i\rangle\}_i$, and product state $|\theta\rangle_{BC...}$. There are at least three variables $x_i$, because $\mathcal{H}_A \neq 2$. With three linear equations and three variables, there is always a solution for the $x_i$. (Note that normalization does not further restrict the solution of these equations, as they only specify sums to ‘zero’ or ‘not zero’.) Therefore, we can always find a product state $|\phi\rangle$ that is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$, but nonorthogonal to $|\psi_3\rangle$. By Lemma 1 this means $|\psi_3\rangle$ is conclusively locally identifiable in $S$, and $S$ is stochastically locally distinguishable. □

**Corollary 3.** Let a multipartite quantum system $Q$ be prepared in one of a set of three pure, linearly independent multipartite quantum states $S = \{ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \}$.

If the space spanned by $S$ cannot be expressed in the form $\bigotimes_{i=1}^n \mathcal{H}_2$, then there is one SLOCC protocol that conclusively identifies all three states.

Proof: By symmetry, the above proof can equally well be applied to $|\psi_1\rangle$ and $|\psi_2\rangle$ as to $|\psi_3\rangle$, so all three states must be conclusively locally identifiable. This means that for each state there is a SLOCC protocol with which they can be locally identified. The local parties can randomly choose to enact one of these three procedures, and the overall SLOCC protocol including this random choice conclusively identifies all three states. □

Surprisingly, the only exceptions to this ‘one for all and all for one’ structure are found amongst the simplest quantum systems - qubits.

**Lemma 3 (Two Qubits).** Let a multipartite quantum system $Q$ be prepared in one of a set of three pure, linearly independent multipartite quantum states $S = \{ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \}$. Let $\mathcal{H}_Q = \bigotimes_{i=1}^n \mathcal{H}_2$. $S$ is SLOCC distinguishable.

Proof: Either at least two of the three members of $S$ are product states, or else at least two of them are entangled states. Whichever is the case, we label the states such that $|\psi_1\rangle$ and $|\psi_2\rangle$ are similar - either they’re both product states, or they’re both entangled. $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly independent and span a two-dimensional subspace of $\mathcal{H}_Q$. Let us call this subspace $\mathcal{H}_a$, and its complementary subspace $\mathcal{H}_a^\perp$. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are product states, it follows from their linear independence that $\mathcal{H}_a$ can be spanned by a pair of orthogonal product states. Its complementary subspace $\mathcal{H}_a^\perp$ must also be spanned by a pair of orthogonal product states in this case. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are entangled states, exactly the same is true - both $\mathcal{H}_a$ and $\mathcal{H}_a^\perp$ must be spanned by a pair of orthogonal product states. Thus $\mathcal{H}_a^\perp$ is spanned by a pair of orthogonal product states.

$|\psi_3\rangle$ is linearly independent of $|\psi_1\rangle$ and $|\psi_2\rangle$, so it has at least some support on $\mathcal{H}_a^\perp$. It therefore has at least some support on one of these two product states spanning $\mathcal{H}_a^\perp$. Let this product state be $|\phi\rangle$. In line with Lemma 1 $|\phi\rangle$ is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$, but nonorthogonal to $|\psi_3\rangle$, and therefore $|\psi_3\rangle$ is conclusively locally distinguishable and $S$ is SLOCC distinguishable. □

By symmetry, if all the states in $S$ are entangled, or if they are all product states, then they are all conclusively locally identifiable as in Corollary 3. If two of them are product states, it is again simple to show they are all conclusively locally identifiable (a consequence of the fact that every set of three orthogonal $2 \otimes 2$ states two of which are product states is perfectly locally distinguishable [11]). But an exception occurs when two of the states are entangled: only the product state can be conclusively locally identified. For example, the set of states:

$$|\psi_1\rangle = \alpha_1 |0\rangle_A |0\rangle_B + \alpha_2 |1\rangle_A |1\rangle_B,$$
$$|\psi_2\rangle = \beta_1 |0\rangle_A |0\rangle_B + \beta_2 |1\rangle_A |1\rangle_B,$$
$$|\psi_3\rangle = |0\rangle_A |1\rangle_B. \quad (3)$$

is SLOCC distinguishable, but only $|\psi_3\rangle$ is conclusively locally identifiable, as neither $|\psi_1\rangle$ nor $|\psi_2\rangle$ can satisfy the necessary condition for conclusive local identifiability established by Lemma 1. This asymmetric property is unique to qubit states.

**Lemma 4 (Many Qubits).** Let a multipartite quantum system $Q$ be prepared in one of a set of three pure, linearly independent multipartite quantum states $S = \{ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \}$. Let $\mathcal{H}_Q = \bigotimes_{i=1}^n \mathcal{H}_2$. $S$ is SLOCC distinguishable.

Proof: By symmetry, we can assume that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to $|\psi_3\rangle$, and that $|\psi_3\rangle$ is orthogonally distributed over $\mathcal{H}_a$ and $\mathcal{H}_a^\perp$. If $|\psi_3\rangle$ is not spanned by $\mathcal{H}_a^\perp$, then $|\psi_3\rangle$ is SLOCC distinguishable. If it is, we can choose a basis $\{|\psi_3\rangle_{a}, |\psi_3\rangle_{a^\perp}\}$ such that

$$|\psi_3\rangle_{a} = \alpha_1 |0\rangle_{a} |0\rangle_{b} + \alpha_2 |1\rangle_{a} |1\rangle_{b},$$
$$|\psi_3\rangle_{a^\perp} = \beta_1 |0\rangle_{a^\perp} |0\rangle_{b} + \beta_2 |1\rangle_{a^\perp} |1\rangle_{b}, \quad (4)$$

is SLOCC distinguishable, but only $|\psi_3\rangle$ is conclusively locally identifiable, as neither $|\psi_1\rangle$ nor $|\psi_2\rangle$ can satisfy the necessary condition for conclusive local identifiability established by Lemma 1. This asymmetric property is unique to qubit states.
In all three possible cases, all triplets of pure states for one of the three states in the Hilbert space of the system with Alice and Bob’s subspaces were combined into one four-dimensional subspace $\mathcal{H}_{AB}$. We can write the states thus:

$$|\psi_1\rangle = \sum_{i,j,...=1}^{2} a_{ij...}|\eta_{ij...}\rangle_{AB}|i\rangle_{AB}C_{ii...},$$

$$|\psi_2\rangle = \sum_{i,j,...=1}^{2} b_{ij...}|\nu_{ij...}\rangle_{AB}|i\rangle_{AB}C_{ii...},$$

$$|\psi_3\rangle = \sum_{i,j,...=1}^{2} c_{ij...}|\mu_{ij...}\rangle_{AB}|i\rangle_{AB}C_{ii...}.  \quad (4)$$

There are $n - 2$ indices $i, j, ...$. The states $\{|ij...\rangle_{CD...}\}$ form an arbitrary canonical basis for the $\bigotimes_{i,j,...=1}^{n-2} \mathcal{H}_{i}$ Hilbert space shared by Carol, Douglas et al. The complex coefficients $a_{ij...}, b_{ij...}$, and $c_{ij...}$ satisfy normalization constraints.

From Lemma 2, we know that a state $|\phi\rangle$ exists that is unentangled under the $\mathcal{H}_{AB} \bigotimes_{i,j,...=1}^{n-2} \mathcal{H}_{2}$ partition and which is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$ but nonorthogonal to $|\psi_3\rangle$. Let us write this state $|\phi\rangle = |\theta\rangle_{AB} \bigotimes |\omega\rangle_{CD...}$. Our choice of canonical basis $\{ |ij...\rangle_{CD...}\}$ for equations 4 was arbitrary, so we can specify retroactively that $|\omega\rangle_{CD...} = |00...0\rangle_{CD...}$. Then we know that that $|\phi\rangle$ satisfies the following equations:

$$\langle \psi_1 | \phi \rangle = a^*_{ij...} \langle \eta_{ij...} | \theta \rangle = 0,$$

$$\langle \psi_2 | \phi \rangle = b^*_{ij...} \langle \nu_{ij...} | \theta \rangle = 0,$$

$$\langle \psi_3 | \phi \rangle = c^*_{ij...} \langle \mu_{ij...} | \theta \rangle \neq 0. \quad (5)$$

Clearly this can be true only if $|\mu_{ij...}\rangle_{AB}$ is linearly independent from both $|\eta_{ij...}\rangle_{AB}$ and $|\nu_{ij...}\rangle_{AB}$. $|\eta_{ij...}\rangle_{AB}$ and $|\nu_{ij...}\rangle_{AB}$ are either linearly independent of one another, or they are identical. If they are identical, we can trivially find a candidate for $|\theta\rangle_{AB}$ that is a product state in $\mathcal{H}_{A} \bigotimes \mathcal{H}_{B}$, and $\mathcal{S}$ is SLOCC distinguishable by Lemma 4. If they are not identical, then $\{ |\eta_{ij...}\rangle_{AB}, |\nu_{ij...}\rangle_{AB}, |\mu_{ij...}\rangle_{AB}\}$ is a set of three pure linearly independent states, and from Lemma 5 there is some product state $|\xi\rangle$ that is nonorthogonal to exactly one of them. In this case, the state $|\xi\rangle_{AB} \bigotimes |\omega\rangle_{CD...}$ is a completely unentangled state in $\mathcal{H}_{2}$ satisfying Lemma 4 for one of the three states in $\mathcal{S}$ (though not necessarily $|\psi_3\rangle$). Therefore that state is conclusively locally identifiable, and $\mathcal{S}$ is SLOCC distinguishable. \square

This is the third and final step in our proof of Theorem 4. In all three possible cases, all triplets of pure linearly independent quantum states have been shown to be stochastically locally distinguishable.

If a set of states is stochastically locally distinguishable then complete ‘which state’ information is potentially locally discoverable. Otherwise it is necessarily hidden from local observation. In spite of the known links between entanglement and SLOCC distinguishability, we have shown that any three states can always be SLOCC distinguished, no matter how entangled. An open question is finding optimal SLOCC protocols, which would allow a quantitative comparison of the local and global situation.

**ACKNOWLEDGEMENTS**

We would like to thank Aidan Roy, Anirban Roy, Barry Sanders and Andrew Scott for useful discussions. JW acknowledges support from the Alberta Ingenuity Fund and the Pacific Institute for the Mathematical Sciences, and thanks DIRO at the Université de Montréal for their hospitality with assistance from the Canadian Institute for Advanced Research. SB acknowledges support from Alberta’s Informatics Circle of Research Excellence (iCORE), the Canadian Network of Centres of Excellence for the Mathematics of Information Technology and Complex Systems (MITACS), the Canadian Institute for Advanced Research, General Dynamics Canada, and the Natural Science and Engineering Research Council of Canada (NSERC).