Continuous variable quantum cryptography using two-way quantum communication

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In recent years, quantum cryptography has been developed into the continuous variable framework where it has been shown to fully exploit the potentialities of quantum optics. In this framework, we introduce novel "transform and measure" protocols which generalize and are proven to outperform the previous "prepare and measure" protocols. In these new protocols the secret information is encoded via random unitary transformations onto a quantum state which is transmitted forward and backward by the trusted parties. Thanks to this multiple quantum communication, Alice and Bob make an iterated use of the uncertainty principle which leads to a security enhancement. In particular, the security threshold is shown to behave like a superadditive quantity over multiple uses of the quantum channel. Our analysis investigates the simplest and non-trivial transform and measure protocols (i.e., the ones based on two-way quantum communication) whose security is tested against Gaussian attacks.

Recently quantum information has discovered the non-trivial advantages offered by continuous variable systems, i.e., quantum systems described by a set of observables, like position and momentum, having a continuous spectrum of eigenvalues. Accordingly, quantum key distribution protocols have been proved to be very powerful for their experimental feasibility and cryptographic protocols based on coherent states have been shown to fully exploit the potentialities of quantum optics. In this framework, we introduce novel "transform and measure" protocols which generalize and are proven to outperform the previous "prepare and measure" protocols. In these new protocols the secret information is encoded via random unitary transformations onto a quantum state which is transmitted forward and backward by the trusted parties. Thanks to this multiple quantum communication, Alice and Bob make an iterated use of the uncertainty principle which leads to a security enhancement. In particular, the security threshold is shown to behave like a superadditive quantity over multiple uses of the quantum channel. Our analysis investigates the simplest and non-trivial transform and measure protocols (i.e., the ones based on two-way quantum communication) whose security is tested against Gaussian attacks.

![Figure 1: One-way quantum cryptography with coherent states.](image)

In both protocols, Alice prepares a coherent state with random amplitude \( \gamma = (Q + iP)/2 \) to be sent to Bob. Then, in protocol (a1), Bob randomly measures quadrature \( Q \) or \( P \) via a homodyne detector, while, in protocol (b1), Bob performs a joint measurement of \( Q \) and \( P \) via a balanced beam splitter and two homodyne detectors.

\[
V = \begin{pmatrix}
VI & \sqrt{V^2 - 1}Z \\
\sqrt{V^2 - 1}I & VI
\end{pmatrix},
\]

where \( Z \equiv \text{diag}(1, -1) \), \( I \) the \( 2 \times 2 \) identity matrix, and \( V \) is a variance characterizing the source and connected to the squeezing parameter \( V = \cosh 2r \) [13]. One can easily show that heterodyning one mode of this EPR source is equivalent to the remote preparation of a coherent state whose amplitude \( \gamma \) is randomly modulated by a Gaussian of variance \( V - 1 \) (see Appendix). The EPR formulation of the protocol (a1) is depicted in Fig.2 where the attack of a potential eavesdropper, Eve, is also shown. Ref. [11] showed that the optimal attack against this protocol is
different protocols are able to exploit them with different appropriate homodyne measurements (Q or P) chosen for each run, Eve will consequently perform the appropriate homodyne measurements (Q or P) on her output modes $E'$ and $E''$. From such measurements, Eve will infer about Alice’s variable (direct reconciliation) or Bob’s variable (reverse reconciliation). An entangling cloner attack can be therefore characterized by two parameters, transmission T and variance W, which can be arranged in the unique quantity

$$\Sigma = W(1-T)T^{-1},$$

representing the variance of the Gaussian noise added to the channel. These quantities are evaluated by Alice and Bob by publishing part of their correlated continuous variables $Q_B$ and $Q_+$, or $P_B$ and $P_-$ (see Fig. 2). In this way, they perform an error analysis of the channel which reconstructs the correlation matrix $V'$ of their shared Gaussian state $\rho_{AB}$ and therefore their mutual information $I_{AB} = I(Q_{B'}, Q_+)$ (see Appendix). Similarly, they can evaluate $I_{AE}$ and $I_{BE}$, and therefore the two key-rates [17, 18] $I_{AB} - I_{AE}$ (direct reconciliation) and $I_{AB} - I_{BE}$ (reverse reconciliation). The security thresholds achieved for high modulation ($V \to +\infty$) are equal to $\Sigma = 1$ for direct reconciliation, and to $\Sigma = T^{-1} > 1$ for reverse reconciliation. In particular, for direct reconciliation, the optimal attack is given by an entangling cloner with $W = 1$, i.e., a beam splitter (lossy channel attack). In such a case, the threshold simply corresponds to $T = 1/2$, i.e., 3 dB of losses [9]. Similar results [10] hold for the other protocol (b1).

(a2) the “homodyne” protocol (a2) extends (a1) to two-way quantum communication. Here, Bob has an EPR source (with variance V), of which he keeps a mode $r$ while he sends the other refer-ence mode $R$ to Alice. Then, Alice randomly displaces this mode in phase-space, i.e., she applies a displacement operator [13] $D(\gamma)$ whose amplitude $\gamma = (Q+ iP)/2$ follows a Gaussian distribution with $\langle Q^2 \rangle = \langle P^2 \rangle = V$ and $\langle QP \rangle = \langle Q \rangle = \langle P \rangle = 0$. Finally, mode $B$ is then sent back to Bob. This mode contains Alice’s signal $\gamma$, since its quadratures are equal to $Q_B = Q_R + Q$ and $P_B = P_R + P$. In order to access this signal, Bob homodynes his modes $r$ and $B$ by choosing to measure their position or momentum at random. For instance, he can decide to measure positions $Q_r$ and $Q_B$, so that he can construct an optimal estimator of $Q_R$ (from $Q_r$) and, then, an estimator $Q^{(B)}_B$ of $Q = Q_B - Q_R$. Symmetrically, he can measure $P_r$ and $P_B$ to infer about $P$. The basis chosen for each run of the protocol will be classically communicated to Alice at the end of all the quantum communications, when the two trusted parties will share pairs of correlated continuous variables $\{Q, Q^{(B)}_B\}$ and $\{P, P^{(B)}_B\}$.

(b2) As for one-way protocols, Bob can attempt a joint measurement of quadratures $Q$ and $P$. This is achieved in the “heterodyne” protocol (b2) which
On the other hand, these new connection possibilities hold also for Bob, who may join his modes \( r \) and \( B \) before the measurement. In fact, the previous protocol can be modified into a sort of dense coding protocol where Bob performs a Bell-type measurement on modes \( r \) and \( B \). This measurement is achieved by inserting these modes into a beam splitter with suitable transmission \( G \) and, then, by homodyning the output ports. At the end of the paper, we will discuss this protocol in more detail.

In a general cryptographic scenario, the trusted parties evaluate Eve’s attack via suitable measurements on the quantum channel, whose outcomes are then used for a subsequent classical analysis of the errors. For these new protocols, Alice and Bob must monitor both the forward and the backward channels. This is achieved if Alice randomly switches from previously described key-distribution stages, also known as key modes or message modes, to “tomographic” stages, also known as control modes [21, 22]. During a control mode, Alice does not encode any information, but detects reference mode \( R \) and sends a new reference mode \( \tilde{R} \) back to Bob, which is then detected in the usual way (i.e., according to the message mode of the protocol). In particular, in (a2) Alice homodynes \( R \) and sends \( \tilde{R} \) using a new EPR source, while, in (b2) and (c2), it is sufficient for Alice to perform heterodyne detection and prepare a returning coherent state. In this way, Alice and Bob are able perform two distinct error analyses, of the forward and backward channels, by comparing the outcomes of their measurements.

As already said, eavesdropping of two-way quantum cryptography is more difficult to study since Eve may adopt strategies which exploit correlations among the forward and the backward channels. Consider then the protocol (a2), and let us derive the optimal attack which can be derived by combining the optimal attacks of the corresponding one-way protocol (a1). If we label the forward and backward channels by \( i = 1, 2 \) respectively, then we must combine two entangling cloners with free parameters \( T_1, W_1 \) and \( T_2, W_2 \) (i.e., added noises \( \Sigma_1 \) and \( \Sigma_2 \)). By homodyning their outputs in the correct basis, Eve
constructs an optimal estimator \( Q^{(E)} \) (or \( P^{(E)} \)) of Alice’s variable, which enables her to share a mutual information \( I_{AE} = (1/2) \ln(V/V_{AE}) \) [22], where the conditional variance \( V_{AE} = V_{Q^{(E)}} = \sum_{P} P_{E} \) quantifies Eve’s remaining uncertainty on Alice’s variable. Similarly, Bob’s estimator \( Q^{(B)} \) (or \( P^{(B)} \)) leaves him with a conditional variance \( V_{AB} = V_{Q^{(B)}} = \sum_{P} P_{E} \). For high modulation \( V \rightarrow +\infty \) and non-trivial attacks \( T_{i} \neq 0,1 \), one computes

\[
V_{AB} = \frac{T_{2}(1-T_{1})W_{1} + (1-T_{2})W_{2}}{T_{2}},
\]

\[
V_{AE} = \frac{T_{2}(1-T_{1})W_{2}^{-1} + (1-T_{2})W_{1}^{-1}}{(1-T_{1})(1-T_{2})}.
\]

The optimal attack can be identified by the minimum of \( V_{AB} V_{AE} \), so that Eve maximizes the perturbation on the channel \( V_{AB} \) while she maximizes the acquired information (inverse of \( V_{AE} \)). Such product takes the minimum value \( V_{AB} V_{AE} = 4 \) for

\[
W_{2} = 1 \quad \text{and} \quad T_{2} = [1 + (1-T_{1})W_{1}]^{-1},
\]

which corresponds to considering an entangling cloner with free parameters \( (T_{1}, W_{1}) \) on the forward channel, followed by a beam splitter with a correlated transmission \( T_{2} = f(T_{1}, W_{1}) \) on the backward channel (see Fig. 4). In order to support the optimality of this attack, consider the singular case \( T_{1} = T_{2} = 1 \). In such a case, Eve is excluded and Bob performs the best possible estimation strategy to evaluate Alice’s relevant transformation. The same strategy would hold for Eve in the other singular case \( T_{1} = T_{2} = 0 \) where Bob is excluded. In the nontrivial cases \( T_{i} \neq 0,1 \), the considered attack mixes these two optimal estimation strategies with a suitable Gaussian transformation.

In order to derive the security threshold we impose \( I_{AB} = I_{AE} \), which is equivalent to \( V_{AB} = V_{AE} \). By using Eqs. (3), (4) and (5), we get \( W_{1} = (1-T_{1})^{-1} \) and \( T_{2} = 2/2 \). These parameters characterize the curve of the threshold attacks which have a total noise equal to

\[
\Sigma = \Sigma_{1} + \Sigma_{2} = 1 + T_{1}^{-1}.
\]

It follows that the two-way security threshold satisfies \( \Sigma > 2 \), and therefore it is more than doubling the threshold \( \Sigma = 1 \) of the corresponding one-way protocol. Thanks to this superadditivity, two-way quantum cryptography brings a non-trivial security improvement. In fact, even when the communication channel is too noisy for one-way quantum key distribution to be secure, it can still be used to provide secure quantum key distribution using our two-way scheme.

In order to further support this superadditivity, we study the powerful attacks based on pure losses. In two-way quantum cryptography a lossy channel attack is realized by using two beam splitters with suitable transmissions \( T_{1} \) and \( T_{2} \) (see Fig. 4). Once the right basis is publicized by Bob, Eve homodynes their output ports \( E_{1}' \) and \( E_{2}' \) to infer about the signal. Since two beam splitters are two entangling cloning with \( W_{1} = W_{2} = 1 \), from Eqs. (3), (4) we get

\[
V_{AB} = \frac{1-T_{1}T_{2}}{T_{2}},
\]

and

\[
V_{AE} = \frac{1-T_{1}T_{2}}{(1-T_{1})(1-T_{2})}.
\]

Then, from \( V_{AB} = V_{AE} \) we get the threshold curve for this kind of attack, to be

\[
T_{2} = (1-T_{1})(1-T_{2}).
\]

The total transmission \( T = T_{1}T_{2} \) has a maximum equal to \( 3 - 2\sqrt{2} \) on this curve. Such a value corresponds to a threshold of about 7.65dB of losses, to be compared with the 3dB limit of the one-way protocol.

More strongly, we prove that this threshold remains the same when we change the nature of the lossy channel attack from individual to collective. In the collective attack, Eve keeps her output probes till the end of the protocol, when she exploits all the classical information exchanged by Alice and Bob to perform a final coherent measurement on all her probes. In such a case, the key rate is bounded by \( I_{AB} - \chi_{E} \) where \( \chi_{E} \) is the Holevo information of the ensemble \( \rho_{E} = \int G(Q)\rho_{E}(Q)dQ \), where \( \rho_{E}(Q) \) is the Eve’s state conditioned to Alice’s encoding \( Q \). and \( G(Q) \) is a Gaussian distribution with variance \( \langle Q^{2} \rangle = V \). For high modulation and \( T_{i} \neq 0,1 \), one can prove (see Appendix) that

\[
\chi_{E} = \frac{1}{2} \ln \left[ \frac{V(1-T_{1})(1-T_{2})}{1-T_{1}T_{2}} \right].
\]
In the same limit, Alice and Bob’s mutual information is given by

\[ I_{AB} = \frac{1}{2} \ln \left( \frac{V}{V_{AB}} \right) - \frac{1}{2} \ln \left( \frac{T_2 V}{1 - T_1 T_2} \right) \]  

(11)

and, therefore, the threshold condition \( I_{AB} = \chi_E \) gives the same curve of Eq. (9). In other words, the security threshold against collective attacks based on pure losses is again 7.65dB [23].

In order to make a simple and direct comparison among the various two-way protocols of Fig. 3 we also test the heterodyne protocol \( (b2) \) and the dense-coding protocol \( (c2) \) against collective attacks based on pure losses. In the protocol \( (b2) \), Bob detects both the quadratures with the aim of doubling the rate of key distribution. However, this improvement would exist only in the ideal case of a lossless channel, while it rapidly vanishes in presence of losses. In fact, Bob has now a less efficient way to decode Alice’s information due to the vacuum entering the unused ports of his beam splitters (see Fig. 3). This vacuum becomes more and more destructive when the signal is progressively damped by the losses on the channel. This is the reason why the threshold of this protocol is found to be worse than before, even if much better than the corresponding one-way protocol. For high modulation (and \( T_i \neq 0,1 \)), one has \( I_{AB} = \ln(T_2 V/2) \) while Eve’s accessible information is bounded by

\[ \chi_E = \ln \left( \frac{eV(1 - T_1)(1 - T_2)}{2(1 - T_1 T_2)} \right) . \]  

(12)

From the condition \( I_{AB} = \chi_E \) one finds the curve

\[ T_2(1 - T_1 T_2) = e(1 - T_1)(1 - T_2) . \]  

(13)

On this curve the total transmission \( T \equiv T_1 T_2 \) has a maximum equal to \( e(e + 4)^{-1} \), corresponding to a threshold of about 3.93dB. Such a value must be compared with the threshold of 1.4dB found for the corresponding one-way protocol \( (b1) \) against collective lossy channel attacks [24]. Also in this case the superadditivity of the security threshold is clearly shown.

The protocol \( (b2) \) has the remarkable property to be fully realizable using coherent states, but this property is lost when we change Bob’s measurement and we consider the Bell-like measurement of the dense coding protocol \( (c2) \). This measurement removes the unwanted vacuum noises entering Bob’s beam splitters and, therefore, enables a more efficient decoding of Alice’s information. By adopting this joint measurement, Bob extends the efficient decoding strategy of the protocol \( (a2) \) from one to both quadratures, and therefore he is able to achieve a security threshold better than before. However, the corresponding increase in the complexity of the protocol is non-trivial. In fact, the beam splitter used for the Bell measurement has a transmission \( G \) equal to 1/2 only in the case of a lossless channel, for which we have exactly the dense coding scheme of Ref. [12]. In the non-trivial cases, one can easily prove that the optimal transmission \( G \) for signal decoding depends on the total transmission \( T \) via the relation \( G = T(1 + T)^{-1} \). Since the exact value of \( G \) is known to Bob only after the error analysis of the channel, he must keep all the states till the end of the protocol, when he will be able to perform the relevant Bell-like measurement. By adopting the optimal value

\[ G = T(1 + T)^{-1} , \]  

(14)

we find

\[ I_{AB} = \ln \left( \frac{T_2 V}{1 - T_1 T_2} \right) , \]  

(15)

for high modulation (and \( T_i \neq 0,1 \)). As expected, this value exactly doubles the mutual information of Eq. (11), relative to the protocol \( (a2) \). On the other hand, Eve’s information \( \chi_E \) is exactly the same of the protocol \( (b2) \) and given in Eq. (12). It follows that the threshold condition \( I_{AB} = \chi_E \) gives the curve

\[ 2T_2 = e(1 - T_1)(1 - T_2) , \]  

(16)

which leads to a maximum of about 0.21 for the total transmission \( T \). Note that such a value corresponds to about 6.75dB, so that protocol \( (c2) \) outperforms protocol \( (b2) \) but does not reach the same performance of protocol \( (a2) \). Actually, the security performances of the homodyne protocol \( (a2) \) against lossy channel attacks are not reproducible by any two-way protocols based on a joint measurement of the quadratures. In fact, even if we allow Bob to perform a final coherent measurement on all his states in order to retrieve the full signal \( (Q,P) \) encoded by Alice, one can verify that his accessible information is bounded by

\[ \chi_B = \ln \left( \frac{eT_2 V}{2(1 - T_1 T_2)} \right) , \]  

(17)

for high modulation and \( T_i \neq 0,1 \) (see Appendix). An upper bound to the security threshold will then be given by \( \chi_B - \chi_E = 0 \). Using Eqs. (12) and (17), one gets the same curve of Eq. (9) and, therefore, the same security threshold 7.65dB of the protocol \( (a2) \). Such a value will then represent an upper bound to all possible thresholds that are achievable by adopting two-way protocols with joint measurements. The possible advantages brought by this kind of protocol comes out for low losses, where they are able to increase the rate of key distribution. In such a case, in fact, their rate is comparable with Eq. (17), and this value roughly doubles the low-loss rate of the two-way protocol \( (a2) \), that is comparable with the mutual information of Eq. (11).
In conclusion, multi-way quantum cryptography represents a new environment to develop both novel and pre-existing quantum key distribution protocols. Here, we have presented the fundamental blocks of this new environment, the simplest non-trivial TM protocols, which are the ones based on two-way quantum communication. We have then studied their security against Gaussian attacks. We have first derived the optimal individual attack which can be constructed from the known optimal strategies and, then, we have tested all the protocols against collective attacks based on pure losses. In each case, we have proved the superadditive behavior of the security threshold which makes the two-way quantum cryptography profitably more secure.

APPENDIX

Estimators and remote state preparation

Consider the general scenario where Alice and Bob share two modes A and B, whose quadratures $\xi \equiv (Q_A, P_A, Q_B, P_B)$ satisfy the canonical commutation relations $[\xi_i, \xi_m] = 2\hbar \eta_{im}$, where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Suppose that modes A and B are described by a bipartite Gaussian state $\rho_{AB}$, with displacement $d \equiv \xi$ set to zero and correlation matrix $V$, where $V_{im} = (\xi_i \xi_m + \xi_m \xi_i)/2$. This real and symmetric matrix must satisfy the Heisenberg principle

$$V + iJ \geq 0,$$

which takes the form $\langle Q^2_A \rangle \langle P^2_A \rangle \geq 1$ for the diagonal elements. All the quantum and/or classical correlations between the modes are described by this matrix which we assume to be completely known to the parties.

Then, suppose that Alice homodynes mode A and Bob homodynes mode B, and they project onto the same quadrature, e.g. $Q$. Thanks to the shared correlations, Alice is then able to infer on Bob’s outcome $Q_B$ from the outcome $Q_A$ of her measurement [11]. In fact, from $Q_A$ Alice can construct the optimal estimator $Q_B^{(A)} = \kappa Q_A$ of the variable $Q_B$ where $\kappa \equiv \langle Q_A Q_B \rangle / \langle Q^2_A \rangle^{-1}$ is directly computable from the correlation matrix. After her estimation, Bob’s variable $Q_B$, with initial variance $V_{QB} = \langle Q^2_B \rangle$, will be reduced to the conditional variable $Q_{B|A} = Q_B - Q_B^{(A)}$ with conditional variance

$$V_{QB|QA} = \langle Q^2_{B|A} \rangle = \langle Q^2_B \rangle - \frac{\langle Q_B^{(A)} Q_B \rangle^2}{\langle Q^2_B \rangle^2} = \langle Q^2_B \rangle - \frac{\langle Q_A Q_B \rangle^2}{\langle Q^2_A \rangle}.$$

Thanks to Alice’s estimation, the Shannon entropy $H(Q_B) = (1/2) \ln V_{QB}$ of Bob’s variable has been reduced to the conditional entropy $H(Q_B|Q_A) = (1/2) \ln V_{QB|QA}$ [22]. Therefore, the mutual information of Alice and Bob will be given by

$$I(Q_B, Q_A) = H(Q_B) - H(Q_B|Q_A) = \frac{1}{2} \ln \frac{V_{QB}}{V_{QB|QA}}.$$

Now, if we do not consider Bob’s measurement, Alice’s local measurement corresponds to a remote state preparation at Bob’s site. In fact, her measurement is a Gaussian quantum operation that projects Bob’s mode onto a Gaussian state which is centered in the point $\{Q_B^{(A)}, 0\}$ of phase space and has uncertainties equal to $V_{QB|QA}$ of Eq. (20) and

$$V_{PB|PA} \geq V_{PB|PA}^{-1},$$

according to the Heisenberg principle of Eq. (19). More generally, Alice can remotely prepare a Gaussian state by making a joint measurement of her quadratures $Q_A$ and $P_A$. For instance, she can perform a heterodyne detection by inserting her mode A into a balanced beam splitter and, then, by detecting quadratures $Q_A$ and $P_-\,\,$ of the output modes ‘$\pm’$ (see Fig. 2). From the outcomes $(Q_+, P_-)$, Alice can construct two optimal estimators $Q_B^{(+)}, P_B^{(-)} = \xi_- P_-$, so that Bob’s variables $Q_B$ and $P_B$ are reduced to the conditional ones $Q_{B|+} = Q_B - Q_B^{(+)}$ and $P_{B|-} = P_B - P_B^{(-)}$ with conditional variances $V_{QB|Q_+}$ and $V_{PB|P_-}$ (computable from the correlation matrices of $\rho_{AB}$ and $\rho_{-B}$ according to Eq. (20)). In other words, Alice remotely prepares a Gaussian state centered in $\{Q_B^{(+)}, P_B^{(-)}\}$ with uncertainties $V_{QB|Q_+}$ and $V_{PB|P_-}$. In particular, if the shared Gaussian state $\rho_{AB}$ is an EPR source with variance V (see Eq. (11)) then $V_{QB|Q_+} = V_{PB|P_-} = 1$, and therefore Alice prepares a coherent state $|\gamma\rangle$ with amplitude

$$\gamma = |Q_B^{(+)} + iP_B^{(-)}|/\sqrt{2}.$$ Due to the probabilistic behavior of the measurement, amplitude $\gamma$ represents a complex random variable over many instances of the process. Such a variable follows a Gaussian distribution with zero mean and second moments given by $\langle Q_B^{(+)} \rangle = \langle P_B^{(-)} \rangle = V - 1$ and $\langle Q_B^{(+)2} \rangle = \langle P_B^{(-)2} \rangle = 0$. Therefore, the physical scheme where Alice and Bob share an EPR source with variance V and Alice heterodynes her mode is equivalent to a black-box where Alice prepares a coherent state whose amplitude is modulated by a Gaussian distribution with variance $V - 1$. In this sense, prepare and measure schemes using coherent states are equivalent to EPR schemes.

Computation of the relevant entropies

Consider the case of a collective lossy channel attack against the protocol (a2), where Eve exploits two beam
In the same limit, the entropy becomes
\[
S_E = g(\nu_-) + g(\nu_+) \to 2 + \ln \left[ \frac{1}{4} \lim_{V \to +\infty} \sqrt{\det \mathbf{V}_{12}} \right] = 2 + \ln \left[ \frac{V^2}{4} (1-T_1)(1-T_2) \right].
\] (33)

The conditional entropy \(S_{E|A}\) can be computed from the symplectic eigenvalues of the matrix \(\mathbf{V}_{E|A}\). It is easy to verify that \(\mathbf{V}_{E|A}\) can be derived from \(\mathbf{V}_E\) by setting \(\Omega(0, V)\) in Eq. (23). Then, repeating the previous steps, one finds
\[
S_{E|A} \to 2 + \frac{1}{2} \ln \left[ \frac{V^3}{16} (1-T_1)(1-T_2)(1-T_1T_2) \right],
\] (34)
so that \(\chi_E\) is equal to Eq. (10).

Consider now a collective lossy channel attack against the protocol (b2). Eve’s entropy \(S_E\) is the same as before, while the partial entropy \(S_{E|A}\) is now conditioned to both Alice’s variables \(Q\) and \(P\). For this reason, such entropy derives from a conditional correlation matrix \(\mathbf{V}_{E|A}\) which is computed from \(\mathbf{V}_E\) by setting \(\Omega(0,0)\) in Eq. (23). For non-trivial attacks and high modulation, one finds
\[
S_{E|A} \to 1 + \ln \left[ \frac{V^2}{2} (1-T_1T_2) \right],
\] (35)
and, therefore, the Holevo information \(\chi_E\) is consequently equal to Eq. (12). Since Eve’s entropies do not depend on Bob’s measurements but only on Alice’s encoding, the result for \(\chi_E\) is the same for the protocol (c2).

In the end, consider again a collective lossy channel attack with beam splitters \(T_1\) and \(T_2\), but now allow Bob to perform a final coherent measurement on all his states, from which he tries to to retrieve the full signal \(\gamma = (Q + iP)/2\) encoded by Alice. Bob’s modes \(r\) and \(B'\) are described by a state \(\rho_B(\gamma)\) which is conditioned to Alice’s encoding \(\gamma\). On average, Bob gets an ensemble \(\rho_B = \int G(\gamma) \rho_B(\gamma) d\gamma\), where \(G(\gamma)\) is a Gaussian distribution with \(\langle Q^2 \rangle = \langle P^2 \rangle = V\) and \(\langle QP \rangle = 0\). The Holevo information of Bob is then equal to \(\chi_B = S_B - S_{B|A}\), where the two Von Neumann entropies \(S_B\) and \(S_{B|A}\) are computable from the correlation matrices of the states \(\rho_B\) and \(\rho_B(\gamma)\) exactly as before. One can verify that \(\rho_B\) has the correlation matrix
\[
\mathbf{V}_B = \begin{pmatrix} \mu \mathbf{I} & \varphi \mathbf{Z} \\ \varphi^* \mathbf{Z}^* & \varsigma + \Omega(V) \mathbf{I} \end{pmatrix}
\] (36)
where
\[
\varphi \equiv \sqrt{T_1 T_2 (V^2 - 1)},
\] (37)
\[
\varsigma \equiv 1 + T_1 T_2 (V - 1),
\] (38)
and
\[
\Omega(V) = T_2 V.
\] (39)
For non-trivial attacks and high modulation, the symplectic eigenvalues of $V_B$ become proportional to $V$ and, therefore, the entropy becomes

$$S_B \to 2 + \ln \left( \frac{1}{4} \lim_{V \to +\infty} \sqrt{\det V_B} \right) = 2 + \ln \left( \frac{T_2 V^2}{4} \right).$$

(40)

On the other hand, the correlation matrix $V_{B|A}$ of $\rho_B(\gamma)$ can be computed by inserting $\Omega(0)$ in Eq. (36). In the usual limit, we have $\nu_- = 1$ and $\nu_+ \to V(1 - T_1 T_2)$ so that

$$S_{B|A} = g(\nu_+) \to 1 + \ln \left( \frac{V}{2} (1 - T_1 T_2) \right). \quad (41)$$

From Eqs. (40) and (41), one easily gets the Eq. (17) for the Holevo information $\chi_B$.

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[23] This value must be again compared with 3dB. Note that also the threshold of the one-way protocol (a1) remains unchanged when we pass to collective lossy channel attacks [24, 25].