HAMILTONIAN THEORY OF BRANE-WORLD GRAVITY

ZOLTÁN KOVÁCS† and LÁSZLÓ Á. GERGELY‡

† Max-Planck-Institut für Radioastronomie,
Auf dem Hügel 69, D-53121 Bonn, Germany
‡ Departments of Theoretical and Experimental Physics, University of Szeged,
Dóm tér 9, H-6720 Szeged, Hungary
zkovacs@mpifr-bonn.mpg.de, gergely@physx.u-szeged.hu

A brane-world universe consists of a 4-dimensional brane embedded into a 5-dimensional space-time (bulk). We apply the Arnowitt-Deser-Misner decomposition to the brane-world, which results in a 3+1+1 break-up of the bulk. We present the canonical theory of brane cosmology based on this decomposition. The Hamiltonian equations allow for the study of any physical phenomena in brane gravity. This method gives new prospects for studying the initial value problem, stability analysis, brane black holes, cosmological perturbation theory and canonical quantization in brane-worlds.

Keywords: canonical gravity, brane-world, embedding variables

The Hamiltonian theory of the brane-world scenario is based on the foliation of the 4-dimensional (4d) world-sheet (the brane) which is embedded into the 5-dimensional (5d) space-time manifold (the bulk \( \mathcal{B} \)). Since the 3-dimensional (3d) space-like slices of the foliation admit tangent bundles of co-dimension 2 with respect to \( \mathcal{B} \), the slices form a two-parameter family of 3-spaces \( \Sigma_{t\chi} \) with \( t, \chi \in \mathbb{R} \) embedded in \( \mathcal{B} \). While the parameter \( t \) represents the many-fingered time in the canonical formalism, a new parameter \( \chi \) defines the position of the brane in the bulk. A common choice is at \( \chi = 0 \).

The 3+1+1 decomposition of the 5d brane-world geometry allows one to express the 5d field equations in terms of 3d quantities. These gravitational variables in these picture are the three metric \( g_{ab} \) describing the intrinsic geometry of the slices \( \Sigma_{t\chi=0} = \Sigma_t \), a vector field \( M^a \) and a scalar field \( M \). The vector and scalar quantities describe the contribution of the bulk-gravity. The extrinsic curvature of the leaves embedded in \( \mathcal{B} \) is given by the second fundamental forms \( K_{ab} \) and \( L_{ab} \), the normal fundamental forms \( K^a = L^a \), and scalars \( K \) and \( L \) associated with the two time-like and space-like normal vector fields \( n^a \) and \( l^a \) of \( \Sigma_t \). The quantities \( K_{ab}, K^a \) and \( K \) are equivalent with the time-derivatives of \( g_{ab}, M^a \) and \( M \), respectively, whereas \( L_{ab} \) and \( L \) contain only pure spatial derivatives of them.\(^1\)

As a result of the decomposition, the 5d Einstein-Hilbert action

\[
S^G \mathcal{[G]} g_{ab} = \int d^5 x L^G = \int d^5 x \sqrt{-g^{(5)} \mathcal{R}}
\]  

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(5) $g_{ab}$ is the bulk metric and (5) $R$ its scalar curvature) can be expressed in terms of the set $(g_{ab}, M^a, M; K_{ab}, K^a, K, L_{ab}, L)$, the lapse function $N$ and the non-vanishing components of the shift vector $N^a$. The component of the shift vector associated with the extra dimension is set to zero by the condition of the confinement of the matter fields on the brane. With the set $(g_{ab}, M^a, M)$ chosen as canonical coordinates of the vacuum bulk gravity, we can express the Lagrangian in the action (1) solely in the terms of canonical coordinates and their time derivatives. Then we introduce the momenta $(\pi^{ab}, p_a, p)$ conjugated to the canonical coordinates, such that the phase space of the 5d vacuum gravity is the set

$$(g_A; \pi^A | A = 1, 2, 3) := (g_{ab}, M^a, M; \pi^{ab}, p_a, p)$$

with the abstract index $A$ defined as $g_1 = g_{ab}, g_2 = M^a$, etc. In order to specify the possible states in the phase space, we first need the evolution equations, then the vacuum constraint equations which restrict these solutions. (In fact they restrict only the initial data. Once they are imposed, the dynamics preserves the constraints.)

The Legendre transformation of the decomposed vacuum Lagrangian yields to the Hamiltonian $H^G$ of the bulk gravity. This is a linear combination of the super-Hamiltonian constraint $H^G$ and supermomentum constraint $H^G_a$:

$$H^G[g_A, \pi^A; N, N^a] = N H^G[g_A, \pi^A] - N^a H^G_a[g_A, \pi^A].$$

When inserting the Lagrangian

$$L^G[g_A, \pi^A, N, N^a] = \pi^A \dot{g}_A - H^G[g_A, \pi^A, N, N^a],$$

into the action (1) and extremizing it with respect to the canonical variables (2), the lapse function and the shift vector, we obtain the equations of motion and the constraints of the 5d vacuum gravity.

The Poisson brackets of any pair of functions on the phase space, as in the field theories, can be defined with the help of the functional derivatives of the functions with respect to the canonical variables or merely via the canonical commutation relations. Then the dynamical equations of the bulk gravity lead to the forms

$$\dot{g}_A(x, \chi) = \{g_A(x, \chi), H^G[N]\},$$

$$\dot{\pi}^A(x, \chi) = \{\pi^A(x, \chi), H^G[N]\}$$

for any $x \in \Sigma_{t\chi}$ and $\chi \in \mathbb{R}$, where the notation $H^G[N]$ includes the smearing of the Hamiltonian density (3) with $N$ and $N^a$.

When matter fields couple to gravity on the brane, we enlarge the phase space of the 5d geometry with the canonical variables of the matter source. After decomposing the stress-energy tensor in the matter action, the Hamiltonian of the matter fields can be derived as well. Then the dynamical and the constraint equations of gravity must be supplemented with the contributions from matter, a procedure resulting in the time-evolution equations and the constraints of the total system of gravity and matter.
The Hamiltonian formalism for brane-world scenarios gives a suitable starting point for canonical quantization. In this approach the quantum state of gravity should be described by a state functional $\Psi(t, x, \chi; g_A)$ over the configuration space $(g_A)$. This functional incorporates not only the intrinsic 3-geometries of the leaves $\Sigma_{t\chi}$, but also the brane-off contributions of the bulk gravity. Dirac constraint quantization imposes either the vacuum constraints $H_{G\bot}$ and $H_{Ga}$, or the constraints of the gravity coupled to matter on the state functional as operator equations, restricting the possible states of the system:

$$\hat{H}^{G\bot}_{G\bot}(t, x, \chi; g_A, \pi_A)\Psi(t, x, \chi; g_A) = 0,$$
$$\hat{H}^{Go}_{Ga}(t, x, \chi; g_A, \pi_A)\Psi(t, x, \chi; g_A) = 0. \quad (4)$$

By inserting the explicit form of the super-Hamiltonian constraint into the first equation, where the canonical momenta $\pi_A$ are represented by the operators $\hat{\pi}^A = -i\delta/\delta g_A$, we obtain a second order functional differential equation of the state functional with respect to the canonical coordinates. The latter is the Wheeler-deWitt equation of the brane-world gravity, which may be simplified by applying the operator restriction of the super-momentum constraint from Eqs. (4).

We expect that the quantization of simple models already done in the context of general relativity, such as the mixmaster universe$^2$ or the Einstein-Rosen waves$^3$, is possible in the brane-world scenario either. The latter general relativistic model illustrates, together with other examples,$^4,5$ that various procedures can be found which transform the constraint equations into new constraints, such that the momentum canonically conjugated to the time variable enters the new super-Hamiltonian only linearly. Hence the quantization of these systems leads to a functional Schrödinger equation instead of the second order functional differential equation. There is hope that similar procedures based on the Hamiltonian formulation of brane-world gravity$^1$ will be successful for various simple brane-world models as well.

References