Superconformal $R$-charges and dyon multiplicities
in $\mathcal{N} = 2$ gauge theories

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Abstract

$\mathcal{N} = (2, 2)$ theories in 1+1D exhibit a direct correspondence between the $R$-charges of chiral operators at a conformal point and the multiplicities of BPS kinks in a massive deformation, as shown by Cecotti and Vafa. We obtain an analogous relation in 3+1D for $\mathcal{N} = 2$ gauge theories that are massive perturbations of Argyres-Douglas fixed points, utilizing the geometric engineering approach to $\mathcal{N} = 2$ vacua within IIB string theory. In this case the scaling dimensions of a certain subset of chiral operators at the UV fixed point are related to the multiplicities of BPS dyons. When the Argyres-Douglas SCFT is realized at the root of a baryonic Higgs branch, this translation from 1+1D to 3+1D can be understood physically from the relation between the bulk dynamics and the $\mathcal{N} = (2, 2)$ worldsheet dynamics of vortices in the baryonic Higgs phase. Under a relevant perturbation, the BPS kink multiplicity on the vortex worldsheet translates to that of the bulk dyonic states. The latter viewpoint suggests the 3+1D version of the Cecotti-Vafa relation may hold more generally, and simple tests provide evidence in favor of this for more generic choices of the baryonic root.

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1. Introduction

The modern Wilsonian interpretation of quantum field theory, with its emphasis on conformal fixed points and the renormalization group (RG) flows which connect them, naturally leads on to questions concerning the general classification of this RG structure. In generic quantum field theories in 3+1D, making progress in this direction appears a daunting problem, as in practice we still have relatively little knowledge of the basic configuration space, namely the space of allowed conformal fixed points. The addition of supersymmetry naturally assists in this regard, and recent developments such as $a$-maximization [1] have allowed significant progress to be made in terms of understanding the properties of $\mathcal{N} = 1$ superconformal points, and some of the flows that connect them [2].

In the more restricted context of field theories in 1+1D, the situation is more clear cut. There are general schemes for classifying fixed points and also powerful constraints, such as the Zamolodchikov $c$-theorem [3], which constrain the allowed RG flows between them. The situation with $\mathcal{N} = (2, 2)$ supersymmetric theories is under even greater control, to the extent that general classification theorems [4] have been discussed for the full space of quantum field theories, conformal or otherwise. One aspect of this control is the elucidation of some interesting relations between properties of superconformal field theories (SCFTs) and certain features of the non-conformal theories to which the CFTs flow under relevant perturbations. In the context of $\mathcal{N} = (2, 2)$ theories, one of the most interesting was obtained by Cecotti and Vafa [4]. It can be written in the form,

$$\text{evals}(\mathcal{M}) = \exp\left[2\pi i \left(q_k - \frac{\hat{c}}{2}\right)\right],$$

where the monodromy matrix $\mathcal{M}$ on the left-hand side, to be discussed further below, is determined by the BPS soliton spectrum in the massive deformation of a UV SCFT, while the right-hand side is given by the $R$-charges $q_k$ of operators in the chiral ring of the undeformed SCFT, and the corresponding central charge $\hat{c}$. In this sense it is a nontrivial UV-IR relation, albeit a special one that relates protected BPS quantities in a supersymmetric theory.

It is tantalizing to think that similar relations may exist in 3+1D. In general, given the significantly richer phase structure and space of allowed relevant deformations, this seems a forlorn hope, but we will argue that in certain cases such constraints must hold, and indeed that the UV-IR relation retains essentially the same interpretation. The specific example we have in mind concerns $\mathcal{N} = 2$ gauge theories that are massive deformations of specific Argyres-Douglas SCFTs [5]. In particular, we will argue that the simplest proof of (1) in 1+1D, for Landau-Ginzburg theories [4], can actually be translated quite directly to the 3+1D theory by making use of the geometric engineering approach to $\mathcal{N} = 2$ vacua in IIB string theory [6, 7, 8, 9]. The superpotential of the Landau-Ginzburg model emerges from the geometry of a local Calabi-Yau singularity, and the associated holomorphic structure required for this particular proof of (1) in 1+1D can be translated to the 3+1D $\mathcal{N} = 2$ gauge theory realized at low energy in this background.

The appearance of the analogue Landau-Ginzburg system in the construction outlined above can be given a clear physical interpretation if this SCFT is realized at the root of the baryonic Higgs branch in $\mathcal{N} = 2$ SCQD with gauge group $U(N)$ and $N_f = N$ fundamental
hypermultiplets, in particular, in recent years a very precise correspondence between the dynamics of these theories and the $\mathcal{N} = (2, 2)$ worldsheet dynamics of BPS vortices has been developed, where the vortices in question are present within the theory in the baryonic Higgs phase itself [10, 11, 12, 13, 14]. In the interior of the vortex, a U(1) subgroup of the unbroken gauge symmetry at the root of the baryonic branch is restored, and it is in this sense that the vortices capture certain features of the full quantum gauge dynamics in the Coulomb phase of the theory. Indeed, beyond providing a precise window to the quantum vacuum structure on the Coulomb branch, there is also an exact map between a subset of BPS states; in particular confined monopoles in the bulk of the Higgs phase translate to BPS kinks in the worldsheet vortex dynamics [15]. In this context, if the baryonic root is tuned near to an Argyres-Douglas point, it has recently been shown that the worldsheet dynamics of the vortex is a Landau-Ginzburg system, specifically a perturbed minimal model [16]. This dynamics precisely realizes the analogue 1+1D system appearing in the geometric engineering construction. Thus, this correspondence allows for a direct translation of (1), interpreted now as a relation within the worldsheet theory of the vortex, to the physics of the bulk gauge theory in 3+1D at the root of the baryonic Higgs branch. In detail, this mapping applies only to the special case where the baryonic root is a massive perturbation of an Argyres-Douglas SCFT, and does not immediately generalize to more generic examples. Nonetheless, the existence of a more general proof of (1) in 1+1D [4] and the general correspondence between vortex dynamics and the bulk gauge theory, suggests that the translation to 3+1D should have more general validity. Indeed some exploratory checks appear to confirm that the relation does hold for more generic choices of the baryonic root.

We will begin in §2 by first discussing the geometric engineering approach to $\mathcal{N} = 2$ gauge theories, and specifically the regime near an Argyres-Douglas point as considered by Shapere and Vafa [9], which allows the 1+1D proof of (1) to be lifted to the bulk theory via the identification of an analogue Landau-Ginzburg superpotential in the complex structure of the geometry. In §3, we turn to the interpretation of this mapping from 1+1D to 3+1D, and describe how the dynamics of vortices in SQCD, recently discussed by Tong [16], provides a physical realization of the analogue Landau-Ginzburg system. This point of view also suggests that (1) may have more general validity in 3+1D and we find that anecdotally it does hold for a more general choice of the baryonic root, where both the bulk and worldsheet physics is somewhat different. We summarize in §4 with some additional remarks and possible generalizations.

2. Perturbed Argyres-Douglas points and BPS states

In testing a relation such as (1) in 3+1D, we require first of all a convenient technique to determine the multiplet structure of BPS states. In $\mathcal{N} = 2$ gauge theories, computing the degeneracies of BPS monopoles and dyons is a rather nontrivial problem in general, and has not been studied in detail away from the weak coupling regime, with the exception of a few special cases with, for example, an SU(2) gauge group. Fortunately, we are concerned here only with those states that become massless at the conformal point, which is therefore a subset of the full BPS spectrum, that which is massive, but light, in a region of the Coulomb branch near the Argyres-Douglas point. The geometric engineering approach to
$\mathcal{N} = 2$ theories within IIB string theory provides a very convenient framework for this analysis [9] as it maps the problem directly to the analogous one of counting BPS states in an analogue 1+1D theory.

The basic geometry in this case is a noncompact Calabi-Yau 3-fold specified by the hypersurface $P(x_i) = 0$ in $\mathbb{C}^4$, where

$$P = F(x, y) + z^2 + w^2,$$

and $F(x, y)$ is a quasi-homogeneous function of $x$ and $y$ describing a Riemann surface $\Sigma$, which is just the Seiberg-Witten curve of the gauge theory [17],

$$\Sigma : F(x, y) = 0.$$  

The holomorphic 3-form is given by

$$\Omega = \prod_i dx_i \frac{dP}{dP},$$

which determines the mass of particles in the effective low-energy gauge theory realized as $D$-branes wrapping cycles in the geometry. In particular for D3-branes wrapping supersymmetric 3-cycles $C$, one finds BPS states with central charge

$$Z = \int_C \Omega.$$  

We will be interested in Calabi-Yau 3-folds with singularities which realize a special class of perturbed Argyres-Douglas points in the gauge theory. To this end we can specify

$$F(x, y) = y^2 + (W'(x))^2,$$

where $W$ is a polynomial of degree $N + 1$,

$$W(x, g_j) = \frac{1}{N + 1} x^{N+1} + \sum_{j=2}^{N} \frac{g_j}{N + 1 - j} x^{N+1-j}.$$  

The interacting superconformal point arises in the infrared when $g_j = 0$, for all $j$, where additional cycles vanish in the geometry leading to a certain subset of the BPS dyon states, realized as wrapped D3-branes, becoming massless at this point.

### 2.1 $R$-charges and the chiral ring

The existence of mutually nonlocal massless degrees of freedom at the conformal point implies that there is no straightforward local Lagrangian description and the Argyres-Douglas fixed points, despite their extended $\mathcal{N} = 2$ supersymmetry, are still quite poorly understood. Fortunately, for our purposes it will be sufficient to know the scaling dimensions of the chiral

\[^1\text{These states in general carry topological and global charges, and we will refer to them generically as "dyons".}\]
primary operators dual to the perturbations $g_j$. This information is BPS-protected and fixed by the chiral ring relations, which are determined by the local structure of the geometry near the conformal point. Indeed, for the present case, this data was determined in [18, 9] using the approach pioneered in [19]. The basic idea is that in the infrared SCFT a $U(1)_R$ symmetry must necessarily be restored, and this symmetry must be visible in the Seiberg-Witten curve. The chiral primaries corresponding to deformations along the Coulomb branch have vanishing $SU(2)_R$ spin and so have dimension $D = R/2$ in terms of the conformal $U(1)_R$ charge. To retain the required $U(1)$ scaling symmetry of the curve which is present at the conformal points, where $D[y] = ND[x]$, one observes from (7) that we require $D[g_j] = jD[x]$, while $D[x]$ can be normalized using the fact that $D[\Omega] = 1$ from (5). One obtains [9],

$$D[g_j] = \frac{j}{N+1}, \quad j = 2, \ldots N. \quad (8)$$

The chiral operators $O_j$ dual to these perturbations then have $D[O_j] = 2 - D[g_j]$ [19, 9] as follows from the dimension of the superpotential.

### 2.2 BPS states and the Cecotti-Vafa relation

The relevant BPS states in this geometry were discussed in [9], and are given by D3-branes wrapping supersymmetric 3-cycles. One constructs the required cycles as 2-spheres, a real subspace of $y^2 + z^2 + w^2 = -(W'(x))^2$, fibered over a real curve in the $x$-plane [6]. The latter curve begins and ends at extremal points of $W(x)$ where the radius of the $S^2$ goes to zero. A possible basis of 3-cycles is then given by choosing the $2N - 1$ intervals between the zeros of $(W'(x))^2$ in a sequence. The intersection of the 3-cycles then gives the Dynkin diagram of $A_{2N-1}$. For these cycles to be supersymmetric, there is the additional condition that along a given path between two zeros of $W'(x)$ the phase of $\int_{S^2} \Omega$ should be constant [9], which ensures that the BPS mass inequality is saturated. This reduced one-form on the $x$-plane is precisely the Seiberg-Witten differential, and in general takes the form [6],

$$\lambda_{SW} = \int_{S^2} \Omega = x \frac{dt}{t}, \quad (9)$$

where $2t = y + (W'(x) + M)$, with $M$ a dimensionful constant, that can be interpreted as setting a fixed scale in the geometry that will be used shortly to consider a scaling limit near the singularity. Denoting the zeros of $W'(x)$ as $x = e_i$, for $i = 1, \ldots, N$, the central charge which determines the mass of the BPS states takes the form,

$$Z_{ij} = \int_{e_i}^{e_j} x \frac{dt}{t} = \int_{e_i}^{e_j} \frac{x W''(x)}{W'(x) + M} dx. \quad (10)$$

We can now go further and take a scaling limit near the Argyres-Douglas point, corresponding to $x/M \to 0$, and after a constant rescaling of $x$ and $g_i$ we find,

$$Z_{ij} = \int_{e_i}^{e_j} \lambda_{SW} \to \int_{e_i}^{e_j} W'(x) dx = W(e_j) - W(e_i). \quad (11)$$
We conclude that the constant-phase condition for the integrand in (11) can be restated as
\[ \int_{x(0)}^{x(t)} dW(x) = \alpha t, \]
where \( \alpha \) is a constant and \( t \in [0, 1] \) parametrizes the real path between
the zeros of \( W'(x) \). In other words, \( W(x) \) must follow a straight line between two extrema
in the \( x \)-plane,
\[ W(x(t)) = W(e_i) + t(W(e_j) - W(e_i)), \]
which we recognize as precisely the condition for a BPS kink trajectory within the Landau-
Ginzburg model with superpotential \( W(x) \) [20], which in this case is a perturbation of the
\( A_{N-1} \) minimal model.

At this point it becomes clear that counting BPS solitons in this setup is entirely equiv-
alent to the problem of counting BPS kinks in the perturbed minimal model,\(^2\) and more
precisely the approach used by Cecotti and Vafa [4] for performing the latter calculation
translates directly to this theory. The answer in both cases is therefore the same and it is
rather remarkable that the same technique may be used to perform the calculation, at least in
this special example. It is worth emphasizing that the manipulations of the Seiberg-Witten
differential leading to (22) are rather particular to this case, and would not apply to a more
generic point on the Coulomb branch.\(^3\)

The answer one obtains for the BPS multiplicity in fact depends on the precise form
of the deformation parameters \( g_i \), due to the presence of marginal stability curves across
which the BPS spectrum can jump. To briefly summarize the results [4], note that near each
critical point \( W(e_i) \) one can define a vanishing cycle \( \Delta_i \) in the space of fields, specified by a
fixed value of \( W(t) \). In the Landau-Ginzburg system, this is the set of all possible solutions
to the linearized Bogomol'nyi equation about the vacuum \( W(e_i) \), but naturally arises here
as the set of vanishing cycles associated with the singularity in the full Calabi-Yau geometry.

As shown in [4], the CFIV index [22] counting the number of BPS multiplets between
two vacua is given by
\[ \mu_{ij} = \Delta_i \circ \Delta_j, \]
where the product is the intersection number of the two cycles. An important aspect of the
arguments demonstrating this relation in [4] is that the changes in the BPS spectrum that
occur on crossing a marginal stability curve translates directly into the Picard-Lefschetz
monodromy that the intersection number can undergo with changes in parameters. The
index also carries a phase associated in 1+1D with the (fractional) fermion number of the
state, \( \exp(2\pi i (f_i - f_j)) \), where in the present case [23, 22],
\[ f_j = -\frac{1}{2\pi} \text{Arg}(W''(e_j)). \]

In particular, since the Kähler metric is real, \( f_j \) is just the phase of the fermion mass (more
generally the phase of the determinant of the mass matrix) in the \( j \)-vacuum. It is natural to
suspect a similar interpretation in 3+1D, where dyon solutions do in general exhibit charge
fractionalization at generic points on the Coulomb branch [24]. We will return to this point
again below.

\(^2\)This correspondence was noted in passing in [9].

\(^3\)The simplification of \( \lambda_{SW} \) in this limit near the Argyres-Douglas point appears related to a recent
discussion of Strebel differentials in \( N = 2 \) gauge theories [21].
From the index, we can build the monodromy matrix in 3+1D,
\[
\mathcal{M}_{4D} = S^{-T}S \quad \text{with} \quad S_{ij} = \delta_{ij} - \mu_{ij}, \quad \text{for} \ i \leq j,
\] (15)
given a deformation leading to a convex vacuum polygon in the \( \mathcal{W} \)-plane. It is the eigenvalues of this matrix which provide the link between the UV and IR properties of the theory. From the discussion above it is clear that these eigenvalues are determined on the one hand by the index counting BPS states in the massive deformation.

On the other hand, the ensuing topological stability of the index allows for an alternative derivation of the eigenvalues of the monodromy matrix, by collapsing all the critical values of \( \mathcal{W} \) to a point, i.e. turning off all the mass deformations. In this conformal limit \( \mathcal{W} \) is quasi-homogeneous and the eigenvalues of \( \mathcal{M} \) can be obtained from a basis of forms dual to the space of vanishing cycles \( \Delta_i \), as described in [4]. In the present case, where \( \mathcal{W} \) is a function of just one holomorphic variable, we have the set of 1-forms \( \omega_k = \phi_k dx \), where \( \phi_k \) is a monomial basis for the chiral ring with \( R \)-charge (or equivalently degree) \( q_k \). Up to changes of basis, these \( R \)-charges are equivalent to (twice) the scaling dimensions in (8), as the latter were inferred directly from \( \mathcal{W} \) in the same way. Writing \( \omega_k = \phi_k dx = \alpha_k dW \), which is well-defined away from the critical points, the functions \( \alpha_k \) form a basis for the dual-space to the cycles \( \Delta_i \). Now on a circular path around a given vacuum \( \mathcal{W}(c_k) \) in the \( \mathcal{W} \)-plane, \( x \) rotates by a phase of \( 2\pi q_k \). Consequently we may read off the phase by which \( \alpha_k \) rotates which gives directly the corresponding eigenvalue of \( \mathcal{M} \) [4].

The result of equating both calculations is the Cecotti-Vafa relation [4],
\[
\text{evals}(\mathcal{M}_{4D}) = \exp \left[ 2\pi i \left( q_k - \frac{c'}{2} \right) \right],
\] (16)
where in this case \( c' = 1 - 2/(N + 1) \). Our primary observation here is that, since the derivation is purely geometrical, its interpretation translates directly from the analogue 1+1D system with superpotential \( \mathcal{W} \) to 3+1D, given the appropriate identification of the BPS states. We have just described how the data on the left-hand side, namely the BPS spectra of dyons, maps directly from that of the kink spectrum in a massive deformation of the \( A_{N-1} \) model in 1+1D. In turn, the spectrum of \( UV \) \( R \)-charges determined directly from the geometry of the CY-fold near the Argyres-Douglas point is specified by the superpotential \( \mathcal{W} \) and consequently by the chiral ring of the 1+1D model. However, an important distinction is that we have only considered a subset of all possible deformations of the Seiberg-Witten curve. Due to the identification in (6), we are only considering half the allowed deformations, namely those that do not disturb the factorized form of the curve, and identifiable with ‘normalizable’ (i.e. mass) perturbations [9], which therefore keep the vevs fixed. We should also note that while \( c' \) can be identified with the central charge of the \( A_{N-1} \) minimal model, it is not the full central charge of the string theory on the CY, as the 1+1D analogue only captures certain parts of the geometry, and the complex dimensions parametrized by \( z \) and \( w \) decouple. This reflects the fact that we are not considering all possible BPS states here but only those that become light specifically at the conformal point. In the next section, we will provide a physical rationale for the appearance of this analogue 1+1D theory within this construction.
Figure 1: A schematic view of the vacuum structure, comprising a slice through the Coulomb branch parametrized by \( \phi_a, a = 1, \ldots, N \), which for \( \phi_a = m_a + \mathcal{O}(\Lambda) \) connects to a baryonic Higgs branch parametrized by the baryon vev \( B \), and the classical vacuum at \( \langle B \rangle = v^N \) admits BPS vortex solutions. The vacuum probed by the interior of a given vortex is a point on the Coulomb branch at the baryonic root and at the quantum level, for a suitable choice of \( m_a \sim \mathcal{O}(\Lambda) \), the baryonic root can be tuned to an Argyres-Douglas SCFT.

3. Interpretation via the vortex worldsheet and generalizations

The direct translation of the Cecotti-Vafa relation from 1+1D to 3+1D discussed above can actually be given a natural physical interpretation if we realize the Argyres-Douglas SCFT at the root of a baryonic Higgs branch. The 1+1D theory that appears as an analogue system in the previous section then arises directly, describing the worldsheet dynamics of BPS vortices which are present in the baryonic phase. This viewpoint also implies that the Cecotti-Vafa relation may have more general validity in 3+1D beyond the special case discussed in §2. In this section, we will expand on this point of view and also provide some anecdotal evidence supporting the general validity of the Cecotti-Vafa relation in theories of this type.

3.1 Argyres-Douglas SCFT at the baryonic root

\( \mathcal{N} = 2 \) SQCD with gauge group U(\( N \)) and \( N_f = N \) fundamental hypermultiplets \((Q, \tilde{Q})\) exhibits a rather complex vacuum structure, comprising a Coulomb branch parametrized, for example, by the vev of the adjoint scalar \( \Phi \) in the vector multiplet, with components \( \phi_a, a = 1, \ldots, N \), and various Higgs branches parametrized by the vevs of the squark multiplets \( Q_1^a \) and \( \tilde{Q}_a \). The Higgs branches connect to the Coulomb branch at special points determined by the mass parameters \( m_i \) of the squarks. The vacuum branch relevant here is the baryonic branch parametrized by a gauge invariant baryon vev, \( B = \epsilon_{a_1 \cdots a_N} Q_1^{a_1} \cdots Q_N^{a_N} \). This baryonic branch connects to the Coulomb branch at the baryonic root, given classically by \( \phi_a = m_a \), which generically is characterized by the presence of additional massless quark hypermultiplets, but is not otherwise special. However, if the mass parameters \( m_a \) are tuned appropriately, then at the quantum level one finds that additional monopole multiplets also become massless at this point. The mutually nonlocal massless degrees of freedom signify an interacting Argyres-Douglas SCFT. A schematic outline of this subset of the moduli space is shown in Fig. 1.

To make contact with the discussion in §2, recall that for \( N_f = N \) the Seiberg-Witten
The curve takes the form \[ y^2 = \prod_{a=1}^{N} (x - \phi_a)^2 - 4\Lambda^N \prod_{k=1}^{N} (x - m_k). \] Specializing to the baryonic root, where \( \prod_{a}(x - \phi_a) = \prod_{k}(x - m_k) + \Lambda^N \) [28, 32, 33], this curve degenerates and one finds that \( N \) quark hypermultiplets become massless, allowing for the baryon vev to be turned on. The degenerate curve can be written as, \[ y^2 = (W'(x))^2, \] with \[ W' = \prod_{k=1}^{N} (x - m_k) - \Lambda^N, \] which is precisely of the form (7), with the deformation parameters \( g_i \) re-expressed in terms of the hypermultiplet masses \( m_k \), such that \( g_1 = \sum_k m_k = 0 \). Moreover, as recently discussed in [16], further tuning to the special point \( g_j = 0, j = 2, \ldots, N \) (i.e. \( m_k = -\exp(2\pi ik/N)\Lambda \) for \( N \)-odd and \( m_k = -\exp(2\pi i(k + 1/2)/N)\Lambda \) for \( N \)-even), leads in addition to a fully degenerate curve \( y^2 = x^{2N} \) and an interacting SCFT in the infrared.

Having realized the conformal point, and small deformations thereof, in this setting we can now make use of the known correspondence between the worldsheet dynamics of vortices on the baryonic branch, where \( \langle B \rangle = v^N \) with \( v \) a Fayet-Iliopoulos term, and the bulk dynamics at the baryonic root [10, 11, 12, 13, 16] to reinterpret the results of §2. As shown schematically in Fig. 1, while vortices exist only in the theory on the baryonic branch for \( v \neq 0 \), in the center of the vortex a U(1) subgroup of the Coulomb phase gauge symmetry at the baryonic root is restored. It is in this sense that the vortex acts as a probe of the theory at special points in the Coulomb phase [16], i.e. in this case at an Argyres-Douglas SCFT.

### 3.2 The \( A_{N-1} \) worldsheet theory

Vortices living on the baryonic Higgs branch are known to inherit a nontrivial dynamical structure on the worldsheet from the fact that they spontaneously break some of the residual flavor symmetry of the theory. In the simplest scenario relevant to the above discussion, this endows the vortex with a set of bosonic moduli parametrizing the coset \( \mathbb{C}P^{N-1} = U(N)/(U(N-1) \times U(1)) \) interpreted as Goldstone modes [10, 11]. Since these states are BPS within the \( \mathcal{N} = 2 \) theory, this bosonic structure is completed to a \( \mathcal{N} = (2,2) \) supersymmetric sigma model. There are in addition further modes associated with the breaking of translational invariance in the directions transverse to the vortex, but these will play no role here.

At the quantum level, the effective superpotential \( \tilde{W} \) for the \( \mathcal{N} = (2,2) \) \( \mathcal{C}P^{N-1} \) theory can be written in terms of a twisted scalar \( \Sigma \) and twisted masses \( \tilde{m}_k \), where

\[
\frac{\partial \tilde{W}}{\partial \Sigma} = \ln \prod_{k=1}^{N} \left( \frac{\Sigma - \tilde{m}_k}{\Lambda} \right),
\]

\[\text{The branch structure in this expression reflects shifts in the } \theta \text{-parameter (giving the phase of } \tilde{\Lambda} \text{), and in } 1+1D \text{ the ambiguity is fixed by minimizing the electric potential associated with } \theta \text{ when moving to this effective description [29, 30, 31, 32].} \]
so that the vacuum constraint \( \widetilde{W}(\Sigma) = 0 \), leading generically to \( N \) massive vacua, matches the singular point \( W'(x) = 0 \) for the bulk curve (18), given the identifications \( x \leftrightarrow \Sigma, \Lambda \leftrightarrow \widetilde{\Lambda} \) and \( m_k \leftrightarrow \widetilde{m}_k \), between the chiral fields and mass scales. Thus the worldsheet dynamics captures the vacuum structure at the baryonic root, as specified by the Seiberg-Witten curve, and indeed the central charge for (a subset of) the bulk dyon states can be written in a form precisely matching that for the BPS kinks which interpolate between the \( N \) vacua of the worldsheet theory [32],

\[
Z_{ji} = \int_{e_i}^{e_j} \widetilde{W}'(x) dx = \widetilde{W}(e_j) - \widetilde{W}(e_i),
\]

(20)

where \( e_i \) are the zeros of \( W' \) and thus also of \( \widetilde{W}' \).

In general the functions \( W \) and \( \widetilde{W} \) are not directly related, except through the structure of their extremal points. However, a more precise correspondence does arise if we take a scaling limit and tune the baryonic root close to an Argyres-Douglas point. Indeed, the \( N \) massive vacua of this theory all merge at a critical point \( \langle \Sigma \rangle = 0 \) when the twisted mass parameters \( \widetilde{m}_k, \) identified with the quark mass parameters in the bulk theory \( m_k = \widetilde{m}_k, \) are set to the conformal point [34, 16]. Generically, the theory has BPS kinks interpolating between the \( N \) vacua, and these all become massless at this special point, where

\[
\left. \frac{\partial \widetilde{W}}{\partial \Sigma} \right|_{\text{crit}} = \ln \left( 1 + \frac{\Sigma^N}{\Lambda^N} \right) \longrightarrow \frac{\Sigma^N}{\Lambda^N} + \ldots .
\]

(21)

Consequently, as recently observed by Tong [16], taking the scaling limit near the critical point \( \Sigma/\Lambda \to 0 \) and retaining only the leading term (in analogy with the corresponding limit taken in (11)), one finds on rescaling \( \Sigma \) and parametrizing the relevant mass deformations in terms of \( \nu_j \),

\[
\widetilde{W}(\Sigma) \longrightarrow W(x = \Sigma, \nu_j) \sim \frac{\Sigma^{N+1}}{N+1} + \sum_{j=2}^{N} \frac{\nu_j}{N+1-j} \Sigma^{N-j+1} + \ldots ,
\]

(22)

which exactly matches the perturbed bulk curve and the analogue superpotential in (7). For \( \nu_j = 0 \) this is the superpotential of a theory which flows in the infrared to the \( \mathcal{N} = (2,2) A_{N-1} \) minimal model with normalized central charge \( \hat{c} = 1 - 2/(N+1) \) [35]. Since \( D[\Sigma] = 1/(N+1) \) at this point, as follows from (22), we may read off the dimensions of the perturbations \( \nu_j \) finding \( D[\nu_j] = j/(N+1) \) in precise agreement with the bulk SCFT [16], while from (20) we also see that the central charges exactly match the bulk prediction (11) in this limit. Note that the monodromy structure for BPS states in \( \mathbb{C}P^{N-1} \) with twisted masses [31, 32] has been truncated in this limit, as the central charge (20) will now exhibit only a finite number of branches in the parameter space \( \{\nu_j\} \). However, this is precisely the truncation we would expect on limiting our attention to the states which are light in the vicinity of the critical point. We conclude that, in this particular case, we can make a precise identification between the analogue 1+1D Landau-Ginzburg theory emerging from the geometry in §2 and the worldsheet dynamics of vortices in the baryonic Higgs phase.
This point of view also suggests that the limiting procedure implicit in the right-arrow symbol in (22) (and thus also in (11)) implies a precise order of operations, namely that one first tunes to the $A_{N-1}$ critical point and then perturbs by turning on $\tilde{\nu}_j$; it is not clear that the same RG trajectory is recovered if these operations are performed in the reverse order. Indeed the $CP^{N-1}$ model is not a relevant perturbation of this CFT, as there are additional irrelevant operators that have been dropped in (22). Thus the perturbations of the critical point, introduced in this limit, need not be related trivially to the original twisted mass perturbations of the $CP^{N-1}$ model.

For completeness, we can also consider the analogous limit in the mirror description of the $CP^{N-1}$ model, namely the $\hat{A}_{N-1}$ affine-Toda theory with superpotential [36],

$$\tilde{W}_M(X_i) = \tilde{\Lambda} \left( \sum_{i=1}^{N-1} e^{X_i} + \prod_{i=1}^{N-1} e^{-X_i} \right) + \sum_{i=1}^{N-1} (\tilde{m}_i - \tilde{m}_N) X_i. \quad (23)$$

The conformal point arises for the same choice of twisted masses, where the vacuum takes the form $\langle X_k \rangle = 2\pi k/N$, and in this case one may verify directly that only a single massless state arises (a point that was an implicit assumption above in assuming the validity of (19)). We can write the effective superpotential for this light mode, $\tilde{\Sigma}$, by perturbing near the conformal point, $e^{X_i} = \langle e^{X_i} \rangle + \tilde{\Sigma}/\Lambda$, obtaining (up to a constant),

$$\tilde{W}_M(\tilde{\Sigma}) \to -\frac{1}{N+1} \frac{\tilde{\Sigma}^{N+1}}{\Lambda^N} + \cdots, \quad (24)$$

consistent with the conformal limit of the $CP^{N-1}$ superpotential. This is in agreement with the fact that the $A_{N-1}$ minimal model is self-mirror, up to a trivial $Z_{N+1}$ orbifold action [36], and provides a nice consistency check on the validity of the effective description near the conformal point.

### 3.3 Generalization to the generic baryonic root

With this physical viewpoint in mind, identifying the analogue 1+1D model with the worldsheet theory on vortices present in the Higgs phase, its natural to enquire whether the correspondence may be more far-reaching. In particular, Cecotti and Vafa were able to prove the relation (1) in a far wider class of theories than just those with a Landau-Ginzburg realization, and it seems reasonable to ask whether it similarly has a greater scope in 3+1D. One obvious candidate concerns a more generic parameter choice for the baryonic root in the $N_f = N$ theory, where the correspondence between bulk and vortex worldsheet dynamics implies a direct relation between the BPS spectra. We will now comment briefly on the possibility that the Cecotti-Vafa relation also extends to 3+1D in this more generic case.

As discussed above, the worldsheet theory for a generic choice of the baryonic root is an $\mathcal{N} = (2, 2)$ sigma model with target space $CP^{N-1}$, arising from the broken flavor symmetries. If we consider the simplest case with twisted mass parameters set to zero, namely a point well inside the marginal stability curve discussed in [32, 34], the theory has $N$ massive vacua at the quantum level, and the BPS spectrum comprises kinks which interpolate between these
vacua. The degeneracies of these 1/2-BPS multiplets are known and, for kinks interpolating between vacua which differ in phase by $2\pi k/N$ units, are given by

$$\mu_{ii+k} = (-1)^{k-1} \left( \frac{N}{k} \right) \implies \text{evals}(M_{2D}) = (-1)^{N-1},$$

where $M_{2D}$ is the corresponding monodromy matrix in 1+1D defined as in (15). The eigenvalues are then $N$-fold degenerate.

The UV $R$-charges of generators of the chiral ring are known in this case to be given by the dimensions of the harmonic forms of $\mathbb{C}P^{N-1}$ [4],

$$q_k = k, \quad \text{for} \quad k = 0, \ldots, N-1.$$  

and we can write $\text{evals}(M_2) = \exp(2\pi i (q_k - \hat{c}/2))$ where $\hat{c} = (N-1)$ is the normalized central charge. This exhibits one of the standard examples of the Cecotti-Vafa relation for sigma models with $\mathbb{Z}_N$ symmetry.

Moving to the bulk theory, the BPS kinks are interpreted in the Higgs phase as monopoles, and so we expect the spectra of these two sets of states to match. This has indeed been verified at the semiclassical level at the root of the baryonic Higgs branch [33]. At this point, the relevant subset of the BPS monopole spectrum is given by a set of bound states of the fundamental monopole with the $N$ quark flavors. At the classical level, this is visible as a set of $N$ fermionic flavor zero modes inherited by the monopole solution so that, within semiclassical quantization, if the ‘bare’ monopole solution is denoted $|0\rangle$, the bound states are obtained by acting with the appropriate fermionic creation operators, corresponding to each fermionic zero mode, $\rho_{\lambda_1}^{\dagger} \cdots \rho_{\lambda_k}^{\dagger} |0\rangle$. Although this analysis is not strictly valid when the quark mass parameters are set to zero, if we extrapolate to this case we find multiplets that lie in the $k$th antisymmetric representation of the SU($N$) flavor symmetry, leading precisely to the result (25) for the multiplicity of BPS states in the sector with $k$ flavor zero modes, with the same result for the monodromy matrix $M_{4D}$. Nonetheless, one should bear in mind that the above analysis is semiclassical and, although the index is a BPS-protected quantity, could be invalidated at the quantum level through the presence of marginal stability curves. However, the fact that the result matches the expectation (25) in this case suggests that no restructuring of this kind occurs for this particular choice of parameters.

We may therefore ask whether the ensuing prediction for the spectrum of $R$-charges, $q_k = k$ for $k = 1, \ldots, N-1$, for operators in the chiral ring matches with expectations. The UV limit is simply the classical limit of the asymptotically free theory, where the $R$-symmetry is restored. The generators of the chiral ring are given by $\mathcal{N} = 2$ completions of $\text{tr}(\Phi^k)$, $\text{tr}(\tilde{Q}\Phi^kQ)$, and baryonic operators in the case with $N_f = N$ that we consider here [37, 38]. In the classical limit we will ignore the operators associated with the gauge multiplet, $\text{tr}(W_\alpha W^\alpha \Phi^k)$, which are in any case presumably part of the $\mathcal{N} = 2$ completion of $\text{tr}(\Phi^k)$. Furthermore, since we are interested only in those deformations which keep the theory at the root of the baryonic Higgs branch, as determined by the classical superpotential $\mathcal{W} = \text{tr}[\tilde{Q}(\Phi - m)Q]$, we can limit our attention to the generators $\text{tr}(\Phi^k)$. Note that mesonic and baryonic perturbations affect only the mass and localization scale of the vortex in this case. Limiting our attention to this reduced subset of the chiral ring, we have the kinematic
constraint, \( \text{rank}(\Phi) \leq N \), and so we can take a basis of such operators to be,

\[
\{ \text{tr}(\Phi^1), \text{tr}(\Phi^2), \ldots, \text{tr}(\Phi^N) \}.
\] (27)

In the UV limit, with \( \text{SU}(2)_R \) spin equal to zero, these operators have \( \text{U}(1)_R \) charge \( Q_j = [\text{tr}(\Phi^j)]_R = 2j \), for \( j = 1, \ldots, N \). Interpreting \( Q_j = q_j + \tilde{q}_j \) in terms of the \((1+1)D\) left- and right-chiral \( \text{U}(1) \) charges, we see that this exactly matches the spectrum of \( R \)-charges for the UV chiral-ring generators in the \( \mathbb{C}P^{N-1} \) model.

The arguments of the preceding section, providing a more direct link between the BPS structure and chiral rings in \( 1+1D \) and \( 3+1D \) do not extend straightforwardly to this example. For a start, the \( \mathbb{C}P^{N-1} \) model is not of Landau-Ginzburg type (even in its mirror formulation as an affine-Toda theory, where the superpotential goes to zero in the UV), and so one must rely on the more general proof by Cecotti and Vafa of the UV-IR correspondence on the vortex worldsheet [4]. Given the apparent success of this anecdotal test of the Cecotti-Vafa relation in the more general setting discussed above, it would be interesting to see if this worldsheet structure also maps in some way to the bulk \( 3+1D \) theory.

4. Concluding remarks

We have discussed a particular UV-IR relation in the context of RG flows in \( \mathcal{N} = 2 \) gauge theories, and more specifically gave arguments that the Cecotti-Vafa relation between conformal scaling dimensions and BPS soliton multiplicities in \( 1+1D \) translates directly to an analogous correspondence in \( 3+1D \). An important aspect of this was the role played by vortex dynamics in the gauge theory, which seems to provide the link between the natural \( 1+1D \) setting for the Cecotti-Vafa relation and its four-dimensional realization. We will conclude in this section with a few remarks on further aspects of the correspondence and possible generalizations.

One point touched on briefly in passing concerns the fractional fermion number, which appears as the phase of the CFIV index counting BPS states in \( 1+1D \), and it is natural to ask whether it has a similar interpretation in the bulk. The natural point of contact is the fractional quark charge of dyonic states, given by \( 2\pi \Delta S_i = \text{Im}(\partial Z_{kl}/\partial m_i) \) for the \( i^{th} \) flavor [40, 41, 42], which is analogous to the Witten effect shifting the electric charge [24], and indeed is equivalent at the baryonic root. For a given extremal point \( e_k \), we have the comparison,

\[
S_i(e_k) = -\frac{1}{2\pi} \text{Arg} \left( \frac{1}{e_k - m_i} \right) \quad \leftrightarrow \quad f_k = -\frac{1}{2\pi} \text{Arg}(W''(e_k)) = -\frac{1}{2\pi} \text{Arg} \left( \sum_i \frac{1}{e_k - m_i} \right),
\] (28)

suggesting that the worldsheet fermion number (14) is identified with an appropriately defined sum over the fractional quark charges. One should bear in mind that the quark states which become massless at the baryonic root are not visible on the worldsheet and these charges must be decoupled in a relation such as (28). Consequently, in the simplest case with \( N = 2 \) where there is just a single quark charge \( S \), we find in the semiclassical limit
for real $m = m_1 - m_2$, $\Delta f = \Delta S = \pm 1/2$, in agreement with [34, 32] reproducing the half integer fermion number of both fundamental monopoles and $\mathbb{C}P^1$ kinks.

This viewpoint suggests a possible reinterpretation of charge fractionalization in 3+1D. In particular, on introducing spatial boundaries in 1+1D, one can view the fermion fractionalization of kinks as arising from part of the integral fermion number residing at the boundaries [43]. This has a physical bulk interpretation in this case.\footnote{I'd like to thank A. Vainshtein for helpful discussions on this point.} In particular, spatial boundaries on the vortex worldsheet arise directly if we consider a configuration on the Higgs branch where the vortex lies between two domain walls [44]. The dyon central charge is independent of the Fayet-Iliopoulos term and thus this effective dimensional reduction of the magnetic flux can be achieved without affecting charge fractionalization which depends only on quantum corrections to the central charge. Consequently, we might expect the residual non-integer fermion number to reside on the bounding domain walls in this case, and it would be interesting to explore this further.

Concerning the generality of relations such as (1) and (28), the initial motivation for this study was the question of whether the Cecotti-Vafa relation had any analogues in the context of $\mathcal{N} = 1$ gauge theories in 3+1D [2], where the structure of SCFTs has recently been elucidated in some detail using new techniques such as $a$-maximization [1] to pin down the spectrum of $R$-charges and scaling dimensions. While it is clear that such a correspondence must be somewhat different to the $\mathcal{N} = 2$ examples discussed here, conjecturally relating chiral charges to BPS domain wall multiplicities, it would be interesting to explore whether soft breaking to $\mathcal{N} = 1$ can shed further light on this.

In this regard, it is also natural to wonder about a direct field-theoretic derivation of relation (1) in 3+1D. Despite the correspondence apparent within geometric engineering, discussed in §2, the general proof of the Cecotti-Vafa relation in 1+1D does not at first sight appear to have a natural physical rationale in 3+1D. It would be particularly interesting were a purely four-dimensional field-theoretic perspective to be placed on the Cecotti-Vafa relation, not least because it may be that many other aspects of the ensuing classification program [4] can be translated to $\mathcal{N} = 2$ gauge theories in a similar way.

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References


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