Fubini vacua as a classical de Sitter vacua

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The Fubini's idea to introduce a fundamental scale of hadron phenomena by means of dilatation non-invariant vacuum state in the framework of a scale invariant Lagrangian field theory is recalled. The Fubini vacua is invariant under the de Sitter subgroup of the full conformal group. We obtain a finite entropy for the quantum state corresponding to the classical Fubini vacua in Euclidean space-time resembling the entropy of the de Sitter vacua. In Minkowski space-time it is shown that the Fubini vacua is mainly a bath of radiation with Rayleigh-Jeans distribution for the low energy radiation. In four dimensions, the critical scalar theory is shown to be equivalent to the Einstein field equation in the ansatz of conformally flat metrics and to the SU(2) Yang-Mills theory in the 't Hooft ansatz. In D-dimensions, the Hitchin formula for the information geometry metric of the moduli space of instantons is used to obtain the information geometry of the free-parameter space of the Fubini vacua which is shown to be a (D + 1)-dimensional AdS space. Considering the Fubini vacua as a de Sitter vacua, the corresponding cosmological constant is shown to be given by the coupling constant of the critical scalar theory. In Minkowski spacetime it is shown that the Fubini vacua is equivalent to an open FRW universe.

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I. INTRODUCTION

The WMAP results [1] combined with earlier cosmological observations shows that we are living in an accelerating universe. The currently observed lumpiness in the temperature of the cosmic microwave background is just right for a flat universe though there are also some evidences that our universe is spatially open [2]. The great simplifying fact of cosmology is that the universe appears to be homogeneous and isotropic along a preferred set of spatial hypersurfaces [3]. Of course homogeneity and isotropy are only approximate, but they become increasingly good approximations on larger length scales, allowing us to describe spacetime on cosmological scales by the Robertson-Walker metric.

Constructing four dimensional de Sitter vacuum from a field theory coupled to gravity has been a long standing challenge. Our main purpose in the present work is to give some evidences for considering the Fubini's approach as a possible solution to the problem. In [4], Fubini obtained a classical vacua of the critical scalar theory which preserves the de Sitter subgroup of the full conformal symmetry of the theory. The idea was to introduce a fundamental scale of hadron phenomena by means of dilatation non-invariant vacuum state in the frame work of a scale invariant Lagrangian field theory. He verified that this program can only be carried out if the vacuum state is not translation invariant. In the following, among the other results, it is shown that the entropy of the Fubini vacua in Euclidean space resembles the conjectured entropy of the de Sitter space and the cosmological constant of the de Sitter vacua corresponding to the Fubini vacua is given classically by the coupling constant of the critical scalar theory. We think that these evidences are enough to consider and study the Fubini's method as a presumable solution for the problem of the cosmological constant.

The present paper is organized as follows. A brief review of the Fubini’s original work is given in section II where we also discuss a possible generalization of the Fubini’s approach applicable as dimensional reduction method, see section II A. The entropy of the Fubini vacua and its properties as a thermal bath is studied in section III. The critical scalar theory and its relation to the Einstein field equation in the ansatz of conformally flat metrics and to SU(2) Yang-Mills theory in the ‘t Hooft ansatz is studied in section IV. Specially in section IV A by applying the Hitchin formula, for a critical scalar theory in D-dimensions the information geometry metric of the free-parameter space of the Fubini classical vacua is shown to be a (D + 1)-dimensional AdS space in which the Fubini classical solution appears as the boundary to bulk propagator. Motivated by the Fubini’s result we study the Fubini vacua as a de Sitter vacua in section V. We show that the cosmological constant is proportional to the coupling constant of the critical theory. In section V A it is shown that in Minkowski spacetime, the Fubini vacua corresponds to an open FRW universe. Section VI is devoted to conclusion and summarizing the results. In appendix A the Hitchin formula for the information geometry metric of the moduli space of the Yang-Mills instantons which we apply to the Fubini

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classical vacua is reviewed. In appendix B the general properties of conformally flat geometries are discussed and the information about the Ricci and the scalar curvature tensors of such backgrounds necessary to read sections IV B and V are given. Finally in appendix C we study the Fubini classical vacua as a metastable stationary trajectory of the critical scalar theory.

II. FUBINI VACUA

In this section we give a brief review the Fubini approach to conformal invariant field theories [4]. The part of his work relevant to the present paper is the construction of a classical vacua for critical scalar theories that is invariant under the de Sitter subgroup of the conformal group:

"The physical idea is to introduce a fundamental scale of hadron phenomena by means of dilatation non-invariant vacuum state in the frame work of a scale invariant Lagrangian field theory. A new unconventional feature is that this program can only be carried out if the vacuum state is not translation invariant. The vacuum is still invariant under a 10-parameter subgroup of the full conformal group . . . ."

We finally propose this vacua as a solution to the problem of cosmological constant. In section II A we discuss the usefulness of the Fubini approach (i.e. spontaneously breaking the translation symmetry) for dimensional reduction purposes.

In $D$-dimensional Minkowski spacetime with Lorenz symmetry $O(1, D-1)$ consider the conformal invariant Lagrangian

$$L = \frac{1}{2} (\partial_{\mu} \phi)(\partial^{\mu} \phi) - \frac{g}{D-2} \phi \frac{D-2}{D-2}.$$  \hspace{1cm} (1)

One is looking for a solution of the field theoretical problem in which the vacuum expectation value of the field $\phi(x)$ is non-vanishing,

$$\left\langle \Omega | \hat{\phi} | \Omega \right\rangle = \phi_0(x).$$  \hspace{1cm} (2)

$\phi_0$ satisfies the equation of motion

$$\Box \phi(x) + g \phi \frac{D-2}{D-2} = 0.$$  \hspace{1cm} (3)

In trying to solve Eq. (3) one shall be guided by invariance considerations. The classical equation of motion (3) is invariant under the full conformal group but it does not admit any non-trivial solution invariant under the full conformal group, as can be seen from the invariance condition,

$$\langle \Omega | [G_i, \phi(x)] | \Omega \rangle = 0,$$  \hspace{1cm} (4)

where $G_i$'s are all conformal generators. If $\phi_0$ is assumed to be translation invariant it should be a constant but for $g \neq 0$, Eq. (3) does not admit any constant solution. The Fubini’s solution to Eq. (3) is not invariant under translation but under the transformation generated by,

$$R_{\mu} = \frac{1}{2} \left( \beta P_{\mu} + \frac{1}{\beta} K_{\mu} \right),$$  \hspace{1cm} (5)

in which $P_{\mu}$ is the translation generator and $K_{\mu} = IP_{\mu}I$, where $I$ is the operator of the inversion transformation $x_{\mu} \rightarrow x_{\mu}/x^2$. $\beta$ is some arbitrary constant and has dimension of a length. One can show that $R_{\mu}$ and the generators of the Lorenz group in $D$-dimensions together form an $O(2, D-1)$ or $O(1, D)$ subgroup of the full conformal group $O(2, D)$ if the coupling constant $g < 0$ or $g > 0$ respectively. Invariance of $\phi_0$ under the generators of the Lorenz group implies that $\phi_0 = \phi_0(x^2)$. Invariance under $R_{\mu}$ gives,

$$\phi_0 = c \left( \frac{\beta^2 + x^2}{2\beta} \right)^{-\frac{(D-2)}{2}},$$  \hspace{1cm} (6)

where $c$ is some constant to be determined by the equation of motion (3). The final result is

$$\phi_0(\vec{x}) = \frac{\alpha}{(\beta^2 + (\vec{x} - \vec{a})^2)^{\frac{D-2}{2}}}, \hspace{1cm} (\vec{x} - \vec{a})^2 = \delta_{\mu\nu}(x-a)^{\mu}(x-a)^{\nu},$$  \hspace{1cm} (7)
where
\[
\alpha = \left( \frac{g}{D(D - 2)\beta^2} \right)^{\frac{D-2}{2}},
\]
(8)
The free parameters $\beta$ and $a^\mu$ are present as far as the broken part of the conformal group contains $D + 1$ generators corresponding to the dilatation operator which amounts to changing the choice of $\beta$ and translation operators which change $a^\mu$'s. In section [IV.A] we will apply the Hitchin formula to obtain the information geometry of the free-parameter space. Interestingly one realizes that the information geometry is $\text{AdS}_{D+1}$ if $g > 0$ and $\text{dS}_{D+1}$ if $g < 0$.

### A. Fubini's approach as a dimensional reduction method

An interesting question is whether there exist a classical vacuum invariant under some translations if not all. In such a vacua all massless fields coupled to the scalar field become massive along those directions that translation symmetry is spontaneously broken while remain massless in other directions. In this way at low-energy limit one observes a lower dimensional space with Poincare symmetry as will be shown in this section.

If we label directions along which the modified vacua $\phi_M^{\mu}$ is invariant by $x_i$, $i = 1, \cdots, D'$, and the remaining spatial directions (along which translation symmetry is spontaneously broken) by $x_a$, $a = 1, \cdots, D - D'$, then it is obvious that $\phi_M^{\mu}$ is independent of $x_i$. From the conformal algebra,
\[
\begin{align*}
[M_{\mu
u}, M_{\rho\sigma}] &= -i \left( \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} \right), \\
[M_{\mu
u}, \left( \begin{array}{c} P_{\rho} \\ K_{\rho} \end{array} \right)] &= i \left( \eta_{\mu\rho} \left( \begin{array}{c} P_{\nu} \\ K_{\nu} \end{array} \right) - \eta_{\nu\rho} \left( \begin{array}{c} P_{\mu} \\ K_{\mu} \end{array} \right) \right), \\
\left( D, \left( \begin{array}{c} P_{\rho} \\ K_{\rho} \end{array} \right) \right) &= -i \left( \frac{P_{\rho}}{K_{\rho}} \right), \\
[P_{\mu}, K_{\nu}] &= 2i(\eta_{\mu\nu} D - M_{\mu\nu}), \\
[M_{\mu\nu}, D] &= 0, \\
[P_{\mu}, P_{\nu}] &= [K_{\mu}, K_{\nu}] = 0,
\end{align*}
\]
and
\[
\begin{align*}
[\phi(x), M_{\mu\nu}] &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\phi(x), \\
[\phi(x), D] &= i \left( x_{\mu}\partial_{\mu} + \frac{D-2}{2} \right) \phi(x), \\
[\phi(x), P_{\mu}] &= i\partial_{\mu} \phi(x), \\
[\phi(x), K_{\mu}] &= i \left[ -x^{2}\partial_{\mu} + 2x_{\mu} \left( x_{\rho}\partial_{\rho} + \frac{D-2}{2} \right) \right] \phi(x),
\end{align*}
\]
(9)

one easily verifies that a nontrivial solution $\phi(x_a)$ of equation of motion, can be assumed to be invariant under $M_{ij}$, $D$ and $P_i$. The Lorentz generators $M_{ij}$ and the translation generators $P_i$ cover the Poincare symmetry group in $D'$ dimension. Consequently a vacuum invariant under transformations generated by these generators breaks spontaneously the full conformal symmetry $O(2, D)$ to Poincare group $O(1, D' - 1)$. The equation of motion for $\phi(x_a)$ is
\[
\Box_a \phi(x_a) + g\phi_a^{\frac{D+2}{2}} = 0.
\]
(11)
To solve this equation one can assume invariance under $M_{ab}$ which gives $\phi(x_a) = \phi(r_0)$ where $r_0^2 = x_ax^a$. Invariance under dilatation $D$, is not helpful, since
\[
[\phi(r_a), D] = i \left( x^b\partial_{b} + \frac{D-2}{2} \right) \phi(r_a) = 0,
\]
(12)
gives
\[
\phi(r_a) \sim r_a^{\frac{D-2}{2}}
\]
(13)
which is singular for $D > 2$. The equation of motion for $\phi(x_a) = \phi(r_a)$ is
\[
\phi'' + \frac{P-1}{r_a} \phi' + g\phi^{\frac{D+2}{2}} = 0,
\]
(14)
where $\phi'$ denotes one time derivation with respect to $r_a$ and $P = D - D'$. We finish this section by giving a solution to this equation for $D = 4$ and $P = 1$. In this case the equation of motion is

$$\phi'' + g\phi^3 = 0,$$

which can be easily integrated to give,

$$\frac{1}{2}g\phi'^2 + \frac{g}{4}\phi^4 = c.$$  

(16)

For $c = 0$ the solution $\phi \sim r_a^{-1}$ is singular at $r_a = 0$. For $c > 0$, defining $c = L^{-4}$ one obtains,

$$\phi = \frac{1}{L} \left( \frac{4}{g} \right)^{1/4} \sin \left( g^{1/4} r_a \right),$$  

(17)

in which $\sin(u|m) = \sin(\phi)$ is the Jacobi elliptic function in which $\phi = \text{am}(u|m)$ is the inverse of Jacobi elliptic function of the first kind. For practical purposes one can assume $\sin(x|1) = \sin(1.2x)$ by a good precision.

### III. FUBINI VACUA AS A THERMAL BATH

In this section assuming a four dimensional spacetime we obtain the quantum state $|\Omega\rangle$ corresponding to the classical vacua $\phi_0$ and study it as thermal bath of free scalar particles. In the Euclidean spacetime we show that the entropy $S$, given by the relation

$$\langle \Omega | \Omega \rangle = e^{-S},$$

(18)

is a finite value,

$$S \sim \frac{1}{g^{1/2}} \left( \frac{\ell}{\beta} \right)^2,$$

(19)

in which $\ell$ is the radius of universe (the volume of box). In the Minkowski spacetime $|\Omega\rangle$ is mainly a bath of radiation but a few percent of its content are tachyons and massive particles. Our method to construct $|\Omega\rangle$ from the vacua $|0\rangle$ which is the empty-space quantum state annihilated by the annihilation operators of the whole (mass) spectrum of (scalar) particles, is to Fourier transform $\phi_0$ to obtain the spectrum of plane waves superposed to construct it. We then create the same composition by operating the creation operators on $|0\rangle$.

#### A. The entropy of Fubini vacua in Euclidean spacetime

Imagining the universe as a spherical box of radius $\ell$ we define a dimensionless parameter $x = r/\ell$. If the scalar fields are assumed to vanish on the boundary of the universe then $\ell$ should be large enough $\ell \gg \beta$ such that $\phi_0$ be a good approximation of the true classical vacua of the theory, since $\phi_0$ vanishes at $r \to \infty$. Instead of $\phi_0(r)$ which mass dimension in four dimensions is $[\phi_0] = L^{-1}$ we work with the dimensionless field $\phi_0(x) = \ell \phi_0(r)$,

$$\phi_0(x) = \frac{\sqrt{8}}{g} \frac{\beta/\ell}{(\beta/\ell)^2 + x^2}.$$  

(20)

By the Fourier transformation

$$\phi(x) = \int \tilde{\phi}(k)e^{i\vec{k} \cdot \vec{x}} d^4k,$$

(21)

one can determine $\tilde{\phi}(k)$, i.e. the number of plane waves $e^{i\vec{k} \cdot \vec{x}}$ with 4-momentum $\vec{k}$, constructing $\phi_0$. By spherical symmetry of $\phi_0$, an Euclidean observer living in the vacua $\phi_0$ detects an isotropic radiation of free particles with spectrum $\tilde{\phi}(k)$. Up to some numerical constant $\tilde{\phi}(k)$ is given as follows,

$$\tilde{\phi}(k) = \frac{\sqrt{8} \beta K_1(\frac{k}{\ell} |k|)}{g \ell |k|},$$

(22)
where $K_n(z)$ is the modified Bessel function of the second kind satisfying the differential equation

$$z^2y'' + zy' - (z^2 + n^2)y = 0.$$  

(23)

Considering the annihilation and creation operators $a_k$ and $a_k^\dagger$ satisfying the identities,

$$a_k^\dagger (0) = |k\rangle, \quad a_k (0) = 0, \quad [a_k, a_k^\dagger (k')] = \delta^4(k-k'),$$

(24)

in which $|0\rangle$ is the vacua corresponding the empty spacetime with norm $\langle 0 | 0 \rangle = 1$, we claim that the quantum state $|\Omega\rangle$ corresponding to the classical vacua $\phi_0$ is given by

$$|\Omega\rangle = \exp \left( \int d^4k \tilde{\phi}(k)^{1/2} a_k^\dagger (k) \right) |0\rangle,$$

(25)

since

$$\frac{\langle \Omega | N(k) | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \tilde{\phi}(k),$$

(26)

where $N(k) = a_k^\dagger (k) a_k (k)$ is the number operator. The above result shows that number of plane waves $|k\rangle$ that can be found in $|\Omega\rangle$ is equal to the number of plane waves $e^{ik.x}$ in $\phi_0$. It can be also shown that the state $|\Omega\rangle$ also satisfies the Fubini’s condition $\langle \Omega | \hat{\phi} | \Omega \rangle = \phi_0$.[[17]

The entropy as given by the identity (18) is given as follows,

$$S \sim \int d^4k \tilde{\phi}(k) \sim \sqrt{\frac{8}{g}} \left( \frac{\ell}{\beta} \right)^2.$$  

(27)

It is easy to show that in six dimensions the Fourier transform of $\phi(x) = \ell^2 \phi_0(r/\ell)$ is given by

$$\tilde{\phi}_6(k) \sim \frac{1}{g} \left( \frac{\beta}{\ell} \right)^3 K_1(\frac{\beta}{|k|} |k|),$$

(28)

and consequently the entropy of the corresponding quantum state is given by

$$S_6 \sim \frac{1}{g} \left( \frac{\ell}{\beta} \right)^4.$$  

(29)

The Fourier transform of $\phi_0$ in three dimensions can not be found analytically. But even by comparing the results given in Eqs. (27) and (29) by the entropy of the D-dimensional empty Euclidean de Sitter space

$$S \sim \frac{\ell^{(D-2)}}{G_N},$$

(30)

a nice similarity can be realized.[18]

**B. Spectrum of the Fubini vacua in Minkowski spacetime**

In Minkowski spacetime $\phi_0$ has a singularity on the hypersurface given by $t^2 - \bar{x}^2 = \beta^2$. The Fourier transform of $\phi(\bar{x}, \tau) = \beta \phi_0(\bar{r}, \tau)$ is given as follows,

$$\tilde{\phi}(\omega, \bar{k}) \sim \left( \frac{1}{g} \right)^{1/2} \beta^{-1} \int d^3xe^{-i\bar{k}.\bar{x}} \int d\tau \frac{e^{i\omega\tau}}{1 - \frac{1}{\ell^2} + \frac{\bar{x}^2}{\bar{x}^2}}$$

$$= \left( \frac{1}{g} \right)^{1/2} \beta^{-1} \int d^3xe^{-i\bar{k}.\bar{x}} \left[ \frac{\pi}{\sqrt{1 + \bar{x}^2}} \sin(\tau |\omega| \sqrt{1 + \bar{x}^2}) \right]$$

$$= \left( \frac{1}{g} \right)^{1/2} \beta^{-1} \left( 2\pi \right)^2 \int_0^\infty dx \frac{x \sin(kx)(\omega \sqrt{1 + x^2})}{\sqrt{1 + x^2}}.$$  

(31)
The above integral is not convergent. The divergency can be analyzed as follows. If we split the integration interval 
\((0, \infty)\) into two subintervals \((0, l]\) and \((l, \infty)\) for some \(l \gg 1\), one verifies that in the second interval the integrand is approximately equal to \(\sin(kx)\sin(|\omega| x)\). Thus \(\tilde{\phi}(\omega, \vec{k})\) is approximately given by,

\[
k \tilde{\phi}(\omega, \vec{k}) \sim \int_0^l dx \sin(kx) \left( \frac{\sin(\omega \sqrt{1 + x^2})}{\sqrt{1 + x^2}} - \frac{\sin(kx)}{x} \right) + \int_l^\infty dx \sin(kx) \sin(\omega x),
\]

where \(l \to \infty\) gives the volume of the box. \(\tilde{\phi}(\omega, \vec{k})\) is mainly given by the second term above which is approximately equal to \(\frac{1}{k} \left( \frac{l}{\pi/k} \right) \delta(\omega - k)\).

In other words the classical vacuum \(\phi_0\) corresponds to radiation of massless scalars \((\omega = k)\) and a few percent amount of various massive and tachyonic scalar fields \((\omega \neq k)\). The details of the full spectrum is given by Eq. (31). The distribution of massless scalar is given by the formula,

\[
n(\omega)d\omega = d\omega \int d^3k \tilde{\phi}(\omega, \vec{k}) \sim \beta^{-1} \omega^2 d\omega,
\]

which is the Rayleigh-Jeans formula for black-body radiation at temperature

\[
T = 1/\beta.
\]

The final result is in agreement with the black-body radiation. Because the Rayleigh-Jeans distribution formula gives the distribution of the black-body radiation for \(\omega \ll T\). On the other hand we obtained Eq. (33) by an integration over \(x > \beta l \gg \beta\) which corresponds to \(k/\beta \ll \beta^{-1}\).

**IV. THE CRITICAL SCALAR THEORY**

In section II the Fubini vacua \(\phi_0\) as the classical vacua of the critical scalar theory, invariant under the de Sitter subgroup of the conformal group was obtained. In this section we at first give \(\phi_0\) as the solution of the nonlinear Laplace equation \([5]\),

\[
\sum_{i=1}^D \partial_i^2 \phi + g \phi \frac{\partial^2 \phi}{\partial x^2} = 0.
\]

Then we show that in four dimensions the critical scalar theory corresponds to the Einstein gravity and \(SU(2)\) Yang-Mills theory in special ansatz respectively and study \(\phi_0\) in those contexts.

The Klein-Gordon equation for \(SO(d+1)\)-invariant solutions \(\phi = \phi(r)\), where \(r^2 = x_\mu x^\mu\) is

\[
\left( \frac{d^2}{dr^2} + \frac{d}{r} \frac{d}{dr} \right) \phi + g \phi^\alpha = 0.
\]

One solution of this equation is

\[
\phi_s(r) = \left( \frac{2(d-1) - \frac{2}{n-1}}{(n-1)} \right)^{\frac{1}{n-1}} \left( \frac{1}{g r^2} \right)^{\frac{1}{n-1}}.
\]

These solutions can be obtained by considering the ansatz \(\phi(r) = \alpha r^\beta\) and solving the wave equation to determine \(\alpha\) and \(\beta\). The above solutions become singular as \(g \to 0\). One can show that Eq. (36) has also solutions like,

\[
\phi_0(\vec{x}) = \frac{\alpha}{(\beta^2 + (\vec{x} - \vec{a})^2)^{\frac{D-2}{2}}}, \quad (\vec{x} - \vec{a})^2 = \delta_{\mu\nu}(x - a)^\mu (x - a)^\nu.
\]

for some constants \(\alpha, \beta\) and \(\gamma\) only for conformally coupled theories, i.e. for \(d = 2, 3, 5\) and \(n = 5, 3, 2\) respectively. In these cases \(\gamma = \frac{D-2}{2}\), and

\[
\beta^2 = \frac{g}{D(D-2)} \alpha^{\frac{D-2}{2}},
\]
where $\alpha$ is some arbitrary real-valued constant. By Wick rotation $t \to it$ one obtains the solutions of wave equation in Minkowski space-time. Using the map $\phi \to \Phi = t^{\frac{2}{\alpha}} \phi$, one can also obtain the corresponding $SO(d)$-invariant solutions in Euclidean $AdS_{d+1}$ space and $dS_{d+1}$ space. For example, a solution for the Klein-Gordon equation for $\phi^4$ model in the $O^-$ region of $dS_4$ space is

$$\phi_{dS_4}(t, \vec{x}) = \frac{\alpha t}{\sqrt{\frac{2}{2} g - t^2 + |\vec{x}|^2}}, \quad \alpha \in \mathbb{R}. \quad (40)$$

A method to obtain solutions given in Eq.(38) and more such solutions for conformally coupled models is as follows. Using the solutions $\phi_s(r)$ given in Eq.(37) one can try to solve the Klein-Gordon equation for $\phi(r) = \phi_s(r) \eta(r)$. The resulting equation for $\eta(s)$ is

$$\left( r \frac{d}{dr} \right) \left( r \frac{d}{dr} \right) \eta + \begin{cases} 4\eta(\eta - 1) = 0, & n = 2, d = 5, \\ 2\eta(\eta^2 - 1) = 0, & n = 3, d = 3, \\ 4\eta(\eta^4 - 1) = 0, & n = 5, d = 2. \end{cases} \quad (41)$$

The solutions given in Eq.(38) are obtained by solving Eq.(41) with vanishing constant of integration.

### A. The 't Hooft ansatz and the information geometry metric

There exists a useful and interesting connection between the $SU(2)$ Yang-Mills theory and the scalar $\phi^4$ theory [6]. This connection is a specific ansatz for the YM potential $W_\mu$ in terms of a scalar field. The ansatz was discovered by 't Hooft in connection with the instanton problem [7, 8]. The 't Hooft ansatz for the YM potential is,

$$W_\mu = \eta_{\mu \nu} \partial^\nu \phi / \phi, \quad (42)$$

where $\eta_{\mu \nu}$ are the 't Hooft tensors. In this ansatz the equation of motion for pure $SU(2)$ gauge theory, reduces to the following equation

$$\frac{1}{\phi} \frac{d}{dr} \Box \phi = \frac{3}{\phi^2} \partial_\mu \phi \Box \phi, \quad (43)$$

which can be integrated once to give

$$\Box \phi + \lambda \phi^3 = 0, \quad (44)$$

where $\lambda$ is an arbitrary integration constant. The $\phi_0$ solution in four dimension corresponds to $k = 1$ instanton. In fact, the $SU(2)$ instanton density is,

$$\mathrm{tr} F^2 \sim \frac{\beta^4}{(\beta^2 + (\vec{x} - \vec{a})^2)^4} \sim \phi_0^4. \quad (45)$$

In [8] $\beta$ in Eq.(45) is considered as the size of the instanton, suggesting to call $\beta$ the size of $\phi_0$.

In D-dimensions, considering $\theta^I = \beta, \sigma^\mu, I = 0, \cdots, D$ in $\phi_0$ as the moduli, the Hitchin information metric of the moduli space, defined as follows [10]:

$$G_{IJ} = \frac{1}{N(D)} \int d^D x L_0 \partial_I (\log L_0) \partial_J (\log L_0), \quad (46)$$

can be shown to describe Euclidean $AdS_{D+1}$ space:

$$G_{IJ} d\theta^I d\theta^J = \frac{1}{\beta^2} (d\beta^2 + d\alpha^2). \quad (47)$$

$N(D)$ is a normalization constant,

$$N(D) = \frac{D^3}{D + 1} \int d^D x L_0, \quad (48)$$

and

$$L_0 = -\frac{1}{2} \phi \nabla^2 \phi_0 - \frac{g}{(2D-2)} \phi_0 \frac{\partial^2}{\partial \mu^2} \phi_0 = \frac{g}{D} \phi_0 \frac{\partial^2}{\partial \mu^2} \phi_0, \quad (49)$$
is the Lagrangian density of the $\phi^4$ theory calculated at $\phi = \phi_0$. See appendix A for details.

$\phi_0$ as a function of $\theta^I$ is a free stable-tachyon field on $E\text{AdS}_{D+1}$ as it satisfies the Klein-Gordon equation given in terms of the metric (47),

$$\left(\beta^2 \partial^2_\beta + (1-D)\beta \partial_\beta + \beta^2 \partial^2_\alpha + \frac{D^2 - 4}{4}\right) \phi_0 = 0.$$ (50)

The tachyon is stable as far as $\frac{D^2 - 4}{4} < m^2 < 0$. $\phi_0$ as a function of $g$ the coupling constant (or $\beta^2$), can not be analytically continued to $g = 0$. For $g = 0$, $\phi_0$ is the Green function of the Laplacian operator i.e. $\nabla^2 \phi(x, a) = \delta^D(x - a)$ and does not satisfy the Klein Gordon equation $\nabla^2 \phi = 0$ for free scalar theory. This shows that $\phi_0$ can not be obtained by perturbation around $g = 0$. In $\mathcal{A}$, the same asymptotic behavior for the instanton density is observed and $\text{tr} F^2$ is interpreted as the boundary to bulk propagator of a massless scalar field on $AdS_5$.

B. The ansatz of conformally flat metrics

In this section we show that in four dimensions the Einstein-Hilbert action for conformally flat ansatz of metrics given by

$$g_{\mu\nu} = \phi^2(x) \eta_{\mu\nu},$$ (51)

in which $\eta_{\mu\nu}$ is the Minkowski metric, reduces to the critical scalar theory in $D = 4$. The Levi-Civita connection in ansatz (51) is given by

$$\Gamma^\mu_{\nu\rho} = \omega_{\rho} \delta^\mu_\nu + \omega_{\nu} \delta^\mu_\rho - \omega^\mu \eta_{\nu\rho},$$ (52)

in which

$$\omega = \ln \phi, \quad \omega_{\mu} = \frac{\partial \phi}{\partial x^\mu}, \quad \omega^\mu = \eta^\mu_{\nu} \omega_{\nu}.$$ (53)

The Riemann tensor $R^\mu_{\nu\rho\sigma} = \left[ (\Gamma^\mu_{\nu\sigma,\rho} + \Gamma^\mu_{\rho\sigma,\nu}) - (\rho \leftrightarrow \sigma) \right]$ is given by,

$$R^\mu_{\nu\rho\sigma} = (\omega_{\nu\rho} \delta^\mu_\sigma + \omega^\mu \eta_{\nu\rho} + \omega_{\rho} \omega_\nu^\mu \delta^\mu_\sigma + \omega^\mu \omega_{\rho} \eta_{\nu\sigma} + \omega^\mu \omega^\rho_{\nu\sigma}) - (\rho \leftrightarrow \sigma).$$ (54)

Consequently

$$R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} = (2-D) \left( \omega_{\nu\sigma} - \omega_{\mu}^{\nu\sigma} \omega^\mu_\nu + \omega_{\nu\sigma} \eta_{\mu\nu}^\alpha \omega^\alpha_\mu \right) - \square \omega \eta_{\nu\sigma},$$ (55)

in which $\square \omega = \omega^\alpha_\alpha$. Finally the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ is given as follows,

$$\phi^2 R = 2(1-D) \square \omega + (D-2)(1-D) \omega_{\alpha}^\alpha \omega^\alpha_\mu.$$ (56)

Since $\omega = \ln \phi$, the scalar curvature can be given in terms of $\phi$,

$$R = 2(1-D) \phi^{-3} \square \phi + (D-1)(4-D) \phi^{-4} \phi_\alpha^\alpha \phi^\alpha_\mu.$$ (57)

Thus in four dimensions the Einstein-Hilbert action (in units $8\pi G = 1$),

$$S = \frac{1}{2} \int d^D x \sqrt{|g|} (R - 2\Lambda),$$ (58)

in ansatz (51), is equivalent to the critical scalar theory in four dimension (up to some boundary term) as can be seen by inserting $\sqrt{|g|} = \phi^D$ and $R$ from Eq. (57) into (58) for D=4,

$$S = \frac{1}{\xi} \int d^4 x \left( -\frac{1}{2} \phi \square \phi - \xi \Lambda \phi^4 \right) = \frac{1}{\xi} \int d^4 x \left( \frac{1}{2} \phi^\alpha_\alpha \phi^\alpha_\alpha - \xi \Lambda \phi^4 \right) - \frac{1}{2\xi} \int \phi^\alpha_\alpha n^\alpha.$$ (59)


where $\xi = \frac{1}{6}$ is the conformal coupling constant in four dimensions. The boundary term is not necessarily vanishing and its value depends on the asymptotic geometry of space-time. The corresponding equation of motion is,

$$\Box \phi + \frac{2}{3} \Lambda \phi^3 = 0, \quad (60)$$

which gives a solution of the Einstein equation. The validity of the last claim is not obvious and should be checked. The Einstein equation in terms of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is given as follows,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (61)$$

It is easy to verify that

$$G_{\mu\nu} = 4\phi^2 - 2\phi_{,\mu} \phi_{,\nu} - 2\phi^2 \eta_{\mu\nu} + 2\phi^{-1} \Box \eta_{\mu\nu} \quad (62)$$

The Einstein equation results in two sets of equations for $\phi$,

$$\phi_{,\mu} \phi_{,\nu} = \frac{1}{2} \phi \delta_{\mu\nu}, \quad \mu \neq \nu, \quad (63)$$

and

$$4\phi^{-2} (\phi_{,\mu}^2 - 2\phi^{-1} \phi_{,\mu} + (2\phi^{-1} \Box \phi - \phi^{-2} \phi_{,\alpha} \phi_{,\alpha} + \Lambda \phi^2) \eta_{\mu\nu} = 0 \quad (64)$$

where $\mu = 0, \cdots, 3$ and there is no summation over $\mu$. It is not obvious that the Fubini classical vacua $\phi_0$ which is a (regular) solution of Eq.(60) (for $\Lambda > 0$), also satisfies the above equations. To check it we first consider the ansatz $\phi = \phi(r)$ for a possible solution to Eqs.(63) and (64) and assume the Euclidean space-time metric. For this ansatz Eq.(63) simplifies to the following equation,

$$\phi \left( \frac{\ddot{\phi}}{r} - \frac{\dot{\phi}}{r} \right) - 2(\dot{\phi})^2 = 0, \quad (65)$$

where $\dot{\phi}(r) = \frac{d}{dr} \phi(r)$ and $\Box \phi = \ddot{\phi} + \frac{1}{r} \dot{\phi}$. Furthermore Eq.(64) simplifies to a linear combination of Eqs.(65) and the equation of motion (60). Therefore the non-trivial check for $\phi_0$ as a solution of the Einstein equation is to examine the validity equation (65) for $\phi = \phi_0$, which can be easily verified.

Consequently the Einstein equation in four dimensions is, roughly speaking, equivalent to the critical scalar theory with coupling constant $g = \frac{2}{3} \Lambda$. This is in complete agreement with the result obtained in the next section in general $D$-dimensions by a different approach see Eq.(74).

V. FUBINI VACUA AS THE METASTABLE DE SITTER VACUA

In section II it was shown that the classical vacua is responsible for spontaneous conformal symmetry breaking to the de Sitter subgroup. Therefore it is natural to expect that the theory governing fluctuations around the vacua $\phi_0$ be a field theory on a de Sitter background [11, 13, 15].

Recall that in general, by inserting $\phi = \Omega^D \phi$ and $\delta_{\mu\nu} = \Omega^{-1} g_{\mu\nu}$ in the action

$$S[\phi] = \int d^D x \frac{1}{2} \delta^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi, \quad (66)$$

one obtains,

$$S[\tilde{\phi}] = \int d^D x \sqrt{g} \left( \frac{1}{2} \delta^{\mu\nu} \partial_{\mu} \tilde{\phi} \partial_{\nu} \tilde{\phi} + \frac{1}{2} \xi R \tilde{\phi}^2 \right), \quad (67)$$

i.e. a scalar theory on conformally flat background given by the metric $g_{\mu\nu} = \Omega \delta_{\mu\nu}$ in which $\Omega > 0$ is an arbitrary $C^\infty$ function. $R$ is the scalar curvature of the background and $\xi = \frac{D-2}{4(D-1)}$ is the conformal coupling constant. For details see [14] or appendix B.

The conformally flat background we are looking for is the one for which the vacua $\phi_0$ becomes a constant value. In other words $\Omega$ should be chosen in such a way that

$$\tilde{\phi}_0 = \Omega^\frac{2-D}{2} \phi_0 = \text{const.}, \quad (68)$$

Consequently, the Einstein equation in four dimensions is, roughly speaking, equivalent to the critical scalar theory with coupling constant $g = \frac{2}{3} \Lambda$. This is in complete agreement with the result obtained in the next section in general $D$-dimensions by a different approach see Eq.(74).
Therefore using Eqs. (7) and (8),
\[ \Omega = \bar{\phi}_0 \frac{D(D - 2)}{g} \left( \frac{\beta}{\beta^2 + r^2} \right)^2. \] (69)

Using Eq. (55) one easily verifies that
\[ R_{\mu\nu} = 4(D - 1) \left( \frac{\beta}{\beta^2 + r^2} \right)^2 \delta_{\mu\nu} = \Lambda g_{\mu\nu}, \] (70)
where the cosmological constant \( \Lambda \) is defined by the following relation,
\[ \Lambda = \frac{g}{\xi D} \bar{\phi}_0 \] (71)

Apparently the value of the cosmological constant in this model depends on the constant value that the classical vacua \( \bar{\phi}_0 \) assumes, see Eq. (68). As far as \( g > 0 \) the cosmological constant is positive and the background is de Sitter as was expected from the Fubini’s result reviewed in section III. Since,
\[ S[\bar{\phi}] = \int d^Dx \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \frac{1}{2} \xi R \bar{\phi}^2 - \frac{g}{2 \beta^2} \bar{\phi} \frac{\partial^2 \bar{\phi}}{\partial r^2} \right), \] (72)
if \( \bar{\phi}_0 \) is required to be the minimum of the effective potential,
\[ V(\bar{\phi}) = \frac{1}{2} \xi R \bar{\phi}^2 - \frac{g}{2 \beta^2} \bar{\phi} \frac{\partial^2 \bar{\phi}}{\partial r^2}, \] (73)
then from Eqs. (70) and (71) one verifies that, \( \bar{\phi}_0 = 1 \) and consequently,
\[ \Lambda = \frac{g}{\xi D}. \] (74)

A. Fubini vacua in 4D Minkowski space-time

In this section we study the Fubini vacua in four dimensional Minkowski spacetime with metric \( \eta_{\mu\nu} = (-, +, +, +) \) and assume that \( g > 0 \) [13].
\[ \phi_0(t, \bar{x}) = \sqrt{\frac{8}{g \beta^2 - (t - a^0)^2 + |\bar{x} - \bar{a}|^2}}, \] (75)
where \( \bar{x} \in \mathbb{R}^3 \). Here on we assume \( a^\mu = 0 \) for simplicity. \( \phi_0 \) is singular on the hyperbola \( t^2 = x^2 + \beta^2 \) and we define its distance to an observer located on the origin to be given by \( \beta \). The Hamiltonian density \( \mathcal{H} \) corresponding to \( \phi_0 \), is,
\[ \mathcal{H} = \frac{16 \beta^2}{g} \frac{t^2 + x^2 - \beta^2}{(-t^2 + x^2 + \beta^2)^2}, \] (76)
which tends to infinity in the vicinity of the singularity. One can show that the most stable \( \phi_0 \) solution is the zero-sized one, corresponding to \( \beta \to \infty \), see appendix C. Therefore the singularity is safe when the scalar theory is coupled to gravity. For \( t < \beta \) one can calculate, say, the total vacuum energy \( H = \int d^3x \mathcal{H} \) corresponding to \( \phi_0 \) which is surprisingly vanishing, \( H = 0 \). The de Sitter background corresponding to the vacua \( \phi_0 \) is given by the metric,
\[ ds^2 = \frac{12 \beta^2}{\Lambda} \frac{1}{(\beta^2 - t^2 + x^2)^2} (-dt^2 + dx^2), \] (77)
where \( \Lambda > 0 \) is the cosmological constant see Eq. (74). This metric can be obtained using the conformal transformation \( \eta_{\mu\nu} \to \Omega \eta_{\mu\nu} \) and using Eq. (59) after a Wick rotation \( t \to it \). A different set of coordinates can be used to describe the
corresponding de Sitter background with FRW metric to see whether it is open, closed or flat. Defining, coordinates \( u, \rho \) and \( z_i, i = 1, 2, 3 \) by the relations \( z_i^2 = 1, t = u \cosh \rho \) and \( x_i = u \sinh \rho z_i \) useful to describe the timelike region \( t > |\vec{x}| \), one obtains,

\[
ds^2 = \frac{12\beta^2}{\Lambda} \frac{1}{(\beta^2 - u^2)^2} \left( -du^2 + u^2(d\rho^2 + \sinh^2 \rho dz_i^2) \right) \tag{78}
\]

we define a time coordinate \( \tau \) by the relation \( d\tau = (\beta^2 - u^2)^{-1} du \). Thus one obtains,

\[
\tau = \begin{cases} 
\frac{1}{\beta} \coth^{-1} \frac{u}{\beta} & u > \beta, \\
\frac{1}{\beta} \tanh^{-1} \frac{u}{\beta} & u < \beta,
\end{cases} \tag{79}
\]

and

\[
ds^2 = \frac{12\beta^2}{\Lambda} \left( -d\tau^2 + \frac{\sinh^2(2\beta \tau)}{4\beta^2}(d\rho^2 + \sinh^2 \rho dz_i^2) \right) \tag{80}
\]

One can call the region \( u < \beta \) which can be observed by observers located on the origin the south pole and the \( u > \beta \) region the north pole, a known terminology in de Sitter geometry. The south pole and north pole in our model are separated by the horizon located at \( u = \beta \), i.e the singularity of \( \phi_0 \). By normalizing \( \tau \) by the normalization factor \( \sqrt{\frac{12}{\Lambda}} \beta \) and defining a new coordinate \( r = \sinh \rho \), one at the end of the day obtains,

\[
ds^2 = -d\tau^2 + a(\tau)^2 \left( \frac{dr^2}{1 + r^2} + r^2 dz_i^2 \right) \tag{81}
\]

in which \( a(\tau) = \sqrt{\frac{3}{\Lambda}} \sinh \sqrt{\frac{3}{\Lambda}} \tau \). This is the Robertson-Walker metric for open de Sitter universe. One can easily calculate the energy density \( \rho \) and the pressure \( p \) of the cosmological stuff corresponding to \( \phi_0 \) using the Friedmann equations for the open universe,

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{1}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p). \tag{82}
\]

One verifies that \( p \) and \( \rho \) satisfy the equation of state for the cosmological constant \( \rho = -p = \Lambda \ (8\pi G = 1) \).

VI. CONCLUSION

The Fubini vacua of the critical scalar theories preserves the de Sitter subgroup of the full conformal symmetry. In \( D \)-dimensions this vacua is determined up to \( D + 1 \) free parameters. The geometry of the free-parameter space is a \( (D + 1) \) dimensional AdS space, where the Fubini classical vacua appears as the boundary to bulk propagator. In Euclidean spacetime, the entropy of the quantum state \( |\Omega\rangle \) corresponding to the classical Fubini vacua, given by the formula,

\[
\langle \Omega | \Omega \rangle = e^{-S}, \tag{83}
\]

in 4 and 6 dimensions is given by,

\[
S = \left( \frac{\ell}{\sqrt{1/\beta}} \right)^{D-2}, \quad D = 4, 6, \tag{84}
\]

in which \( g \) is the coupling constant of the scalar theory, \( \ell \) is the size of the universe assumed as a spherical box and \( \beta \) is the free parameter of the Fubini vacua.

In Minkowski space-time, \( \beta \) appears to be equivalent to \( 1/T \) where \( T \) is the temperature of the background, and one verifies that the Fubini vacua is equivalent to a bath of radiation. The radiation mainly consists of massless scalars with Rayleigh-Jeans distribution for frequencies \( \omega \ll T \).
Since the Fubini vacua is invariant under the de Sitter subgroup of the full conformal group, it is reasonable to construct the corresponding de Sitter background in which the Fubini vacua is a constant. The cosmological constant in this case is given by,

$$\Lambda = \frac{g}{\xi D}$$  \hspace{1cm} (85)$$

where $\xi$ is the conformal coupling constant. A similar result can be obtained in four dimensions where the Einstein field equation simplifies to the critical scalar theory in the ansatz of conformally flat metrics. In Minkowski spacetime the corresponding de Sitter space is equivalent to an open FRW universe.

A generalization of the Fubini's approach was considered in section II A by looking for a classical vacua with some translation symmetries. In such a vacua massless fields gain mass in directions along which the translation symmetry is broken. Thus at low-energies a lower-dimensional space with Poincare symmetry will be observed.

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**APPENDIX A: THE GEOMETRY OF THE MODULI SPACE IN FUBINI VACUA**

Here we give a detailed calculation of Hitchin information metric on the moduli space of $\phi_0$ [10, 11]:

$$\phi_0 = \left( \frac{D(D - 2)}{g} \right)^{\frac{D-2}{2}} \left( \frac{\beta}{\beta^2 + (x - a)^2} \right)^{\frac{D-2}{D}}. \hspace{1cm} (A1)$$

From Eq. (49) one verifies that

$$L_0 = g D \left( \frac{D(D - 2)}{g} \right)^{\frac{D}{2}} \left( \frac{\beta}{\beta^2 + (x - a)^2} \right)^D. \hspace{1cm} (A2)$$

Therefore

$$\partial_\beta \log L_0 = D \left( \frac{1}{\beta} - \frac{2\beta}{\beta^2 + (x - a)^2} \right), \hspace{1cm} \partial_a \log L_0 = \frac{2D(x - a)}{\beta^2 + (x - a)^2}. \hspace{1cm} (A3)$$

Using these results and after some elementary calculations one can show that,

$$G_{ij} = \frac{1}{N(D)} \int d^D x \partial_i \log L_0 \partial_j \log L_0 = \frac{4K(D)}{N(D)\beta^2} \delta_{ij} \int d^D y \frac{y^2}{(1 + y^2)^{D+2}};$$

$$G_{i\beta} = G_{i\beta} = \frac{1}{N(D)} \int d^D x \partial_i \log L_0 \partial_\beta \log L_0 = 0,$$

$$G_{\beta\beta} = \frac{1}{N(D)} \int d^D x (\partial_\beta \log L_0)^2 = \frac{DK(D)}{N(D)\beta^2} \int d^D y \frac{1}{(1 + y^2)^D} \left( 1 - \frac{2}{(1 + y^2)^2} \right). \hspace{1cm} (A4)$$

where $K(D) = g^{1-D/2}D^{D/2+1}(D - 2)^{D/2}$. By performing the integrations and using Eq. (48), one obtains,

$$G_{I,J} = \frac{1}{\beta^2} \delta_{I,J}, \hspace{1cm} (A5)$$

in which $\delta_{I,J} = 1$ if $I = J$ and vanishes otherwise.
APPENDIX B: GENERAL PROPERTIES OF CONFORMALLY FLAT BACKGROUNDS

In this appendix we briefly review free scalar field theory in \( D + 1 \) dimensional (Euclidean) curved space-time \([11, 14]\). The action for the scalar field \( \phi \) is

\[
S = \int d^D x \sqrt{g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R) \phi^2),
\]

for which the equation of motion is

\[
(\Box - m^2 - \xi R) \phi = 0, \quad \Box \phi_0 = |g|^{-1/2} \partial_\mu \left( |g|^{1/2} g^{\mu\nu} \partial_\nu \phi_0 \right).
\]

(With \( h \) explicit, the mass \( m \) should be replaced by \( m/h \).) The case with \( m = 0 \) and \( \xi = \frac{D-2}{4(D-1)} \) is referred to as conformal coupling.

The curvature tensor \( R^\nu_{\nu\sigma\tau} \) in term of Levi-Civita connection,

\[
\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} (\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}),
\]

is given as follows,

\[
R^\nu_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\alpha} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\nu\alpha} \Gamma^{\alpha}_{\rho\sigma},
\]

The Ricci tensor \( R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} \) and the curvature scalar \( R = g^{\sigma\tau} R_{\nu\sigma} \).

The metric of a conformally flat space-time can be given as \( g_{\mu\nu} = \Omega \delta_{\mu\nu} \), where \( \Omega \) is some function of space-time coordinates. One can easily show that,

\[
R_{\mu\nu} = \frac{2-D}{2} \partial_\mu \partial_\nu (\log \Omega) - \frac{1}{2} \delta_{\mu\nu} \nabla^2 (\log \Omega) + \frac{D-2}{4} \left( \partial_\mu (\log \Omega) \partial_\nu (\log \Omega) - \delta_{\mu\nu} \partial_\rho (\log \Omega) \partial_\rho (\log \Omega) \right),
\]

and

\[
\Omega R = (1-D) \nabla^2 (\log \Omega) + \frac{(1-D)(D-2)}{4} \delta_{\mu\nu} \partial_\rho (\log \Omega) \partial_\rho (\log \Omega).
\]

By inserting \( \tilde{\phi} = \Omega^{-\frac{2-D}{4}} \phi \) in the action \( S[\tilde{\phi}] = \int d^D x \frac{1}{2} \delta^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \), one obtains,

\[
S[\tilde{\phi}] = \int d^D x \left( \frac{1}{2} \Omega^{\frac{2-D}{2}} \delta^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{1}{2} \left( \Omega^{\frac{2-D}{2}} \nabla^2 \Omega^{\frac{2-D}{4}} \right) \tilde{\phi}^2 \right) = \int d^D x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} + \frac{1}{4} \xi \tilde{\phi}^2 \right).
\]

To obtain the last equality the identities \( g_{\mu\nu} = \Omega \delta_{\mu\nu} \) and \( \xi \sqrt{g} R = -\Omega^{\frac{2-D}{2}} \nabla^2 \Omega^{\frac{2-D}{4}} \) are used. Consequently the free massless scalar theory on \( D \)-dimensional Euclidean space, is (classically) equivalent to some conformally coupled scalar theory on the corresponding conformally flat background.

A \( D \)-dimensional de Sitter (dS) space may be realized as the hypersurface described by the equation \(-X_0^2 + X_1^2 + \cdots + X_D^2 = \ell^2\). \( \ell \) is called the de Sitter radius. By replacing \( \ell^2 \) with \(-\ell^2 \) the hypersurface is the \( D \)-dimensional anti de Sitter (AdS) space. (A)dS spaces are Einstein manifolds with positive (negative) scalar curvature. The Einstein metric \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \), satisfies \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \), where \( \Lambda = \frac{(D-2)(D-1)}{2g^2} \) is the cosmological constant.

APPENDIX C: THE METASTABLE LOCAL MINIMA OF THE ACTION

In this section we discuss the stability of the classical \( \phi_0 \) in 4D for different values of the free parameter \( \beta \) by studying the variation of the action by small field variations around it \([12, 13]\).

By recasting the action in terms of new fields \( \tilde{\phi} = \phi - \phi_0 \) one obtains,

\[
S[\phi] = S[\phi_0] + S_{\text{free}}[\tilde{\phi}] + S_{\text{int}}[\tilde{\phi}],
\]
where $S[\phi_0] = \int d^4x L_0 = \frac{8\pi^2}{3g}$, and

$$S_{\text{free}}[\phi] = \int d^4x \left( \frac{1}{2} \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} M^2(x) \phi^2 \right)$$  \hspace{1cm} (C2)

in which,

$$M^2(x) = -3g\phi_0^2 = -24 \frac{\beta^2}{(\beta^2 + (x - a)^2)^2}.$$  \hspace{1cm} (C3)

These equations show that $\phi_0$ is a metastable local minima of the action. This can also be verified explicitly by numerical analysis of variations of the action. For this purpose it is enough to show that there are field variations $\phi_0 \rightarrow \phi_\eta = \phi_0 + \epsilon \eta$ for $C^1$ functions $\eta$ vanishing as $x \rightarrow \infty$ such that $\delta S = c_\eta e^\eta + O(e^3)$ for some real positive constant $c_\eta$. For simplicity one can assume $\eta = \left( \frac{1}{1+x^2} \right)^n$, $g = 8$, $b = 1$ and calculate $\delta S = S[\phi_0] - S[\phi_0]$ for some integers $n$. One recognizes that $c_\eta > 0$ for $n > 5$, though it is negative for $0 < n \leq 5$. A good sign for metastability of the action at $\phi_0$.

Another interesting observation is that bubbles with larger size are less stable than those with smaller size. This can be checked noting that the size of a bubble is proportional to $\delta S$. In fact if we calculate the variation of action at the stationary point $\phi_0(\beta)$ for different values of the moduli $\beta_1$ and $\beta_2$, under variation $\delta \phi$, from Eqs. (C2) and (C3) one verifies that,

$$\Delta S = \delta S|_{\beta_1} - \delta S|_{\beta_2} \sim \int d^4x \left( \phi_0(\beta_2)^2 - \phi_0(\beta_1)^2 \right) \delta \phi^2 + O(\delta \phi^3).$$  \hspace{1cm} (C4)

For simplicity we assume that $a_i^4 = 0$, $i = 1, 2$. Therefore $\Delta S$ is proportional to,

$$(\beta_1^2 - \beta_2^2) \int_0^\infty \frac{x^3(-x^4 + \beta_1^2 \beta_2^2)}{(\beta_1^2 + x^2)(\beta_2^2 + x^2)^2} \delta \phi_0^2.$$  \hspace{1cm} (C5)

For $\delta \phi$ with compact support, i.e. $\delta \phi = 0$ if $|x| > \sqrt{\beta_1 \beta_2}$ the integral above is positive therefore $\Delta S \sim (\beta_1^2 - \beta_2^2)$. As far as $\phi_0$ is a metastable local minima there exist $\delta \phi$ with compact support such that $\delta S|_{\beta_i} > 0$ $i = 1, 2$. Consequently if $\beta_1 > \beta_2$ then $\delta S|_{\beta_1} > \delta S|_{\beta_2}$. One can convince herself/himself that for some $\delta \phi$ one obtains $\delta S|_{\beta_2} < 0$ while $\delta S|_{\beta_1} > 0$. Consequently one concludes that there is a transition $\beta_2 \rightarrow \beta_1$ induced by say, thermal fluctuations. In addition the stability increases as $\beta \rightarrow \infty$.

[16] When comparing to the original work [4] it should be noted that the Minkowski metric we use is $(+,-,\cdots,+)$. While Fubini assumes the metric $(+,+\cdots,+)$.}
To show it one has to do some slight modifications. Consider $r$ as the time coordinate and define the Fourier transformation by $\phi_0(r) = \int_{-\infty}^{\infty} dk e^{ikr} \tilde{\phi}(k)$. For second quantization consider $a(k)$ as the annihilation or creation operator for negative and positive values of $k$ respectively. The modification of definition (25) and the algebra (24) is straightforward. Defining the operator $\hat{\phi} = \int_{-\infty}^{\infty} a(k)e^{ikr}$ it is easy to verify the Fubini condition (2).

In section IV B it is shown that $\phi_0$ gives a solution of the Einstein equation with positive cosmological constant. See also section V.