We develop a general technique for solving the Riemann-Hilbert problem in presence of a number of "heavy charges" and a small one thus providing the exact Green functions of Liouville theory for various non trivial backgrounds. The non invariant regularization suggested by Zamolodchikov and Zamolodchikov gives the correct quantum dimensions; this is shown to one loop in the sphere topology and for boundary Liouville theory and to all loop on the pseudosphere. The method is also applied to give perturbative checks of the one point functions derived in the bootstrap approach by Fateev Zamolodchikov and Zamolodchikov in boundary Liouville theory and by Zamolodchikov and Zamolodchikov on the pseudosphere, obtaining complete agreement.

1. Introduction

Liouville theory has attracted a lot of interest as an example of quantum conformal field theory\(^1\) and for its applications to model string theory and to brane theory. Remarkable results have been obtained within the bootstrap approach\(^2,4\), which starting from some assumptions provides exact results for a few interesting correlation functions.

Here we address the problem to recover the conformal quantum Liouville field theory from the functional integral procedure understood in the usual sense in which one starts from a stable background and then one integrates over the quantum fluctuations. As it is well known, a quantum field theory is specified not only by an action but also by a regularization and renormalization procedure.

Both on the sphere topology formulated on the Riemann sphere, on the pseudosphere and obviously in the conformal boundary case, the Liouville action has to be supplemented by boundary terms. For definiteness we shall illustrate here the conformal boundary case. The action in presence of sources is given by

\[
S_{\Gamma, N}[\phi] = \lim_{\varepsilon_n \to 0} \left\{ \int_{\Gamma_n} \left[ \frac{1}{\pi} \partial_\zeta \phi \partial_\zeta \phi + \mu e^{2b \phi} \right] d^2 \zeta + \oint_{\partial \Gamma} \left[ \frac{Q}{2\pi} \phi + \mu B e^{b \phi} \right] d\lambda \right\} + \frac{1}{2\pi i} \sum_{n=1}^{N} \alpha_n \int_{\partial_\gamma_n} \phi \left( \frac{d_\zeta}{\zeta - \zeta_n} - \frac{d_\zeta^*}{\zeta^* - \zeta_n} \right) - \sum_{n=1}^{N} \alpha_n^2 \log \varepsilon_n^2
\]
where the integration domain $\Gamma_\varepsilon = \Gamma \setminus \bigcup_{n=1}^{N} \gamma_n$ is obtained by removing $N$ infinitesimal disks $\gamma_n = \{ |\zeta - \zeta_n| < \varepsilon_n \}$ from the simply connected domain $\Gamma$ and $\phi \approx -\alpha_n \log |\zeta - \zeta_n|^2$ for $\zeta \to \zeta_n$. $Q = 1/b + b$ and $k$ is the extrinsic curvature of the boundary $\partial \Gamma$, defined as

$$k = \frac{1}{2i} \frac{d}{d \lambda} \left( \log \frac{d \zeta}{d \lambda} - \log \frac{d \bar{\zeta}}{d \lambda} \right), \quad \zeta(\lambda) \in \partial \Gamma \quad (2)$$

where $\lambda$ is the parametric boundary length, i.e. $d \lambda = \sqrt{d \zeta d \bar{\zeta}}$. It is possible to write action (1) as the sum of a classical part and quantum action. One notices that due to $Q \neq 1/b$ the above written action is not exactly invariant under conformal transformations. In $^8,^9$ it was found that if one starts from $Q = 1/b$ and adopts an invariant regularization procedure one does not reach a theory invariant under the full conformal group. This is similar to the result of $^6$. The reason is that in such an approach the cosmological term $e^{2b\phi}$ acquires weight $(1 - b^2, 1 - b^2)$ instead of $^8 (1, 1)$ as required by the full infinite dimensional conformal invariance.

The regularization suggested at the perturbative level in $^4$ in the case of the pseudosphere provides the vertex functions with the correct quantum dimensions at the first perturbative order $\Delta_\alpha = \alpha (1/b + b - \alpha)$. In $^10$ is was explicitly proven that such a result stays unchanged to all orders perturbation theory. In particular the weight of the cosmological term becomes $(1, 1)$ as required by the invariance under local conformal transformations. These calculations correspond to a double perturbative expansion in the coupling constant and in the charge of the vertex function.

Here we use a more powerful approach which allows to resum infinite classes of graphs $^9$. We start from the background generated by finite charges, i.e. “heavy charges” in the terminology of $^3$. This means that we consider the vertex operators $V_{\alpha_n}(z_n) = e^{2\alpha_n \phi(z_n)}$ with $\alpha_n = \eta_n/b$ and $\eta_n$ fixed in the semiclassical limit $b \to 0$. This has the remarkable advantage to give the resummation of infinite classes of usual perturbative graphs. In order to do that however one needs the exact Green function on a non trivial background.

In the case of a single heavy charge, by solving a Riemann-Hilbert problem in presence of the given heavy charge and an infinitesimal one we are able to compute such exact Green function on such a background in closed form in terms of incomplete Beta functions and such a Green function is used to develop the subsequent perturbative expansion in the coupling constant $b$.

After such a result is accomplished one is faced with the non trivial task of computing a functional integral constrained by the boundary conditions imposed by action (1).

The background generated by a single charge is stable only in presence of a negative value of $b^2 \mu_B$. We compute the Green function on such a background satisfying the correct conformally invariant boundary conditions and such a Green function is regularized at coincident points by simply subtracting the logarithmic divergence. For the sphere and conformal boundary case one obtains the correct
quantum dimensions to one loop in such background improved perturbation theory. The presence of a negative boundary cosmological constant imposes to work with the fixed boundary length \( l \) constraint and to compare our results with the ones given in\(^5\) also the fixed area \( A \) constraint is introduced. It is possible to factorize the functional integral in a term resulting from the boundary length and area constraints and an unconstrained functional on functions satisfying the correct conformal invariant boundary condition. We compute such functional integral through the technique of varying the charges and the invariant ratio \( A/l^2 \). The one loop result on the one source background obtained in this way is\(^11\)

\[
Z(\eta; A, l) = e^{-S_0(\eta; A, l)/b^2} \frac{\beta}{8\pi^2} \frac{l}{b^2A} \frac{e^{2\eta\gamma_E} \Gamma(2\eta)}{\sqrt{1 - 2\eta}} \left(1 + O(b^2)\right)
\]  

where \( S_0(\eta; A, l)/b^2 \) is the classical action without the bulk and boundary cosmological terms, computed on the one source background. Eq.(3) agrees with the expansion of the fixed area and boundary length one point function derived through the bootstrap method in\(^5\) and for which there was up to now no perturbative check.

Applying similar techniques in the pseudosphere case one obtains for the one point function

\[
\langle V_{\eta/b}(0) \rangle = e^{-S_{cl}(\eta)/b^2} \frac{e^{2\eta\gamma_E}}{\Gamma(1 - 2\eta)(1 - 2\eta)^{3/2}} \left(1 + O(b^2)\right)
\]

where \( S_{cl} \) is the full classical action and the one loop expression for the two point function due to a finite charge and an infinitesimal one. Eq.(4) agrees with the expansion of the bootstrap result, while the expression for the two point function is consistent with the existing results of the standard perturbation approach and agrees with the exact two point function when one vertex is given by the degenerate field \( V_{-1/(2\beta)} \). On the other hand adopting the invariant regularization for the Green function at coincident points one finds an expression which disagrees with the degenerate two point function \( \langle V_{-1/(2\beta)}(x) V_{\varepsilon/b}(y) \rangle \) on the pseudosphere.

References

5. V. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov, [hep-th/0001012]
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