The generalized second law of gravitational thermodynamics is examined in scenarios where the dark energy dominates the cosmic expansion. For quintessence and phantom fields this law is fulfilled but it may fail when the dark energy is in the form of a Chaplygin gas. However, if a black hole is allowed in the picture, the law can be violated if the field is of phantom type.

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1. Introduction

Nowadays, the overwhelming observational evidence suggests that the Universe is undergoing an accelerated expansion driven by some form of energy (dubbed “dark energy”) that violates the strong energy condition, \( \rho + 3p > 0 \). What is more, models in which this field also violates the dominant energy condition (DEC), \( \rho + p > 0 \), -dark energy of “phantom” type- appears marginally favored by the data. This short Communication briefly summarizes our research about the validity of generalized second law (GSL) of gravitational thermodynamics in spatially flat, dark energy dominated, FLRW universes.\(^1\)\(^2\) This law asserts that the entropy of matter and fields within the event horizon plus the entropy of the horizon is a never decreasing quantity. Event horizons, of radius \( R_H = a(t) \int_t^{\infty} dt'/a(t') \), are unavoidable features of ever-accelerating cosmologies and are widely assumed to possess thermodynamic properties as temperature and entropy, though this has been shown in a rigorous manner for the de Sitter horizon only.\(^3\)

2. The GSL in accelerated universes

We begin by considering a phantom dominated universe with equation of state parameter \( w = \text{constant} < -1 \). Its horizon radius \( R_H = n/[(1 + n)H] \), with \( n = -2/[3(1 + w)] \), decreases with expansion whence its associated temperature, \( T_H = n/[(1 + n)2\pi R_H] \) augments with time while its entropy, \( S_H = \pi R_H^2 \), diminishes. However, the entropy of the phantom, assumed in thermal equilibrium with the horizon, calculated through Gibbs’ equation\(^4\) exactly compensates the decrease of the horizon entropy, \( S = -S_H \), whereby the GSL is satisfied. (It is noteworthy that, in agreement with previous authors,\(^5\) the entropy of the phantom fluid is negative. However, strange as it may be, this is not forbidden by any thermodynamical law). A parallel study for non-phantom dark energy dominated cosmologies, with \(-1 < w =\)
constant $< -1/3$ (e.g., “quintessence”), yields identical result only that, now, $S$ is positive and decreasing, and $S_H$ augments.

As an example of a phantom-dominated expansion with variable $w$ we consider the model of Sami and Toporensky\textsuperscript{6} which features a scalar field with negative kinetic energy and potential $V = V_0 \phi^\alpha$, where $0 < \alpha \leq 4$. A similar analysis shows that $\dot{S} + \dot{S}_H \geq 0$, the equality sign holding just for $t \to \infty$, i.e., when $R_H$ vanishes. Again, $S$ is negative and increasing and such that $|S|/S_H \geq 1$.

The Chaplygin gas,\textsuperscript{7} of equation of state $p = -A/\rho$, corresponds to a non-phantom dark energy field with $w = -Aa^6/(Aa^6 + B)$. Here $A$ and $B$ are positive-definite constants. In this case, the radius of the horizon increases with expansion up to the de Sitter value, $H^{-1}$, for $t \to \infty$, as it should -see, Fig. 3 of Ref.\textsuperscript{1}, and the GSL is fulfilled for $a \geq [2.509 B/A]^{1/6}$ but it may be violated at earlier times.

We next consider the impact of a small black hole within the event horizon in phantom dominated universes on the validity of the GSL. By “small” we mean that the black hole mass is much lower than the phantom energy inside the horizon, i.e., $M/E_\phi \ll 1$, so that neither the scale factor nor the event horizon radius gets significantly modified. As is well known, Schwarzschild black holes, immersed in a phantom environment, are bound to lose mass by accreting phantom energy at a rhythm $\dot{M} = -16\pi M^2 \dot{\phi}^2$ -see Ref.\textsuperscript{8} Therefore, the black hole entropy, $S_{BH} = 4\pi M^2$, will necessarily decrease. Thus, for scenarios with $w = \text{constant} < -1$ it follows that $\dot{S} + \dot{S}_{BH} + \dot{S}_H < 0$, i.e., the GSL is violated.

It remains to be seen whether it will be also violated when $w$ varies with time. To this end we consider again the model of Sami and Toporensky.\textsuperscript{6} In this scenario, $\dot{S}_{BH} = -8\pi \alpha H^2 M^3 x^{-1}$, with $x \equiv 4\pi \dot{\phi}^2/\alpha$, and the GSL is satisfied provided the black hole mass does not exceed the critical value,\textsuperscript{2}

$$M_{cr} = \sqrt{\frac{3}{8V_0}} (4\pi)^{-\frac{1}{2}} \frac{\alpha^\frac{1}{4}}{x^{\frac{1}{4}}} \times \left\{ e^{\frac{\alpha}{4}} \Gamma \left( \frac{4 - \alpha}{4}, x \right) \left[ \Gamma \left( \frac{4 - \alpha}{4}, x \right) e^{\frac{\alpha}{2x}} \left( 2 + \frac{\alpha}{2x} \right) - 2x^{-\alpha/4} \right] \right\}^{\frac{1}{3}},$$

which decreases with time at fixed $\alpha$ much slowly than $M$. This implies that, regardless the initial mass of the black hole, sooner or later we will have $M > M_{cr}$ while the condition $M/E_\phi \ll 1$ still holds -see Fig. 1 in Ref.\textsuperscript{2} As a consequence the GSL will be violated. At some point further ahead the said condition will no longer be met and our analysis will break down. From this point on we can say nothing about the validity of the GSL.

3. Discussion

The assumption of thermal equilibrium between the dark energy and the event horizon may seem artificial. However, this condition must be fulfilled for the entropy concept to be meaningful. In other words, the entropy is an exclusive property of
equilibrium systems whence the entropy of two systems cannot be meaningfully added unless they are into equilibrium with one another.\textsuperscript{4}

In view of the failure of the GSL in phantom dominated scenarios when black holes are present different reactions may arise: (i) Some phantom energy fields might be physical but not those considered in this Communication. Indeed, several predictions lending support to phantom fields may have come from an erroneous interpretation of the observational data.\textsuperscript{9} (ii) The GSL was initially formulated for systems complying with the DEC, so there is no reason why it ought to be satisfied by systems that violate it. (iii) Strictly speaking, a general proof of the GSL even for systems complying with the DEC is still lacking,\textsuperscript{10} therefore we should not wonder at its failure in some particular cases.

It is for the reader to decide which of these alternatives, if any, is more to his/her liking.

Yet, one may argue that it is unclear that black holes retain their thermodynamic properties (entropy and temperature) in presence of a field that does not comply with the DEC. In such an instance, one may think, that there is no room for the black hole entropy in the expression for the GSL. However, the latter is often formulated by replacing $S_{BH}$ by the black hole area. Again, this variant of the GSL will fail in the two cases of above. To sum up, if eventually phantom energy is shown to be a physical reality, it will pose a serious threat to the generalized second law.

References