Fluctuations of CMBR in accelerating universe

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Abstract. The influence of the observed relict vacuum energy on the fluctuations of CMBR going through cosmological matter condensations is studied in the framework of the Einstein-Strauss-de Sitter vakula model. It is shown that refraction of light at the matching surface of the vakula and the expanding Friedman universe can be very important during accelerated expansion of the universe, when the velocity of the matching surface relative to static Schwarzschildian observers becomes relativistic. Relevance of the refraction effect for the temperature fluctuations of CMBR is given in terms of the redshift and the angular extension of the fluctuating region.

Keywords: Cosmology, fluctuations of CMBR, accelerating universe, Rees-Sciama effect

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INTRODUCTION

Temperature fluctuations of the Cosmic Microwave Background Radiation (CMBR), recently measured by COBE, WMAP, etc., are observed on the level of $\Delta T/T \sim 10^{-5}$ [1]. These fluctuations can be explained in two ways. First, by the Sachs–Wolfe effect [2], i.e., as an imprint of energy density fluctuations related to the CMBR temperature fluctuations at the cosmological redshift $z \sim 1300$ during the era of recombination, when effective interaction of matter and CMBR is ceased [3]. Second, by the Rees–Sciama effect [4], i.e., as a result of influence of large-scale inhomogeneities (large galaxies or their clusters, and large voids) evolved in the expanding universe due to the gravitational instability of matter at the era characterised by $z \lesssim 10$. In the case of spherically symmetric clusters and voids, the Rees–Sciama effect was considered in detail by Mészáros and Molnár [5]. They describe the clusters by the standard Einstein–Strauss vakula model, while the voids they model in an approximative way that does not meet the full general-relativistic junction conditions. Further, they do not consider the effect of refraction of light at the boundary surface matching the cluster (void) with the expanding universe. However, this effect could be of great importance in an accelerating universe, indicated by many of recent cosmological tests predicting present value of the vacuum energy density $\rho_{\text{vac}} \sim 0.7 \rho_{\text{crit}}$ ($\rho_{\text{crit}} \equiv 3H^2/8\pi G$ is the critical energy density corresponding to the flat universe predicted by the inflationary paradigm [6, 1]). The vacuum energy density (or energy of a quintessence field) is related to the (effective) cosmological constant by

$$\Lambda = \frac{8\pi G}{c^2}\rho_{\text{vac}}.$$  \hspace{1cm} (1)

Here, we present a study of the influence of the relict repulsive cosmological constant, indicated by observations to be equal $\Lambda \approx 10^{-56}$ cm$^{-2}$, on the Rees–Sciama effect. We use the Einstein–Strauss–de Sitter (ESdS) vakula model in which the inhomogeneity is represented by a spherically symmetric cluster immersed into the Friedman dust-filled universe (see Fig. 1). We determine temperature fluctuations of the CMBR passing the vakula described by the ESdS model and give estimations of the relevance of the effect of refraction at the matching surface.

We use the geometric units with $c = G = 1$.

1. EINSTEIN–STRAUSS–DE SITTER VAKUOLA MODEL

In the construction of the ESdS model with a $\Lambda > 0$, we remove a spherical ball of dust of the mass $M$ from the dust-filled universe and replace it by the Schwarzschild–de Sitter spacetime of the same mass $M$. Its expanding boundary surface coincides at a fixed value of the comoving Robertson-Walker (RW) coordinate $\chi_E = \chi_b$ with expanding surface $\chi = \chi_b = \text{const}$ of the Friedman universe (see Fig.1). The Schwarzschild–de Sitter (SdS) spacetime can be completely vacuum, i.e., a black-hole spacetime, or, as
FIGURE 1. ESdS vakuola model. A schematic picture of a cluster represented as a spherically symmetric inhomogeneity immersed in the dust filled Friedman universe. The observer O receives two photons coming through the vakuola, and third one coming directly.

used frequently, it has a spherical source represented by a part of an internal dusty Friedman universe with parameters different than those of the external Friedman universe outside of the vacuum SdS spacetime, and characterized by $\chi_I < \chi_E$. For simplicity, we shall not consider influence of the source of the internal part of the ESdS vakuola model on the CMBR fluctuations.

In the standard Schwarzschild coordinates, the vacuum SdS spacetime of mass $M$ is described by the line element

$$\text{d}s^2 = -\text{d}t^2 + A^{-2}(r) \text{d}r^2 + r^2 \text{d}\Omega,$$

(2)

The Friedman universe is described by the FRW geometry. In the comoving coordinates its line element reads

$$\text{d}s^2 = -\text{d}T^2 + R^2(T) \left[ \text{d}\chi^2 + \Sigma_k^2(\chi) \text{d}\Omega \right],$$

(3)

where

$$\Sigma_k(\chi) = \begin{cases} 
\sin \chi & \text{for } k = 1, \\
\chi & \text{for } k = 0, \\
\sinh \chi & \text{for } k = -1.
\end{cases}$$

(4)

The RW metric describes the external Friedman universe at $\chi \geq \chi_E = \chi_b$, while at $\chi < \chi_b$ it is replaced by the expanding part of the SdS spacetime. The particles with $\chi = \chi_b$ follow radial geodesics of the SdS spacetime.

The evolution of the Friedman universe is given by the evolution of the scale factor $R$ and the energy density $\rho$ in dependence on the cosmic time $T$. The scale factor fulfils the Friedman equation

$$\left( \frac{\text{d}R}{\text{d}T} \right)^2 = \frac{8\pi\rho}{3R^3} + \frac{\Lambda}{3} R^2 - k,$$

(5)

and the energy density $\rho$ satisfies the energy conservation equation in the form

$$\frac{8\pi\rho}{3} R^3 = \text{const} = R_0.$$

(6)

It is necessary to synchronize the proper time of a dust particle on the matching hypersurface (MH hereinafter) $\chi = \chi_b$ as measured from the both sides of the MH. Therefore, the proper time $\tau$ of a test particle following the radial geodesic, as measured in the SdS spacetime, must be equal to the cosmic time $T$, as measured in the FRW spacetime. The junction conditions have the following form [7]

$$r_b = R(T) \Sigma_k(\chi_b), \quad \tilde{R} = R_0 \Sigma_k(\chi_b), \quad \tilde{R} \sqrt{\tilde{R}/2M} = R_0,$$

(7)

where the parameter $\tilde{R}$ is related to the covariant energy $E_b$ of the test particles along the radial geodesic, representing the MH by the relation

$$E_b = \sqrt{1 - \frac{2kM}{\tilde{R}}}.$$

(8)

The internal 3-geometry of the MH as measured from the FRW universe side is given by the line element

$$\text{d}s^2_3 = -dT^2 + R^2(T) \Sigma_k^2(\chi_b) \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right).$$

(9)

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From the side of the SdS spacetime it is given by the line element
\[ ds^2 = -dT^2 + r_s^2(T) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \] (10)

Both geometries are identical due to the junction conditions. One can show that the same statement holds for the extrinsic curvature of the MH [8].

2. THE GEODESICS INTERSECTING THE MATCHING HYPERSURFACE

Let us consider geodesics crossing the MH. We have to find the relation between the directional angle as measured by the comoving Friedman observers, \( \psi_F \), and the directional angle as measured by the Schwarzschild–de Sitter static observers, \( \psi_S \). The segments of the geodesics in the FRW and the SdS geometry must be smoothly connected on the MH. We are looking for the Lorentz transformation relating the comoving Friedman and the static SdS observers on the MH.

In the RW metric, the geodesic equations can be integrated and expressed in the form [8]
\[ p^T = \frac{dT}{d\lambda} = \left( m^2 + \frac{p^2}{r_s^2} \right)^{1/2}, \quad p^\lambda = \frac{d\lambda}{d\lambda} = \pm \frac{1}{r_s} \left( p^2 - L^2 \frac{k}{\Sigma^2} \right)^{1/2}, \] (11)
\[ p^\theta = \frac{d\theta}{d\lambda} = \pm \frac{1}{r^2 \Sigma_k} \left( L^2 + \frac{\ell^2}{\sin^2 \theta} \right)^{1/2}, \quad p^\phi = \frac{d\phi}{d\lambda} = \frac{\ell}{r^2 \Sigma_k \sin^2 \theta}, \] (12)
where \( \lambda \) is an affine parameter and \( m \) is mass of the particle; the proper time \( \tau = m\lambda \). The constants of motion are
\[ \ell = p_\phi, \quad L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}, \quad p^2 = p_\phi^2 + \frac{L^2}{\Sigma_k^2}, \] (13)
where \( \ell(L) \) represent the azimuthal (total) angular momentum. Geodesic equations in the SdS spacetime are in the integrated form expressed by the formulae
\[ p^T = \frac{dv}{d\lambda} = E \mathcal{S}^{-2}(t), \quad p^r = \frac{dr}{d\lambda} = \pm \left( E^2 - V^2_{\text{eff}} \right)^{1/2}, \] (14)
\[ p^\theta = \frac{d\theta}{d\lambda} = \pm \frac{1}{r^2} \left( L^2 + \frac{\ell^2}{\sin^2 \theta} \right)^{1/2}, \quad p^\phi = \frac{d\phi}{d\lambda} = \frac{\ell}{r^2 \sin^2 \theta}, \] (15)
where the effective potential
\[ V^2_{\text{eff}} = \mathcal{S}^2(r) \left( m^2 + \frac{L^2}{r^2} \right). \] (16)

The constants of motion \( \ell \) and \( L \) have the same meaning as in the FRW case. The covariant energy \( E = -p_t \).

Let us consider coordinate systems with coincidentally oriented coordinate axes, moving mutually in the direction of the radial axis. The orthonormal base vectors are related by the standard Lorentz transformation
\[ \mathbf{e}_{(\mu')} = \Lambda_{\mu'}^\nu \mathbf{e}_{(\nu)} \] (17)
with the Lorentz matrix
\[ \Lambda_{\mu'}^\nu = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (18)

The orthonormal basis of the static SdS observers is given by the relations
\[ \mathbf{e}_{(t)} = \mathcal{S}^{-1}(r) \frac{\partial}{\partial t}, \quad \mathbf{e}_{(r)} = \mathcal{S}(r) \frac{\partial}{\partial r}, \] (19)
\[ \mathbf{e}_{(\theta)} = r^{-1} \frac{\partial}{\partial \theta}, \quad \mathbf{e}_{(\phi)} = (r \sin \theta)^{-1} \frac{\partial}{\partial \phi}, \] (20)
while in the case of the comoving FRW observers it is given by
\[ \mathbf{e}_{(T)} = \frac{\partial}{\partial T}, \quad \mathbf{e}_{(\chi)} = R^{-1} \frac{\partial}{\partial \chi}. \] (21)

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can be expressed in the FRW and SdS spacetimes by the relation

Denoting the directional angles (related to the outward radial direction defined for observers at the radius, where the MH is located)

In Table 1, we give the critical angles of the total refraction

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present the dependence

formulae

We obtain the parameter of the Lorentz transformation from the fact that the 4-velocity of the test particles comoving with the MH can be expressed in the FRW and SdS spacetimes by the relation

The velocity parameter of the Lorentz shift, giving the speed of the expansion of the MH as measured by the static SdS observers,

and the Lorentz factor are then given by the relations

Therefore, we arrive at the Lorentz transformation parameter in the form

The velocity parameter of the Lorentz shift, giving the speed of the expansion of the MH as measured by the static SdS observers, and the Lorentz factor are then given by the relations


\[ V(r_b) = \sqrt{1 - \frac{\alpha^2(r_b)}{\alpha^2_b}}, \quad \gamma = \cosh \alpha = \left[1 - V(r_b)^2\right]^{-1/2}. \]

### 3. Refraction of Light at the Matching Hypersurface

Denoting the directional angles (related to the outward radial direction defined for observers at the radius, where the MH is located momentarily) of a photon entering (leaving) the FRW universe from (into) the SdS vakuola as \( \psi^{\pm}_F \), \( \psi^{\pm}_S \) (\( \psi^{\pm}_F \), \( \psi^{\pm}_S \)), we arrive at the formulae

\[ \cos \psi^{\pm}_F = \frac{\cos \psi^{\pm}_S \mp V_r}{1 \mp V_r \cos \psi^{\pm}_S}, \quad \sin \psi^{\pm}_F = \frac{\sin \psi^{\pm}_S \sqrt{1 - V_r^2}}{1 \pm V_r \cos \psi^{\pm}_S}. \]

For photons entering the FRW universe from the SdS spacetime, the detailed analysis [11] shows that

\[ \psi^{+}_F > \psi^{+}_S \quad \text{for} \quad \psi^{+}_S \in [0, \pi/2], \]

i.e., for such photons the refraction angle is always larger than the impact angle. The total reflection occurs for angles

\[ \psi^{+}_S > \psi^{+}_S(T) \equiv \arccos V_r. \]

In Table 1, we give the critical angles of the total refraction \( \psi^{\pm}_S(T) \) for some values of the MH expansion velocity. In Fig. 2a, we present the dependence \( \psi^{+}_F = \psi^{+}_S(T) \quad \text{for} \quad \psi^{+}_F \in [0, \pi/2] \), i.e., the refraction angle is again always larger than the impact angle, and the total reflection occurs for

\[ \psi^{+}_F > \psi^{+}_S(T) \equiv \arccos V_r. \]

### 3.1. The Expansion Velocity of the Matching Hypersurface

We shall consider the simplest case of the MH expansion velocity for the spatially flat universe \( (k = 0) \):

\[ V_r = \sqrt{\frac{2M}{r_b} \Lambda^2_b + \frac{\Lambda^2_b}{3}}. \]
FIGURE 2. Plot A: The refraction angle $\psi^F$ for fixed speed parameter $V$, as a function of the impact angle $\psi^S$ from the interval $[0, \pi/2]$. The shaded area corresponds to the total reflection of the light. Plot B: The dependence $V_r = V_r(r_b)$ for fixed mass $M = 7 \cdot 10^{18} M_{\odot}$ of vakuola and three representative values of $\Lambda = 0$, $10^{-52} m^{-2}$ and $5 \cdot 10^{-52} m^{-2}$ and vakuola radius $r_b$ from the interval $[10 Mpc, 500 Mpc]$. The function $V(r_b)$ reaches its local minimum as it approaches the static radius $r_s$. Introducing a dimensionless cosmological parameter $y \equiv \frac{1}{\sqrt{3}} \Lambda M^2$, we find the local extremum of $V_r(r_b)$ ($dV_r/dr_b = 0$) located at so called static radius of the SdS spacetime

$$r_s M \equiv y^{-1/3},$$

(31)

where the gravitational attraction of the central mass condensation (or a black hole) is just balanced by the cosmic repulsion [9]. We can see that with $r_b$ growing $V_r(r_b)$ falls down for $r_b < r_s$, it reaches its minimum at the static radius ($r_b = r_s$), where

$$V_{r(\text{min})} = V_r(r_b = r_s) = \frac{3M}{r_s} = 3y^{1/3},$$

(32)

while the expansion speed is accelerated at $r_b > r_s$, approaching velocity of light ($V_r \to 1$) when $r_b$ approaches the cosmological horizon of the SdS region ($r_b \to r_c$) (see Fig.2b). Notice that for $y \ll 1$, the cosmological horizon is approximately given by

$$\frac{r_c}{M} \sim y^{1/2}. $$

(33)

For the exact formulae giving loci of the event horizons $r_c$ and $r_b$ in the SdS spacetimes see [9].

4. INFLUENCE OF THE REFRACTION EFFECT ON TEMPERATURE FLUCTUATIONS OF THE CMBR

We shall study the influence of the refraction effect on the CMBR in the framework of the ESdS model using the simplified approach developed by Mészáros and Molnár (for more detailed model, considering also deflection of light by the mass condensation, see [10]). We do not consider the model of void used in [5], since it is not self-consistent from the point of view of general relativity. It was shown in [5] that the temperature fluctuations are fully determined by the length of the photon ray spanning in the vakuola region, i.e., it is determined by the angle $\psi_S$ giving the impact angle of photon on the MH. The effect of refraction can be incorporated into the model by substituting the angle $\psi^F_S$ influenced by the refraction effect directly into the formula determining the temperature fluctuation. For simplicity, we shall consider here photon trajectories which do not enter the internal Friedman region, and, as usual in the model, we abandon deflection of light in the SdS spacetime. The impact angle $\psi^F_S$ then has to be related to the view angle $\beta$ of observer through the angle of refraction $\psi^F$ (see Fig. 3).

The temperature fluctuation (frequency shift) of a CMBR photon due to transversing the vakuola is given by the relation [5]

$$\Delta T = \frac{2\pi^{3/2} y^3}{H^3} \left\{ \frac{\Omega}{2} \sin^2 \psi \cos \psi + \frac{1 + 2\Omega}{3 \cos^3 \psi} \right\},$$

(34)
where $\psi = \psi^+_S$ determines the length of the ray in the vakuola; $Y = R(\eta)\chi$ is the actual physical extension of the vakuola and $H = \dot{R}/R$ is the actual value of the Hubble parameter; $R(\eta)$ is the scale factor, $\dot{R} \equiv dR/dT$, $\eta$ is the conformal time defined by $d\eta = dT/R$.

Refraction effect will change the length of light ray spanning the vakuola region (see Fig. 3). Of course, for vanishing refraction effect, there is $\psi^+_S = \psi^+_F$ in agreement with [5]. Using formula (34) we find the temperature fluctuation with the refraction effect included to be given by the relation

$$\Delta T_r = \frac{2c^3Y^3}{H^3} \left[ \frac{\cos \psi^+_F + V(r_b)}{1 + V(r_b) \cos \psi^+_F} \right]$$

$$\times \left\{ \frac{\Omega}{2} \left[ \frac{\sin \psi^+_F}{(1 + V(r_b) \cos \psi^+_F)} \right]^2 + \frac{1 + 2\Omega}{3} \left[ \frac{\cos \psi^+_F + V(r_b)}{1 + V(r_b) \cos \psi^+_F} \right]^2 \right\}.$$  (35)

The relevance of the refraction effect is given by the difference of the temperature fluctuations $\Delta T_r$ and $\Delta T$:

$$\Delta \equiv \Delta T_r - \Delta T = \frac{2c^3Y^3}{H^3} \left\{ \frac{\Omega}{2} \left[ \frac{\cos \psi^+_F + V_r}{1 + V_r \cos \psi^+_F} \right] \left[ \frac{\sin \psi^+_F}{\gamma(1 + V_r \cos \psi^+_F)} \right]^2 - \cos \psi^+_F \sin^2 \psi^+_F \right\}$$

$$+ \frac{1 + 2\Omega}{3} \left[ \frac{\cos \psi^+_F + V_r}{1 + V_r \cos \psi^+_F} \right]^3 - \cos^3 \psi^+_F \right\}.$$  (36)

In the limit of non-relativistic velocities, $V_r \ll 1$, the relations (26) imply

$$\cos \psi^+_S \sim \cos \psi^+_F + V_r \sin^2 \psi^+_F, \quad \sin \psi^+_S \sim \sin \psi^+_F (1 - V_r \cos \psi^+_F),$$  (37)

so that up to the first order of $V_r$, the temperature difference is given by the formula

$$\Delta \equiv \Delta T_r - \Delta T \sim \frac{2c^3Y^3}{H^3} V_r \cos^2 \psi^+_F \sin^2 \psi^+_F \left( 1 + \frac{\Omega}{2} \tan^2 \psi^+_F \right).$$  (38)

Clearly, as we expected intuitively, the influence of the refraction effect vanishes linearly with $V_r \to 0$.

The relevance of the refraction effect in terms of the viewing angle $\beta$ follows directly from the sine rule (see Fig. 3)

$$\sin \psi^+_F = \frac{X_0 + X_b}{X_b} \sin \beta$$  (39)

and from the relation between the Schwarzschild coordinate $r_b$ and the Robertson–Walker comoving coordinate $X_b$ given by

$$r_b = R(t_b)X_b = \frac{R_0}{1 + z}X_b.$$  (40)
FIGURE 4. Plot A: Relevance of refraction is plotted as a function of angle $\beta$ for three representative values of velocity $V = 0.2$, 0.5 and 0.9. Plot B: Relevance of refraction is plotted as a function of velocity $V$ of MH for three representative values of angle $\beta = 4^\circ, 6^\circ$ and $9.5^\circ$. Both plots are drawn for $\Omega = 1$, $\chi_0 = 10$, $\chi_b = 2$ (see Fig 3).

where $R_0$ is recent value of $R$, and $z$ is the cosmological redshift, being the measure of the cosmic time. Introducing new variables

$$A(\beta) = \frac{\sqrt{1 - \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right)^2} + V_r}{1 + V_r \sqrt{1 - \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right)^2}},$$

$$B(\beta) = \frac{\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta}{1 + V_r \sqrt{1 - \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right)^2}},$$

$$C(\beta) = \sqrt{1 - \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right)^2} \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right),$$

the temperature difference (36) can be expressed as a function of $\beta$ in the form

$$\Delta T_r - \Delta T = \frac{2c^3Y^3}{H^3} \left\{ \frac{\Omega}{2} A(\beta) B^2(\beta) - C(\beta) \right\} + \frac{1 + 2\Omega}{3} \left\{ A^3(\beta) - \left(\frac{\chi_0 + \chi_b}{\chi_b} \sin \beta\right)^2 \right\}^{\frac{3}{2}}.$$ 

The relevance of the refraction effect is illustrated by Fig. 4. By analysing the relation (43), we can show that for any value of $\beta$ the influence of the refraction on the temperature fluctuations $\Delta T_r - \Delta T$ monotonically grows with $V_r$ growing.

5. CONCLUDING REMARKS

Studying the fluctuations of CMBR in an accelerating universe, we have shown, how the influence of the refraction effect grows with the velocity of the MH. Note that in the standard Friedman models with $\Lambda = 0$, the velocity of the MH falls in the expanding universe and the refraction effects are suppressed. However, in the accelerated universe, the velocity grows after MH crosses the static radius of the SdS spacetime, and the refraction effect becomes significant. Such effect could serve as another test of the presence of the cosmological constant; it could have strong observational consequences in future, when the velocity of the MH becomes to be relativistic. We conclude that there are two basic phenomena related to the importance of the refraction effect in the ESdS model explaining the temperature fluctuations of CMBR.

1. The total reflection phenomenon implies that some part of the vakula region will not be visible to the external observer. This part will be enlarged with expansion velocity $V_r$ growing.
2. The refraction effect on the temperature fluctuations (in the case of spatially flat universe) will fall, if the boundary of the MH $r_b$ approaches the static radius $r_s$ of the Schwarzschild-de Sitter region, and it starts to grow after crossing the static radius. The effect becomes to be extremely strong when $r_b$ approaches the cosmological horizon $r_c$ and $V_r \rightarrow 1$. 

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We can expect that in the accelerated universe the influence of the relict vacuum energy on the fluctuations of CMBR due to the Rees–Sciama effect could be very important, especially the refraction effect has the tendency to rise up the fluctuations. At present, we make our model more precise, and we estimate conditions under which currently observable effects could be expected.

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REFERENCES