Relativistic momentum and kinetic energy, and $E = mc^2$

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(Dated: December 22, 2006)

Abstract

Based on relativistic velocity addition and the conservation of momentum and energy, I present derivations of the expressions for the relativistic momentum and kinetic energy, and $E = mc^2$. 
The standard formal way that expressions for the relativistic momentum $p$ and the relativistic kinetic energy $T$, and the mass-energy relationship $E = mc^2$ are derived in upper-level undergraduate textbooks is by first introducing Lorentz transformations and 4-vectors, and then defining the 4-momentum vector $p^\mu = m dx^\mu / d\tau$ ($\mu = 0, 1, 2, 3$), where $\tau$ is the proper time. It is then postulated, backed up by extensive experimental observations, that in an isolated system all the components of $p^\mu$ are conserved. The spatial components of $p^\mu$ reduce to $mv$ in the non-relativistic limit, and hence correspond to the components of the relativistic momentum. The temporal component reduces in the non-relativistic limit to $mc^2 + mv^2/2$, and therefore is identified as the total energy, composed of the rest-mass and kinetic energies. Logically, there is of course nothing wrong with this approach. Pedagogically, however, it is probably helpful to have more intuitive derivations. To this end, through the years many have been published.

This paper describes relatively simple and concise derivations of the relativistic forms of $p$ and $T$, and $E = mc^2$, based on (i) conservation of momentum and energy in the collisions of two particles, and (ii) the velocity addition rules. Momentum and energy conservation should be familiar concepts to students, and the velocity addition rules can be quite simply derived from the constancy of the speed of light in all inertial reference frames. In each derivation, collisions are viewed in the center-of-momentum frame of reference, $S_{cm}$, in which both particles have momenta that are equal in magnitude and opposite in direction, and in the laboratory frame of reference, $S_{lab}$, in which one of the particles initially is at rest. Imposition of the conservation laws gives the desired expressions.

To simplify the algebra, velocities in this paper are expressed in units of $c$, the speed of light. Hence velocities are dimensionless, and $c = 1$. To obtain the standard dimensional expressions, replace all velocities in the expressions given here by $v \rightarrow v/c$, and multiply all masses by $c^2$ in order to obtain energy. Also, in this paper primes on variables denote “after collision.”
II. THE DERIVATIONS

First, let us recall the relativistic velocity transformation rules. Let \( \tilde{S} \) be an inertial frame moving with velocity \((u, 0)\) with respect to frame \(S\). If a particle has velocity \((v_x, v_y)\) in frame \(S\), the components of its velocity in frame \(\tilde{S}\) are

\[
\tilde{v}_x = \frac{v_x - u}{1 - v_x u}, \quad (1a)
\]
\[
\tilde{v}_y = \frac{v_y \sqrt{1 - u^2}}{1 - v_x u}. \quad (1b)
\]

A. Relativistic momentum

From dimensional analysis and the vector nature of momentum, the momentum of a particle of mass \(m\) travelling with velocity \(v\) must have the form

\[
p = m\gamma_v v, \quad (2)
\]

where \(\gamma_v\) is an unknown function to be determined. Since \(p = mv\) for non-relativistic velocities, \(\gamma_{v\rightarrow0} = 1\).

Consider the case where the particles are identical, hence \(m_1 = m_2 = m\). Let the motion of the particles be in the \(x-y\) plane and their initial velocities in \(S_{\text{cm}}\) be \(\pm (v, 0)\). Assume that the particles barely graze each other, so that in the collision each particle picks up a very small \(y\)-component of the velocity of magnitude \(\delta v\) in \(S_{\text{cm}}\). [See Fig. 1(a).] Their speeds in \(S_{\text{cm}}\) do not change because the collision is elastic, and hence their velocities after the collision are \(\pm (\sqrt{v^2 - (\delta v)^2}, \delta v) = \pm (v \sqrt{1 - (\delta v/v)^2}, \delta v) \approx \pm (v, \delta v)\), to first order in \(\delta v\). Because \(\delta v\) is assumed to be very small, we ignore all terms of order \((\delta v)^2\) and higher.

Now consider the collision in the laboratory frame of reference \(S_{\text{lab}}\) that is moving with velocity \((-v, 0)\) with respect to \(S_{\text{cm}}\). [See Fig. 1(b).] The pre-collision velocities of the particles \(S_{\text{lab}}\), using Eqs. (1) on the \(S_{\text{cm}}\) velocities \(\pm (v, 0)\), are \(v_{1,\text{lab}} = (w, 0)\), where

\[
w = \frac{2v}{1 + v^2}, \quad (3)
\]

and \(v_{2,\text{lab}} = (0, 0)\). After the collision, transforming the post-collision \(S_{\text{cm}}\) velocities \(\pm (v, \delta v)\)
to the $S_{\text{lab}}$ frame we obtain, to first order in $\delta v$,

$$v'_{1,\text{lab}} \approx \left( w, \frac{\delta v \sqrt{1 - v^2}}{1 + v^2} \right), \quad (4a)$$

$$v'_{2,\text{lab}} \approx \left( 0, -\frac{\delta v \sqrt{1 - v^2}}{1 - v^2} \right). \quad (4b)$$

The $y$-component of the total momentum before the collision is zero, and hence by conservation of momentum, after the collision

$$(p'_{1,\text{lab}} + p'_{2,\text{lab}})_y = m \gamma |v'_{1,\text{lab}}| v'_{1,\text{lab},y} + m \gamma |v'_{2,\text{lab}}| v'_{2,\text{lab},y} = 0. \quad (5)$$

Since $v'_{\text{lab},y}$ terms are of order $\delta v$ and we are ignoring terms of order $(\delta v)^2$, it is sufficient in Eq. (5) to evaluate $|v'_{1,\text{lab}}|$ and $|v'_{2,\text{lab}}|$ to zeroth order in $\delta v$ (i.e., ignoring $\delta v$ altogether). Substituting $|v'_{1,\text{lab}}| \approx w$, $|v'_{2,\text{lab}}| \approx 0$, and the $y$-components of the velocities from Eqs. (4a) and (4b) into Eq. (5) and using $\gamma \rightarrow 0 = 1$ yields

$$\frac{\gamma w}{1 + v^2} - \frac{1}{1 - v^2} = 0. \quad (6)$$

This, together with Eq. (3) gives

$$\gamma_w = \frac{1 + v^2}{1 - v^2} = \left(1 - \left[ \frac{2v}{1 + v^2} \right]^2 \right)^{-1/2} = (1 - w^2)^{-1/2}. \quad (7)$$

**B. Relativistic kinetic energy**

Dimensional analysis and the scalar property of kinetic energy imply that its form is

$$T = m G(v), \quad (8)$$

where $m$ is the mass of the particle, $v = |v|$ is its speed and the function $G(v)$ is to be determined.

Consider an elastic head-on collision between two particles, of mass $m$ and $M \gg m$, with speeds in $S_{\text{cm}}$ of $v$ and $V$, respectively. In $S_{\text{cm}}$, the particles simply reverse directions, and the motion is one-dimensional. [See Fig. 2(a).] Assume that the mass $M$ is so large that in frame $S_{\text{cm}}$ its speed $V \ll 1$, and hence we can use the non-relativistic expressions for the momentum and kinetic energy of mass $M$. The magnitudes of the momenta of $m$ and $M$ are equal in $S_{\text{cm}}$, implying

$$m \gamma_v v = MV. \quad (9)$$
The $S_{cm}$ frame pre- and post-collision velocities of mass $m$ are $v_{cm} = v$ and $v'_{cm} = -v$ respectively, and of mass $M$ are $V_{cm} = V$ and $V'_{cm} = -V$, respectively. Transforming these to the $S_{lab}$ frame, which is moving at velocity $-V$ with respect to $S_{cm}$ [see Fig. 2(b)], gives $v_{lab} = (v + V)/(1 + vV), \quad v'_{lab} = (-v + V)/(1 - vV), \quad V_{lab} = 0$ and $V'_{lab} = 2V/(1 + V^2)$. By conservation of kinetic energy in an elastic collision in the $S_{lab}$ frame and Eq. (8),

$$
m G(|v_{lab}|) = m G(|v'_{lab}|) + \frac{M}{2} |V'_{lab}|^2. \quad (10)
$$

Expanding $|v_{lab}|$, $|v'_{lab}|$ and $|V'_{lab}|$ to first order in $V$,

$$
|v_{lab}| \approx (v + V)(1 - vV) \approx v + V(1 - v^2), \quad (11a)
$$

$$
|v'_{lab}| \approx (v - V)(1 + vV) \approx v - V(1 - v^2), \quad (11b)
$$

$$
|V'_{lab}| \approx 2V(1 - V^2) \approx 2V, \quad (11c)
$$

and substituting these into the Taylor expansions of the $G$ terms about $v$ in Eq. (10) gives, to first order in $V$,

$$
m \left( G(v) + \left[ \frac{dG(u)}{du} \right]_v V(1 - v^2) \right) = m \left( G(v) - \left[ \frac{dG(u)}{du} \right]_v V(1 - v^2) \right) + 2MV^2. \quad (12)
$$

Substituting $2MV^2 = 2m\gamma_v vV$ [from Eq. (9)] into Eq. (12) leads to

$$
\left[ \frac{dG}{du} \right]_v \frac{v}{1 - v^2} = \frac{v}{(1 - v^2)^{3/2}}, \quad (13)
$$

which upon integration yields

$$
G(v) - G(0) = \left[ \frac{1}{(1 - u^2)^{3/2}} \right]_{u=v}^{u=0} = \gamma_v - 1. \quad (14)
$$

Since the kinetic energy vanishes when $v$ is zero, $G(0) = 0$, and hence Eqs. (8) and (14) imply that (reintroducing $c$) $T = m(\gamma_v - 1)c^2$.

C. $E = mc^2$

Consider the initial situation as in Sec. II B except that the speed $V$ of mass $M$ can be relativistic, and after collision the two particles merge into one composite particle. In $S_{cm}$, $M\gamma_v V = m\gamma_v v$, and after the collision the composite particle is stationary. In $S_{lab}$ which is moving with velocity $-V$ with respect to $S_{cm}$, before the collision particle $M$ is stationary.
and particle $m$ moves with velocity $v_{\text{lab}} = (v + V)/(1 + vV)$, and after the collision the composite particle moves with velocity $V$.

The total momentum in $S_{\text{lab}}$ before the collision is $P_{\text{lab}} = m\gamma_{v}v_{\text{lab}} = m\gamma_{v}\gamma_{V}(u + V)$. If the mass of the composite particle does not change, then the momentum of the composite particle after the collision in $S_{\text{lab}}$ would be $(M + m)\gamma_{V}V \neq P_{\text{lab}}$ in general, violating conservation of momentum. Therefore, the mass of the composite particle must change by $\Delta m$ such that momentum is conserved in $S_{\text{lab}}$; i.e.,

$$m\gamma_{v}\gamma_{V}(v + V) = (M + m + \Delta m)\gamma_{V}V.$$  \hspace{1cm} (15)

Substituting $m\gamma_{v}v = M\gamma_{V}V$ on the left hand side and cancelling $\gamma_{V}V$ on both sides gives

$$\Delta m = m(\gamma_{v} - 1) + M(\gamma_{V} - 1).$$ \hspace{1cm} (16)

From Sec. II B, the right hand side of Eq. (16) is equal to $-\Delta T$, the total change in kinetic energy in $S_{\text{cm}}$ (since particles $m$ and $M$ start with speeds $v$ and $V$, respectively, and both are stationary at the end). By conservation of total energy, $\Delta E + \Delta T = 0$, where $\Delta E$ is the energy associated with the change in mass. Hence, $\Delta E = -\Delta T = \Delta m$ or (reintroducing $c$, and making the plausible assumption that a zero mass object with zero velocity has zero energy) $E = mc^{2}$. Finally, combining the results of Sections II B and II C gives the total energy of a particle of mass $m$ moving with speed $v$, $E + T = E_{\text{total}} = m\gamma_{v}c^{2}$.

III. CONCLUDING REMARKS

It should be noted that these derivations do not guarantee that the momentum and total energy are conserved in all inertial reference frames or in all collisions. They only show the forms that the momentum, kinetic energy and energy–mass relation must have, given momentum and energy conservation. Once these expressions are known, when the 4-momentum is introduced its components will be recognized as the total energy and momentum. The covariance of the momentum 4-vector can then be used to demonstrate momentum and total energy conservation in all inertial frames. Conservation of momentum can be shown to be a consequence of conservation of energy, and, as befitting an experimental science, the
conservation of energy ultimately depends on experimental observations.

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4 Here, the terms “vector” and “scalar” are used in the non-relativistic (*i.e.*, not the 4-vector)
The term $2M V^2$ in Eq. (12) is actually first order in $V$, because $M$ is of order $V^{-1}$ [see Eq. (9)].

FIG. 1: Grazing collision between two particles of equal mass, in (a) center-of-momentum and (b) laboratory frames of reference. Dashed and solid lines indicate before and after the collision, respectively.

FIG. 2: Head-on collision between particles of mass \( m \) and \( M \gg m \), in (a) center-of-momentum and (b) laboratory frames of reference. Dashed and solid lines indicate before and after the collision, respectively.