The Lamb shift \( (2P_{1/2} - 2S_{1/2}) \) in the muonic helium ion \( (\mu^4He)^+ \) is calculated with the account of contributions of orders \( \alpha^3 \), \( \alpha^4 \), \( \alpha^5 \) and \( \alpha^6 \). Special attention is given to corrections of the electron vacuum polarization, the nuclear structure and recoil effects. The obtained numerical value of the Lamb shift 1381.716 meV can be considered as a reliable estimate for the comparison with experimental data.

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I. INTRODUCTION

The ion of muonic helium \( (\mu^4He)^+ \) is the bound state of the negative muon and alpha particle. This simple atom is short-lived. The lifetime is determined by the muon decay in a time \( \tau_{\mu} = 2.19703(4) \times 10^{-6} \). The increase of the lepton mass in going from the electron to the muon hydrogenic atoms \( (m_{\mu}/m_e = 206.7682838(54) \) \cite{1}) leads to the enhancement of the nuclear structure effects, the electron vacuum polarization corrections and the recoil contributions to the fine and hyperfine structure of the energy spectrum. So, the experimental investigation of the Lamb shift in muonic hydrogen, muonic helium ions could be useful to obtain more precise values of the nuclear charge radii (the proton, helion, alpha particle) \cite{2,3}. In the case of muonic hydrogen the Lamb shift measurement is carried out at present at PSI (Paul Sherrer Institute) \cite{4,5}. The measurement of the Lamb shift \( (2P_{3/2} - 2S_{1/2}) \) in the ion of muonic helium \( (\mu^4He)^+ \), which was carried out many years ago at CERN \cite{6}, gave the value 1527.5 ± 0.3 meV. At a later time the experimental study of the \( (2P - 2S) \) splitting in the \( (\mu^4He)^+ \) \cite{7} found no resonance effect at the wavelength interval 811.4 ≤ \( \lambda \) ≤ 812.0 nm with a greater than 95% probability. So, at present there is the need of new experiment which could resolve the existing experimental problem.

Theoretical investigation of the Lamb shift \( (2P - 2S) \) in muonic helium ions was performed many years ago in Refs.\cite{8,9,10,11} on the basis of the Dirac equation (see other references in review article \cite{10}). Their calculation took into account different QED corrections with the accuracy 0.01 meV. High order corrections over the fine structure constant \( \alpha \) to the Lamb shift \( (2P - 2S) \) in the electron hydrogenic atom were obtained in the last years in the analytical form. Modern status of these calculations is presented in Ref.\cite{2}. The aim of the present work is to calculate the Lamb shift \( (2P - 2S) \) in the ion of muonic helium \( (\mu^4He)^+ \) with the account of contributions of orders \( \alpha^3 \), \( \alpha^4 \), \( \alpha^5 \) and \( \alpha^6 \) on the basis of quasipotential method in quantum electrodynamics \cite{12,13,14}. We consider such effects of the electron vacuum polarization, the recoil and nuclear structure corrections which are crucial to attain...
the high accuracy. With the exception of the nuclear polarizability contribution, we calculate all corrections in the interval \((2P_{1/2} - 2S_{1/2})\) with a precision 0.001 meV. Our purpose consists in the improvement of the earlier performed calculations [9, 10] and derivation the reliable estimate for the \((2P_{1/2} - 2S_{1/2})\) Lamb shift, which can be used for the comparison with experimental data. Modern numerical values of fundamental physical constants are taken from Ref.[1]: the electron mass \(m_e = 0.510998918(44) \times 10^{-3}\) GeV, the muon mass \(m_\mu = 0.1056583692(94)\) GeV, the fine structure constant \(\alpha^{-1} = 137.03599911(46)\), the mass of alpha particle \(m_\alpha = 3.72737917(32)\) GeV.

II. EFFECTS OF VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our approach to the investigation of the Lamb shift \((2P - 2S)\) in the muonic helium ion \((\mu_4^+He)\) is based on the use of quasipotential method in quantum electrodynamics [14, 15, 16], where the two-particle bound state is described by the Schrödinger equation. The basic contribution to the muon and \(\alpha\)-particle interaction operator is determined by the Breit Hamiltonian [17]:

\[
H_B = \frac{p^2}{2\mu} - \frac{Ze}{r} - \frac{p^4}{8m_1^3} - \frac{p^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(r) - \frac{Z\alpha}{2m_1m_2r} \left( p^2 + \frac{\mathbf{r}(\mathbf{p})\mathbf{p}}{r^2} \right) + \frac{Z\alpha}{2} \left( \frac{1}{4m_1^2} + \frac{1}{4m_2^2} \right) (\mathbf{L}\sigma_1) = H_0 + \Delta V_B,
\]

where \(H_0 = \frac{p^2}{2\mu} - \frac{Ze}{r}, m_1, m_2\) are the muon and \(\alpha\)-particle masses, \(\mu = m_1m_2/(m_1 + m_2)\).

The wave functions of \(2S-\) and \(2P\)-states are equal:

\[
\psi_{200}(r) = \frac{W_0^{3/2}}{2\sqrt{2\pi}} e^{-\frac{Wr}{2}} \left( 1 - \frac{Wr}{2} \right), \quad \psi_{2lm}(r) = \frac{W_0^{3/2}}{2\sqrt{6}} e^{-\frac{Wr}{2}} Wr Y_{lm}(\theta, \phi), \quad W = \mu Z\alpha. \quad (2)
\]

The ratio of the Bohr radius of muonic helium to the Compton wavelength of the electron \(m_e/W = 0.34\), so, the basic contribution of the electron vacuum polarization (VP) to the Lamb shift is of order \(\alpha(Z\alpha)^2\) (see Fig.1(a)).

Accounting the modification of the Coulomb potential due to vacuum polarization in the coordinate representation

\[
V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty d\xi \rho(\xi) \left( -\frac{Z\alpha}{r} e^{-2m_e\xi r} \right), \quad \rho(\xi) = \frac{\sqrt{\xi^2 - 1}(2\xi^2 + 1)}{\xi^4},
\]

we present its contributions to the shifts of \(2S-\), \(2P\)-states and the Lamb shift \((2P - 2S)\) in the form:

\[
\Delta E_{VP}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi} \int_1^\infty \rho(\xi)d\xi \int_0^\infty xdx \left( 1 - \frac{x}{2} \right)^2 e^{-x(1 + \frac{2m_e\xi}{W})} = -2077.231 \text{ meV}, \quad (4)
\]

\[
\Delta E_{VP}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi} \int_1^\infty \rho(\xi)d\xi \int_0^\infty x^3dx e^{-x(1 + \frac{2m_e\xi}{W})} = -411.449 \text{ meV}, \quad (5)
\]

\[
\Delta E_{VP}(2P - 2S) = 1665.782 \text{ meV}. \quad (6)
\]
The muon one-loop vacuum polarization correction is known in analytical form [2]. We included corresponding value to the total shift in section 5. The two-loop vacuum polarization effects in the one-photon interaction are shown in Fig.1(b,c,d). To obtain the contribution of the amplitude in Fig.1(b) to the interaction operator, it is necessary to use the following replacement in the photon propagator:

$$\frac{1}{k^2} \to \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi)d\xi \frac{1}{k^2 + 4m_e^2\xi^2}. \quad (7)$$

In the coordinate representation the diagram with two sequential loops gives the following particle interaction operator:

$$V_{C-V_P-VP}(r) = \frac{\alpha^2}{9\pi^2} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \left( -\frac{Z\alpha}{r} \right) \frac{1}{(\xi^2 - \eta^2)} \left( \xi^2 e^{-2m_e\xi r} - \eta^2 e^{-2m_e\eta r} \right). \quad (8)$$

Averaging (8) over the Coulomb wave functions (2), we find the contribution to the Lamb shift of order \(\alpha^2(Z\alpha)^2\):

$$\Delta E_{V_P-V_P}(2P - 2S) = 3.800 \text{ meV}. \quad (9)$$

Higher order \(\alpha^2(Z\alpha)^4\) correction is determined by the amplitude with two sequential electron (VP) and muon (MVP) loops. Corresponding potential can be written as:

$$\Delta V_{VP-MVP}(r) = -\frac{4(Z\alpha)\alpha^2}{45\pi^2m_e^2} \int_1^\infty \rho(\xi)d\xi \left[ \pi\delta(r) - \frac{m_e^2\xi^2}{r} e^{-2m_e\xi r} \right]. \quad (10)$$

Its contribution to the shift \((2P - 2S)\) is equal

$$\Delta E(2P - 2S) = 0.002 \text{ meV}. \quad (11)$$

The particle interaction potential, corresponding to the two-loop amplitudes in Fig.1(c,d) with the second order polarization operator, takes the form:

$$\Delta V_{2\text{-loop } V_P}^C = -\frac{2Z\alpha}{3\pi} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{f(v)dv}{(1 - v^2)} e^{-\frac{2m_e r}{\sqrt{1 - v^2}}}. \quad (12)$$
FIG. 2: Effects of the three-loop vacuum polarization in the one-photon interaction (a,b) and in the third order perturbation theory (c).

where the spectral function

\[ f(v) = v \left\{ (3-v^2)(1+v^2) \left[ Li_2 \left( -\frac{1-v}{1+v} \right) + 2Li_2 \left( \frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{1-v} \ln v \right] \right. \]

\[ + \left. \left[ \frac{11}{16} (3-v^2)(1+v^2) + \frac{v^4}{4} \ln \frac{1+v}{1-v} + \frac{3}{2} v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \right\}. \]  

The potential \( \Delta V_{2\text{-loop}}^{C V P}(r) \) gives larger contribution as compared with (8) both to the hyperfine structure and Lamb shift \((2P-2S)\):

\[ \Delta E_{2\text{-loop}}^{V P}(2P-2S) = 9.681 \text{ meV}. \]  

Numerical values of corrections (9), (14) and the accuracy of the calculation show that it is important to consider the three-loop contributions of the vacuum polarization (see Fig.2). Part of corrections from the diagrams of three-loop vacuum polarization to the potential in the one-photon interaction can be derived as the relations (8), (12) (the sequential loops in Fig.2(a,b)). Corresponding contributions to the potential and the splitting \((2P-2S)\) are the following:

\[ V_{V P-2\text{-loop}}^{C V P}(r) = -\frac{Z\alpha}{r} \frac{\alpha^3}{(3\pi)^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta)d\eta \int_1^\infty \rho(\zeta) d\zeta \times \]

\[ \left. e^{-2m_c \xi r} \left( \frac{\zeta^4}{(\xi^2 - \zeta^2)(\eta^2 - \xi^2)} + e^{-2m_c \eta r} \left( \frac{\xi^4}{(\zeta^2 - \xi^2)(\eta^2 - \xi^2)} + e^{-2m_c \eta r} \frac{\eta^4}{(\xi^2 - \eta^2)(\zeta^2 - \eta^2)} \right) \right, \right. \]  

\[ V_{V P-2\text{-loop}}^{C V P}(r) = -\frac{4\mu\alpha^3(Z\alpha)}{9\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \frac{f(\eta)d\eta}{\eta} \left( \frac{1}{r(\eta^2 - \xi^2)} \right) \left( \eta^2 e^{-2m_c \eta r} \eta^2 - \xi^2 - \xi^2 e^{-2m_c \xi r} \right), \]  

\[ \Delta E_{V P-2\text{-loop}}^{V P}(2P-2S) = 0.009 \text{ meV}, \]  

\[ \Delta E_{V P-2\text{-loop}}^{V P}(2P-2S) = 0.046 \text{ meV}. \]
There exists a number of the diagrams that express three-loop corrections in the polarization operator. They were first calculated for the \((2P - 2S)\) Lamb shift in Refs.\([18, 19]\). The largest contribution to the energy spectrum comes from the sixth-order vacuum polarization diagrams with one electron loop (\(\Pi^{(3)}\) corrections \([18]\)). The estimate of their contribution to the Lamb shift in \((\mu \frac{3}{2}He)^+\) is included in Table I. Analysis of the contribution of three-loop vacuum polarization in third order perturbation theory in Fig.2(c) shows that we can neglect it accounting the declared accuracy of the calculation.

Additional one-loop vacuum polarization diagram is presented in Fig.3. It gives in the energy spectrum the correction of fifth order over \(\alpha\) (the Wichmann-Kroll correction) \([20, 21]\). The particle interaction potential can be written in this case in the integral form:

\[
\Delta V^{WK}(r) = \frac{\alpha (Z\alpha)^3}{\pi r} \int_0^\infty \frac{d\zeta}{\zeta^4} e^{-2m_e\zeta r} \left[ -\frac{\pi^2}{12} \sqrt{\zeta^2 - 1} \theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f_{WK}(x) \right].
\]

The exact form of the spectral function \(f^{WK}\) is presented in Refs.\([2, 20, 21]\). Numerical integration in Eq.(19) with the wave functions (2) gives the following contribution to the Lamb shift:

\[
\Delta E^{WK}(2P - 2S) = 0.135 \text{ meV}.
\]

III. RELATIVISTIC CORRECTIONS WITH THE VACUUM POLARIZATION EFFECTS

The electron vacuum polarization effects lead not only to corrections in the Coulomb potential (3), but also to the modification of the other terms of the Breit Hamiltonian (1). The one-loop vacuum polarization corrections in the Breit interaction were obtained in Refs.\([22, 23]\):

\[
\Delta V^{B}_{1,VP}(r) = \frac{Z\alpha}{8} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left[ 4\pi \delta(r) - \frac{4m_e^2\zeta^2}{r} e^{-2m_e\zeta r} \right],
\]

\[
\Delta V^{B}_{2,VP} = -\frac{Z\alpha m_e^2\zeta^2}{r m_1 m_2} e^{-2m_e\zeta r}(1 - m_e\zeta r),
\]
\[
\Delta V_{3,VP}^B = -\frac{Z\alpha}{2m_1m_2} \frac{e^{-2m_e\xi r}}{p_i} \frac{\delta_{ij} + \frac{r_i r_j}{r^2}(1 + 2m_e\xi r)}{r} p_j, \tag{24}
\]
\[
\Delta V_{4,VP}^B = \frac{Z\alpha}{r^3} \left( \frac{1}{4m_1^2} + \frac{1}{2m_1m_2} \right) e^{-2m_e\xi r}(1 + 2m_e\xi r)(L\sigma_1). \tag{25}
\]

In the first order perturbation theory (PT) the potentials \(\Delta V_{i,VP}^B(r)\) give necessary contributions of order \(\alpha(Z\alpha)^4\) to the shift \((2P - 2S)\):

\[
\Delta E_{1,VP}^B(2P - 2S) = -0.894 \text{ meV}, \tag{26}
\]
\[
\Delta E_{2,VP}^B(2P - 2S) = 0.012 \text{ meV}, \tag{27}
\]
\[
\Delta E_{3,VP}^B(2P - 2S) = 0.022 \text{ meV}, \tag{28}
\]
\[
\Delta E_{4,VP}^B(2P - 2S) = -0.088 \text{ meV}. \tag{29}
\]

Potentials \(\Delta V_{2,VP}^B, \Delta V_{3,VP}^B, \Delta V_{4,VP}^B\) take into account the recoil effects over the ratio \(m_1/m_2\). We have included in Table I the summary correction of order \(\alpha(Z\alpha)^4\), which is determined by the relations (26)-(29). The next to leading order \(\alpha^2(Z\alpha)^4\) correction appears in the energy spectrum from the two-loop modification of the Breit Hamiltonian. We consider the term of the leading order over \(m_1/m_2\) in the potential (the function \(f(v)\) is determined by Eq.(13)):

\[
\Delta V_{2\text{-}loop}^B V_P(r) = \frac{\alpha^2(Z\alpha)}{12\pi^2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \int_0^1 \frac{f(v)dv}{1 - v^2} \left[ 4\pi\delta(r) - \frac{4m_e^2}{1 - v^2} e^{-\frac{4m_e}{\sqrt{1-v^2}}} \right]. \tag{30}
\]

Corresponding shift is the following:

\[
\Delta E_{2\text{-}loop}^B V_P(2P - 2S) = -0.003 \text{ meV}. \tag{31}
\]

Other two-loop contributions to the Breit potential are omitted because they give the energy corrections which lie outside the accuracy of the calculation in this work.

In the second order perturbation theory (SOPT) we have a number of the electron vacuum polarization contributions in the leading orders \(\alpha^2(Z\alpha)^2\) and \(\alpha(Z\alpha)^4\), shown in the diagrams of Fig.4 (c,b):

\[
\Delta E_{SOPT}^V = <\psi|\Delta V_C^C G \Delta V_C^C|\psi > + 2 <\psi|\Delta V_B^C G \Delta V_C^C|\psi > \tag{32}
\]

The second order perturbation theory corrections in the energy spectrum of hydrogen-like system are determined by the reduced Coulomb Green function \(\tilde{G}\) (RCGF), whose partial expansion has the form \([24]\):

\[
\tilde{G}_n(r, r') = \sum_{l,m} \tilde{g}_{nl}(r, r')Y_{lm}(n)Y_{lm}^*(n'). \tag{33}
\]

The radial function \(\tilde{g}_{nl}(r, r')\) was presented in Ref.\([24]\) in the form of the Sturm expansion in Laguerre polynomials. For the calculation of the Lamb shift \((2P - 2S)\) in muonic helium it is convenient to use the compact representation for the RCGF of \(2S-\) and \(2P-\) states, which was obtained in Ref.\([22]\):

\[
\tilde{G}(2S) = \frac{Z\alpha}{4x_1x_2} e^{-\frac{x_1 + x_2}{2}} \frac{1}{4\pi} g_{2S}(x_1, x_2), \tag{34}
\]
theory (SOPT). The dashed line shows the Coulomb photon.

![Diagram of vacuum polarization effects]

FIG. 4: Effects of the one-loop and two-loop vacuum polarization in the second order perturbation theory (SOPT). The dashed line shows the Coulomb photon.

\[
g_{2S}(x_1, x_2) = 8x_\ < -4x_\ < ^2 + 8x_\ > + 12x_\ < x_\ > - 26x_\ < x_\ > - 26x_\ < x_\ > + 23x_\ < x_\ > - 26x_\ < x_\ > - 4x_\ < - 26x_\ < x_\ > - 26x_\ < x_\ > + 36x_\ < x_\ > + 26x_\ > + 23x_\ > - 26x_\ > - 4x_\ > - 26x_\ > + 36x_\ > + 26x_\ > - 26x_\ > - 4x_\ > - 26x_\ > + 36x_\ > + 26x_\ > = 0 \quad (35)
\]

\[
x_\ < ^3 x_\ > + 2x_\ < x_\ > - x_\ < ^2 x_\ > + 4e^x (1 - x_\ < ) (x_\ > - 2)x_\ > + 4(x_\ > - 2)x_\ < (x_\ > - 2)x_\ > \times
\]

\[
\times [-2C + Ei(x_\ < ) - \ln(x_\ < ) - \ln(x_\ > )] \]

\[
\tilde{G}(2P) = -\frac{Z \alpha^2}{36x_1^2 x_2^2} e^{-x_1+x_2} x_1 x_2 g_{2P}(x_1, x_2), \quad (36)
\]

\[
g_{2P}(x_1, x_2) = 24x_\ < + 36x_\ < x_\ > + 36x_\ < x_\ > + 24x_\ > + 36x_\ < x_\ > + 36x_\ < x_\ > + 49x_\ < x_\ > + 3x_\ < ^3 x_\ > - 3x_\ < ^3 x_\ > - 36x_\ < x_\ > + 24x_\ > + 36x_\ > - 24x_\ > - 36x_\ > - 49x_\ > - 3x_\ > ^3 x_\ > - 3x_\ > ^3 x_\ > = 0 \quad (37)
\]

where \( x_\ < = \min(x_1, x_2), \ x_\ > = \max(x_1, x_2), \ C = 0.57721566... \) is the Euler constant. As a result the two-loop vacuum polarization contributions in the first term of Eq.(32) can be presented first in the integral form. The subsequent numerical integration gives the following results:

\[
\Delta E_{SOPT}^{VPVP}(2S) = -\frac{\mu \alpha^2 (Z \alpha)^2}{72 \pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times \quad (38)
\]

\[
\times \int_0^\infty \int_0^\infty \left(1 - \frac{x}{2}\right) e^{-x(1 - \frac{\text{max}}{\text{max}})} dx_1 dx_2 \int_0^\infty \left(1 - \frac{x'}{2}\right) e^{-x'(1 - \frac{\text{max}}{\text{max}})} dx_1' dx_2' g_{2S}(x, x') = -1.901 \text{ meV},
\]

\[
\Delta E_{SOPT}^{VPVP}(2P) = -\frac{\mu \alpha^2 (Z \alpha)^2}{7776 \pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times \quad (39)
\]

\[
\times \int_0^\infty e^{-x(1 - \frac{\text{max}}{\text{max}})} dx_1 \int_0^\infty e^{-x'(1 - \frac{\text{max}}{\text{max}})} dx_1' dx_2' g_{2P}(x, x') = -0.194 \text{ meV},
\]
The second term in (32) has the similar structure (see Fig.4(b)). The transformation of the different matrix elements is carried out with the use of the algebraic relation of the form:

\[
<\psi|\frac{\mathbf{p}^4}{(2\mu)^2}\sum_m&\frac{|\psi_m><\psi_m|\Delta V_C^{VP}|\psi>}{E_2 - E_m} = \frac{\mathbf{p}^4}{(2\mu)^2}\sum_m\frac{|\psi_m><\psi_m|\Delta V_C^{VP}|\psi>}{E_2 - E_m} \\
= <\psi|(E_2 + \frac{Z\alpha}{r})(\hat{H}_0 + \frac{Z\alpha}{r})\sum_m&\frac{|\psi_m><\psi_m|\Delta V_C^{VP}|\psi>}{E_2 - E_m} = (40)
\]

Omitting further details of the calculation of numerous matrix elements in Eq.(32), we present the summary contribution from the second term in (32) to the shift \((2P - 2S)\) as follows:

\[
\Delta E_{SOPT}^{B,VP}(2P - 2S) = 0.746 \text{ meV.} \tag{41}
\]

![FIG. 5: The three-loop vacuum polarization corrections in the second order perturbation theory.](image)

The three-loop vacuum polarization contributions to the energy spectrum in the second order perturbation theory are presented in Fig.5. Respective potentials are obtained earlier in relations (3), (8), (12). Considering the accuracy of the calculation we can restrict our analysis by the shifts of \(2S\) level, which can be written in the form:

\[
\Delta E_{SOPT}^{V,VP}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{108\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_1^\infty \rho(\xi) d\xi \int_0^\infty dx (1 - \frac{x}{2}) \times (42)
\]

\[
\int_0^\infty dx'\left(1 - \frac{x'}{2}\right) e^{-x'(1 + \frac{2m\xi}{W})} \int_0^\infty dx'\left(1 - \frac{x'}{2}\right) e^{-x'(1 + \frac{2m\xi}{W})} g_{2S}(x, x') = -0.011 \text{ meV;}
\]

\[
\Delta E_{SOPT}^{2,loop}^{V,VP}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{18\pi^3} \int_0^1 f(v) dv \int_1^\infty \rho(\xi) d\xi \times (43)
\]

\[
\times \int_0^\infty dx \left(1 - \frac{x}{2}\right) e^{-x(1 + \frac{2m\xi}{\sqrt{1 - v^2}})} \int_0^\infty dx' \left(1 - \frac{x'}{2}\right) e^{-x'(1 + \frac{2m\xi}{\sqrt{1 - v^2}})} = -0.017 \text{ meV;}
\]

Yet another contributions of the second order PT exist (see Fig.4(d,e,f)), which have the general structure similar to Eqs.(38), (39). They appear after the replacements \(\Delta V_C^{VP} \rightarrow \Delta V^B\) and \(\Delta V_C^{VP} \rightarrow \Delta V_C^{VP,VP}\) in the basic amplitude shown in Fig.4(c). The estimate of this contribution of order \(\alpha^2(Z\alpha)^4\) to the shift \((2P - 2S)\) can be obtained if we take into account in the Breit potential the leading order term in the parameter \(m_1/m_2\). Its numerical value is

\[
\Delta E_{SOPT}^{V,VP;\Delta V^B}(2P - 2S) = 0.008 \text{ meV.} \tag{44}
\]
The two-loop vacuum polarization contribution is determined also by the amplitude in Fig. 4(a). In order for its calculation to be made we can use Eqs. (21) and (3). In the leading order in the ratio \( m_1/m_2 \) we have again the potential (22), which leads to the following correction of order \( \alpha^2(Z\alpha)^4 \):

\[
\Delta E_{S_{OPT}}^{V_{P}\Delta V_{P}}(2P - 2S) = -0.006 \text{ meV}. \tag{45}
\]

IV. NUCLEAR STRUCTURE AND VACUUM POLARIZATION EFFECTS

![Diagram](image)

Fig. 6: The leading order nuclear structure and vacuum polarization corrections.

The influence of the nuclear structure on the muon motion in the ion (\( \mu^+_{4He} \)) is determined in the leading order by the charge radius of alpha particle \( r_\alpha = 1.676(8) \text{ fm} \) \[25, 26\] (Fig. 6(a)):

\[
\Delta E_{str}(2P - 2S) = -\frac{\mu^3(Z\alpha)^4}{12} < r_\alpha^2 > = -295.848 \text{ meV}. \tag{46}
\]

The next to leading order correction of order \( (Z\alpha)^5 \) is described by the one-loop exchange diagrams (Fig. 7). Introducing the charge form factor \( F(k^2) \) of the alpha particle, we can express it in the integral form:

\[
\Delta E_{str}^{2\gamma}(nS) = -\frac{\mu^3(Z\alpha)^5}{\pi n^3} \delta_{l0} \int_0^\infty \frac{dk}{k} V(k), \tag{47}
\]

\[
V(k) = \frac{2(F^2 - 1)}{m_1 m_2} + \frac{8m_1[-F(0) + 4m_2F'(0)]}{m_2(m_1 + m_2)k} + \frac{k^2}{2m_1^3} \times \tag{48}
\]

\[
\times \left[ 2(F^2 - 1)(m_1^2 + m_2^2) - F^2 m_1^2 \right] + \frac{\sqrt{k^2 + 4m_1^2}}{2m_2^3(m_1^2 - m_2^2)k} \times
\]

\[
\times \left\{ k^2 \left[ 2(F^2 - 1)m_2^2 - F^2 m_1^2 \right] + 8m_1^4 F^2 + \frac{16m_1^4 m_2^2(F^2 - 1)}{k^2} \right\} -
\]

\[
-\frac{\sqrt{k^2 + 4m_1^2} m_1}{2m_2^3(m_1^2 - m_2^2)k} \left\{ k^2 \left[ 2(F^2 - 1) - F^2 \right] + 8m_2^2 F^2 + \frac{16m_2^4(F^2 - 1)}{k^2} \right\}. \]
To perform numerical integration in Eq.(47) we use the following dipole parameterization of the charge form factor:

\[ F(k^2) = \frac{\Lambda^4}{(k^2 + \Lambda^2)^2}, \quad \Lambda^2 = \frac{12}{<r_\alpha^2>}. \tag{49} \]

Numerical value of the Lamb shift \((2P - 2S)\) is equal

\[ \Delta E_{str}^{2P}(2P - 2S) = 7.196 \text{ meV}. \tag{50} \]

\[ \begin{align*}
\text{a} & \\
\text{b} &
\end{align*} \]

FIG. 7: The nuclear structure corrections of order \((Z\alpha)^5\). The thick points in the diagrams are the nuclear vertex operators.

The essential increase of corrections (46), (50) in the comparison with muonic hydrogen is conditioned by two reasons. In the first place, the charge radius of the \(\alpha\) - particle increases so that \(\mu^2 < r_\alpha^2 > = 0.76\). Secondly, the charge of the \(\alpha\)-particle \(Z = 2\), what leads to the additional factor \(2^5\) in Eq.(50). The particle interaction amplitudes containing the nuclear structure and vacuum polarization effects must be considered to obtain total value of the Lamb shift. In the leading order such amplitude are presented in Fig.6(b). Corresponding interaction operator can be written as

\[ \Delta V_{str}^{VP}(r) = \frac{2}{3\pi} Z\alpha < r_\alpha^2 > \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left[ \delta(r) - \frac{m_e^2 \xi}{\pi r} e^{-2m_e \xi r} \right]. \tag{51} \]

Its contributions to the shifts of the \(2S-\) and \(2P-\) levels are determined by following expressions:

\[ \begin{align*}
\Delta E_{str}^{VP}(2S) &= \frac{\alpha(Z\alpha)^4 < r_\alpha^2 > \mu^3}{36\pi} \int_1^\infty \rho(\xi) d\xi \left[ 1 - \frac{4m_e^2 \xi}{W^2} \right] \int_0^\infty x dx (1 - \frac{x}{2})^2 e^{-x(1 + \frac{2m_e}{W})} ] = 0.937 \text{ meV}, \tag{52} \\
\Delta E_{str}^{VP}(2P) &= -\frac{\alpha(Z\alpha)^4 \mu^3 < r_\alpha^2 > m_e^2}{108\pi} \int_1^\infty \xi^2 \rho(\xi) d\xi \int_0^\infty x^3 e^{-x(1 + \frac{2m_e}{W})} dx = -0.023 \text{ meV}, \tag{53} \\
\Delta E_{str}^{VP}(2P - 2S) &= -0.960 \text{ meV}. \tag{54} \\
\end{align*} \]

The contribution of the same order \(\alpha(Z\alpha)^4\) is given by the amplitude in the second order perturbation theory in Fig.6(c):

\[ \begin{align*}
\Delta E_{str,SOPT}^{VP}(2P - 2S) &= -\frac{\alpha(Z\alpha)^4 \mu^3 < r_\alpha^2 >}{36\pi} \int_1^\infty \rho(\xi) d\xi \times \tag{55} \\
\end{align*} \]
\begin{equation}
\times \int_{0}^{\infty} dx e^{-x(1 + \frac{2m_e \xi}{W})} (1 - \frac{x}{2}) \left[ 4x(x - 2)(\ln x + C) + x^3 - 13x^2 + 6x + 4 \right] = -1.506 \text{ meV}.
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig8.png}
\caption{The nuclear structure and two-loop vacuum polarization effects in the one-photon interaction. The thick points in the diagrams are the nuclear vertex operators.}
\end{figure}

The two-loop vacuum polarization corrections with the account of the nuclear structure are presented in Fig. 8(a,b,c). The interaction potentials constructed by means of Eqs. (7), (8), (12), (51), are determined by the integral relations:

\begin{equation}
\Delta V_{\text{str}}^{V^P-V^P}(r) = \frac{2}{3} Z \alpha < r_e^2 > \left( \frac{\alpha}{3\pi} \right)^2 \int_{1}^{\infty} \rho(\xi)d\xi \int_{1}^{\infty} \rho(\eta)d\eta \times \left[ \pi \delta(r) - \frac{m_e^2}{r(\xi^2 - \eta^2)} \left( \xi^4 e^{-2m_e \xi r} - \eta^4 e^{-2m_e \eta r} \right) \right].
\end{equation}

\begin{equation}
\Delta V_{\text{str}}^{2\text{-loop } V^P}(r) = \frac{4}{9} Z \alpha < r_e^2 > \left( \frac{\alpha}{\pi} \right)^2 \int_{0}^{1} f(v)dv \left[ \pi \delta(r) - \frac{m_e^2}{r(1 - v^2)} e^{-\frac{2m_e r}{\sqrt{1 - v^2}}} \right].
\end{equation}

The sum of corrections (55) and (56) to the Lamb shift \((2P - 2S)\) is equal:

\begin{equation}
\Delta E_{\text{str}}^{V^P,V^P}(2P - 2S) = -0.005 \text{ meV}.
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig9.png}
\caption{The nuclear structure and two-loop vacuum polarization effects in the second order perturbation theory. The sick points in the diagrams are the nuclear vertex operators.}
\end{figure}
We have included in the Table I the two-loop vacuum polarization and nuclear structure contribution of order $\alpha^2(Z\alpha)^4 \Delta E_{V P, V P}^{str, SOP T}$ in the second order PT, shown in Fig.9(a,b,c,d). In the sixth order over $\alpha$ there exists also the nuclear structure correction coming from the two-photon exchange diagrams with the electron vacuum polarization insertion (see Fig.10). We find it value by means of Eq.(47):

$$\Delta E_{str, V P}^{2\gamma}(nS) = -\frac{2\mu^3(Z\alpha)^5}{\pi^2 n^3} \int_0^\infty k V(k) dk \int_0^1 \frac{v^2(1-v^2)dv}{k^2(1-v^2) + 4m_e^2}.$$  \hfill (59)

$$\Delta E_{str, V P}^{2\gamma}(2P - 2S) = 0.120 \text{ meV}. \hfill (60)$$

![Diagram](image_url)

FIG. 10: The nuclear structure and electron vacuum polarization effects in the two-photon exchange diagrams. The sick points in the diagrams are the nuclear vertex operators.

V. RECOIL CORRECTIONS, MUON SELF-ENERGY AND VACUUM POLARIZATION EFFECTS

The investigation of the different order corrections to the Lamb shift of electronic hydrogen ($2P - 2S$) was performed during many years. Modern analysis of the advances in the solution of this problem is presented in review articles [2, 3]. The most part of the results was obtained in the analytical form, so they can be used directly in the muonic helium ion. In this section we analyse different contributions up to the sixth order over $\alpha$ in the energy spectrum ($\mu 1^2 \text{He}^+$) and derive their numerical values in the Lamb shift ($2P - 2S$).

The recoil correction of order ($Z\alpha$)$^4$ in the Lamb shift appears in the matrix element of the Breit potential with the functions (2) [2, 27]:

$$\Delta E_{rec}^{(Z\alpha)^4}(2P - 2S) = \frac{\mu^3(Z\alpha)^4}{48m_e^2} = 0.074 \text{ meV}. \hfill (61)$$

The recoil correction of the fifth order over ($Z\alpha$) is determined by the relation [2, 27]:

$$\Delta E_{rec}^{(Z\alpha)^5} = \frac{\mu^3(Z\alpha)^5}{m_1m_2\pi^3 n^3} \left[ \frac{2}{3} \delta_{l0} \ln \frac{1}{Z\alpha} - \frac{8}{3} \ln k_0(n, l) - \frac{1}{9} \delta_{l0} - \frac{7}{3} a_n - \frac{2}{m_1^2 - m_2^2} \delta_{l0} \left( m_2^2 \ln \frac{m_1}{\mu} - \frac{m_1}{m_2} \ln \frac{m_2}{\mu} \right) \right], \hfill (62)$$

where $\ln k_0(n, l)$ is the Bethe logarithm:

$$\ln k_0(2S) = 2.811769893120563, \hfill (63)$$
\[ \ln k_0(2P) = -0.030016708630213, \]  
\[ a_n = -2\ln \frac{2}{n} + (1 + \frac{1}{2} + \ldots + \frac{1}{n}) \delta_{l0} + \frac{(1 - \delta_{l0})}{l(l + 1)(2l + 1)}. \]  

It gives the following numerical result:

\[ \Delta E_{\text{rec}}^{(Z\alpha)^5} (2P - 2S) = -0.433 \text{ meV}. \]  

The recoil correction in the sixth order over \((Z\alpha)\) was obtained analytically in Ref.\[28\]:

\[ \Delta E_{\text{rec}}^{(Z\alpha)^6} (2P - 2S) = \frac{(Z\alpha)^6 m_1^2}{8 m_2} \left( \frac{23}{6} - 4 \ln 2 \right) = 0.004 \text{ meV}. \]  

The energy contributions coming from the radiative corrections in the lepton line, the Dirac and Pauli form factors, the muon vacuum polarization have the form \[2, 29\]:

\[ \Delta E_{\text{MVP,MSE}}^{(Z\alpha)^6} (2S) = \frac{\alpha(Z\alpha)^4 \mu^3}{8\pi} \left[ \frac{4}{3} \ln \frac{m_1}{\mu(Z\alpha)^2} - \frac{1}{3} \ln k_0(2S) + \frac{38}{45} \right] + \frac{\alpha}{\pi} \left( -\frac{9}{4} \zeta(3) + \frac{3}{2} \pi^2 \ln 2 - 10 \frac{\pi^2}{27} - 2179 \frac{1}{648} \right) + 4\pi Z_\alpha \left( \frac{427}{384} - \frac{\ln 2}{2} \right) = 10.939 \text{ meV}, \]

\[ \Delta E_{\text{MVP,MSE}}^{(Z\alpha)^6} (2P) = \frac{\alpha(Z\alpha)^4 \mu^3}{8\pi} \left[ -\frac{4}{3} \ln k_0(2P) - \frac{m_1}{6\mu} \right] - \frac{\alpha m_1}{3\pi \mu} \left( \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2 + \frac{\pi^2}{12} + \frac{197}{144} \right) = -0.168 \text{ meV}. \]

Omitting explicit form of the radiative-recoil corrections of orders \(\alpha(Z\alpha)^5\) and \((Z^2\alpha)(Z\alpha)^4\) from the Tables 8-9 \[2\], we present here their numerical value in the Lamb shift \((2P - 2S)\) of muonic helium \((\mu^4 He)^+\):

\[ \Delta E_{\text{rad-rec}} (2P - 2S) = -0.040 \text{ meV}. \]

The nuclear structure corrections of orders \((Z\alpha)^6\) and \(\alpha(Z\alpha)^5\) were studied in Refs.\[30, 31\] for arbitrary hydrogenic atom. Let us present their numerical values in the case of muonic helium:

\[ \Delta E_{\text{str}}^{(Z\alpha)^6} (2S) = -\frac{(Z\alpha)^6}{12} \mu^3 \langle r_\alpha^2 \rangle \left[ \langle \ln m_1 Z\alpha r \rangle + C - \frac{27}{16} \right] = 0.081 \text{ meV}, \]

\[ \Delta E_{\text{str}}^{(Z\alpha)^6} (2P) = \frac{(Z\alpha)^6}{64} \mu^3 \langle r_\alpha^2 \rangle = 0.012 \text{ meV}, \]

\[ \Delta E_{\text{str}}^{(Z\alpha)^5} (2P - 2S) = -0.069 \text{ meV}, \]

\[ \Delta E_{\text{str}}^{(Z\alpha)^5} (2P - 2S) = 0.070 \text{ meV}. \]

The diagram in Fig.11(b) gives the contribution to the energy spectrum, which can be expressed in terms of the slope of the Dirac form factor \(F'_1\) and the Pauli form factor \(F_2\):

\[ \Delta E_{\text{rad+VP}} (nS) = \frac{\mu^3 (Z\alpha)^4}{n^3 m_1^2} \left[ 4m_1^2 F'_1(0) + F_2(0) \right], \]
FIG. 11: Radiative corrections with the vacuum polarization effects.

\[ C_{jl} = \delta_{l0} + (1 - \delta_{l0}) \frac{j(j + 1) - l(l + 1) - \frac{3}{4}}{l(l + 1)}. \]  

(76)

The two-loop contribution to form factors \( F_1'(0) \) and \( F_2(0) \) was calculated in Ref. [32]:

\[ m_1^2 F_1'(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{1}{9} \ln \frac{m_1}{m_e} - \frac{29}{108} \ln \frac{m_1}{m_e} + \frac{1}{9} \zeta(2) + \frac{395}{1296} \right], \]  

(77)

\[ F_2(0) = \left( \frac{\alpha}{\pi} \right)^2 \left[ \frac{1}{3} \ln \frac{m_1}{m_e} - \frac{25}{36} + \frac{\pi^2 m_e}{4 m_1} - \frac{4 m_e^2}{m_1^2} \ln \frac{m_1}{m_e} + \frac{3 m_e^2}{m_1^2} \right]. \]  

(78)

Then the correction to the Lamb shift \((2P - 2S)\) is equal

\[ \Delta E_{\text{rad}+\text{VP}}(2P - 2S) = -0.031 \text{ meV}. \]  

(79)

To estimate the muon self-energy and electron vacuum polarization contribution in Fig.11(a), we use the relation obtained in Ref. [22]:

\[ \Delta E_{\text{MSE}}^{\text{VP}}(2P - 2S) = -0.107 \text{ meV}. \]  

(81)

The hadron vacuum polarization (HVP) contribution can be taken into account on the basis of numerical result obtained for muonic hydrogen in Refs.[33, 34]. The HVP correction and the contribution of nuclear polarizability calculated in Refs.[35, 36] are included in Table I.

VI. SUMMARY AND CONCLUSION

In this work, various corrections of orders \( \alpha^3, \alpha^4, \alpha^5 \) and \( \alpha^6 \) have been calculated for the Lamb shift \((2P - 2S)\) in muonic helium ion \((\mu_2^4\text{He})^+\). Contrary to earlier performed investigations of the energy spectra of light muonic atoms in Refs.[8, 9], we have used the
three-dimensional quasipotential approach for the description of two-particle bound state. Our analysis of the different contributions to the Lamb shift included the terms of two groups. The first group contains the specific corrections for muonic helium, connected with the electron vacuum polarization effects, nuclear structure and recoil effects in the first and second orders PT. The contributions of this group are calculated numerically for the first time. The necessary order corrections of the second group include analytical results known from the corresponding calculation in the electronic hydrogen Lamb shift. Recent advances in the physics of the energy spectra of simple atoms are presented in the review articles [2, 3] which we use in this study. Numerical values of all corrections are written in the Table I, which contains also basic references on the earlier performed investigations (other references can be found in Ref.[2]). Total value 1381.716 meV of the Lamb shift \((2P - 2S)\) in muonic helium ion from the Table I is in good agreement with the results of Refs. [8, 9]. The difference of our results from Refs.[8, 9] is connected both to the calculation of new contributions of higher order and slightly different numerical value of the charge radius of \(\alpha\)-particle \(r_{\alpha}\) used in this work. The authors of Refs.[8, 9] used the value of charge radius \(r_{\alpha} = 1.674(12)\) fm.

Let us summarize the basic particularities of the calculation performed above.

1. Numerical value of the parameter \(m_e/\mu Z \alpha = 0.34\) in muonic helium \((\mu^4He)^+\) is sufficiently large, so the electron vacuum polarization effects play essential role in the interaction operator. We have considered the one-loop, two-loop, three-loop VP contributions.

2. The nuclear structure effects are expressed in the Lamb shift of muonic helium both in terms of the charge radius of \(\alpha\) - particle in the leading and next to leading orders and by means of the charge form factor of the alpha particle in the two-photon exchange amplitudes.

3. The estimation of the alpha particle polarizability contribution to the Lamb shift is used from Refs.[35, 36]. The nuclear structure and polarizability give the largest theoretical uncertainty in the total value of the Lamb shift \((2P - 2S)\). For instance, in the leading order \((Z\alpha)^4\) the theoretical error, connected with the uncertainty in the value of the alpha particle charge radius \(r_{\alpha} = 1.676(8)\) fm comprises \(\pm 2.8\) meV. The total result of this work for the \((2P - 2S)\) Lamb shift can be used for the comparison with the future experimental data and determination more precise value of the alpha particle charge radius.

Acknowledgments

The author is grateful to R.N.Faustov for useful discussions. This work was supported by the Russian Foundation for Basic Research (grant No. 06-02-16821).

TABLE I: Lamb shift \((2P_{1/2} - 2S_{1/2})\) in muonic helium ion \((\mu^4He)^+\).

<table>
<thead>
<tr>
<th>Contribution to the splitting</th>
<th>(\Delta E(2P - 2S)), meV</th>
<th>Formula, Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1VP contribution of order (\alpha(Z\alpha)^2) in one-photon interaction</td>
<td>1665.782</td>
<td>(6)</td>
</tr>
<tr>
<td>Two-loop VP contribution of order (\alpha^2(Z\alpha)^2)</td>
<td>13.481</td>
<td>(9), (14)</td>
</tr>
<tr>
<td>VP and MVP contribution in one-photon interaction</td>
<td>0.002</td>
<td>(11)</td>
</tr>
<tr>
<td>Three-loop VP contribution in one-photon interaction</td>
<td>0.110</td>
<td>(17), (18), [18]</td>
</tr>
<tr>
<td>The Wichmann-Kroll correction</td>
<td>0.135</td>
<td>(20)</td>
</tr>
<tr>
<td>Relativistic and VP corrections of order (\alpha(Z\alpha)^4) in the first order PT</td>
<td>-0.948</td>
<td>(26)-(29)</td>
</tr>
<tr>
<td>Relativistic and two-loop VP corrections of order (\alpha^2(Z\alpha)^4) in the first order PT</td>
<td>-0.003</td>
<td>(31)</td>
</tr>
<tr>
<td>Two-loop VP contribution of order (\alpha^2(Z\alpha)^2) in the second order PT</td>
<td>1.707</td>
<td>(38)-(39)</td>
</tr>
<tr>
<td>Relativistic and one-loop VP corrections of order (\alpha(Z\alpha)^4) in the second order PT</td>
<td>0.746</td>
<td>(41)</td>
</tr>
<tr>
<td>Three-loop VP contribution in the second order PT of order (\alpha^3(Z\alpha)^2)</td>
<td>0.028</td>
<td>(42)-(43)</td>
</tr>
<tr>
<td>Relativistic and two-loop VP corrections of order (\alpha^2(Z\alpha)^4) in the second order PT</td>
<td>0.002</td>
<td>(44)-(45)</td>
</tr>
<tr>
<td>Nuclear structure contribution of order ((Z\alpha)^4)</td>
<td>-295.848</td>
<td>(46), [2]</td>
</tr>
<tr>
<td>Nuclear structure contribution of order ((Z\alpha)^5) from 2(\gamma) amplitudes</td>
<td>7.196</td>
<td>(50)</td>
</tr>
<tr>
<td>Nuclear structure and VP contribution in 1(\gamma) interaction of order (\alpha(Z\alpha)^4)</td>
<td>-0.960</td>
<td>(54)</td>
</tr>
<tr>
<td>Nuclear structure and VP contribution in the second order PT of order (\alpha(Z\alpha)^4)</td>
<td>-1.506</td>
<td>(55)</td>
</tr>
<tr>
<td>Nuclear structure and two-loop VP contribution in 1(\gamma) interaction of order (\alpha^2(Z\alpha)^4)</td>
<td>-0.005</td>
<td>(58)</td>
</tr>
<tr>
<td>Nuclear structure and two-loop VP contribution in the second order PT of order (\alpha^2(Z\alpha)^4)</td>
<td>-0.007</td>
<td>Fig.9</td>
</tr>
<tr>
<td>Nuclear structure contribution of order (\alpha(Z\alpha)^5) from 2(\gamma) amplitudes with VP insertion</td>
<td>0.120</td>
<td>(60)</td>
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Table I (continued).

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recoil correction of order $(Z\alpha)^4$</td>
<td>0.074</td>
<td>(61),[2, 27]</td>
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<tr>
<td>Recoil correction of order $(Z\alpha)^5$</td>
<td>-0.433</td>
<td>(66),[2, 27]</td>
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<tr>
<td>Recoil correction of order $(Z\alpha)^6$</td>
<td>0.004</td>
<td>(67),[2]</td>
</tr>
<tr>
<td>Muon self-energy and MVP contribution</td>
<td>-11.107</td>
<td>(68)-(69),[2]</td>
</tr>
<tr>
<td>Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$</td>
<td>-0.040</td>
<td>(70), Tables 8-9 [2]</td>
</tr>
<tr>
<td>Nuclear structure corrections of orders $(Z\alpha)^6$, $\alpha(Z\alpha)^5$</td>
<td>0.001</td>
<td>(73),(74),[2, 30, 31]</td>
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<td>Muon form factor $F_1'(0)$, $F_2(0)$ contributions</td>
<td>-0.031</td>
<td>(79),[2, 22, 32]</td>
</tr>
<tr>
<td>Muon self-energy and VP contribution</td>
<td>-0.107</td>
<td>(81),[2, 22]</td>
</tr>
<tr>
<td>HVP contribution</td>
<td>0.223</td>
<td>[33, 34]</td>
</tr>
<tr>
<td>Nuclear polarizability contribution</td>
<td>3.100</td>
<td>[35, 36]</td>
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<tr>
<td>Total contribution</td>
<td>1381.716</td>
<td></td>
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</table>