A Bootstrapping Approach for Generating Maximally Path-Entangled Photon States

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We propose a bootstrapping approach to generation of maximally path-entangled states of photons, so called “NOON states”, achievable within the current experimental technology. Strong atom-light interaction of cavity QED can be employed to generate NOON states with about 100 photons; which can then be used to boost the existing experimental Kerr nonlinearities based on quantum coherence effects to facilitate NOON generation with arbitrarily large number of photons. We also offer an alternative scheme that uses an atom-cavity dispersive interaction to obtain sufficiently high Kerr-nonlinearity necessary for arbitrary NOON generation.

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NOON states, path-entangled states of N photons of the form \(|N\_0:0N\rangle\equiv(\langle N\rangle\_a|0\rangle\_b + |0\rangle\_a|N\rangle\_b)/\sqrt{2}\), are important for quantum lithography [1] and Heisenberg limited interferometry with photons [2]. Several theoretical proposals exist for NOON-state generation; nevertheless, experimentally it seems to be a formidable task since, so far, NOON states with only three and four photons have been generated [3, 4]. There exists a proposal that simulates a six-photons NOON state by post-selection [5] and hence may not be directly useful for the quantum lithography application. In this context it is imperative to develop practical strategies for generation of high-NOON states. In this letter, we propose a number of routes, achievable within the current experimental technology, to increase the maximum number of photons entangled in the NOON form.

To recollect, existing proposals for generation of maximally path-entangled states of photons use either the Linear Optical Quantum Computing (LOQC) approach or some kind of optical nonlinearity. We note that LOQC approaches would be unsuitable in this quest as the larger the required number of particles in the entangled state the lower is the success probability of the proposal [3]. Thus, the routes using optical nonlinearities seem to be promising in the longer run. Nevertheless, experimental nonlinearities are not as strong as one would require them for the present task, even in the case where quantum coherence effects such as Electromagnetically Induced Transparency, are employed as discussed later. Here we propose a bootstrapping approach such that existing experimental nonlinearities could be boosted in order to eventually have a large number of particles in the generated NOON state.

The bootstrapping technique proposed here involves preparation of a NOON state with a small number of photons (up to 100), which can be used to boost the conditional nonlinearities necessary to obtain NOON states with an arbitrarily large number of photons \(N\) provided, the \(N\)-photon Fock states are available. The approach is primarily based on the scheme described in Fig. 1(a) proposed by Gerry and Campos [2]. The scheme involves two Mach-Zehnder interferometers (MZI) coupled via a cross-Kerr nonlinearity. Presence of a single photon in mode \(c\) is required to give phase shift of \(\pi\) to photons in mode \(b\); this phase shift is however not within the reach of current experimental cross-Kerr nonlinearities, to obtain NOON states in the modes \(a\) and \(b\) at the output. The best current experimental cross-Kerr phase shifts are of the order of 0.1 radians, obtained via atom-light interaction in systems using quantum coherence effects [8, 9]. This suggests that further enhancement in the nonlinearity—by roughly two orders of magnitude—is necessary for NOON state generation. It is, however, important to note that the scheme requires a conditional phase shift of either 0 or \(\pi\), which would require a NOON state (of about hundred photons) as opposed to a simple Fock state of \(N\) photons entering at the input of MZI-2. The necessary setup is shown in Fig. 1(b). The presence of a large number of photons would boost the quantum-coherence-based cross-Kerr nonlinearity; enough to obtain a phase shift of 0 or \(\pi\) if all the \(N\) photons occupy mode \(d\) or \(c\), all at once. Once the enhanced nonlinearity is used, measurement of the number of photons \(n_{d1}\) and \(n_{d2}\) would give the output state \([(−1)^{n_{d2}}|N\_0\rangle_b + i(−1)^{n_{d1}}|0\_aN\_b\rangle\_b)/\sqrt{2}\] which can be trivially corrected for the relative phase of the two components to obtain a NOON state.

As pointed out earlier, the foremost step is to generate a NOON state with about hundred photons. We now discuss our proposal for this low-NOON generation via a new type of photon gun. The low-NOON gun, as we call it, is based on strong atom-light interactions offered by cavity QED and collective enhancement of the interaction strength. The system required to obtain the low-NOON state of photons is an ensemble of cold alkali atoms—roughly a few hundred of them trapped inside an optical cavity—such that the spatial extent of the cloud is much smaller than the wavelength of the light interacting with it. The schematic of the device and the detailed level structure of the atoms being targeted is shown in Fig. 2. The operational steps required are described below. Step 1: single-step generation of the GHZ state of atoms in two of its internal state via the Hamiltonian given in Eq. (1).

\[
H_{\text{GHZ}} = \hbar\eta \sum_j \hat{S}_j^+ \hat{S}_j^- = \hbar\eta \left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - \hat{S}_z^2 + \hat{S}_z \right], \tag{1}
\]

where \(\hat{S}_j^+ = |a\rangle\langle b|\), \(\hat{S}_j^- = |a\rangle\langle b|\), \(\hat{S}_z = |a\rangle\langle a| - |b\rangle\langle b|\) and \(\eta\) is the Raman Rabi frequency, signifying the coupling between the two states \(|a\rangle\) and \(|b\rangle\), achieved through a quantum field of a cavity and a time-varying classical field. The approach is
well studied \cite{10} and can be used, with an initial state of all the atoms to be the superposition $|\{a, b\}\rangle\sqrt{2}$, to generate GHZ states in the basis given by $|\{+\rangle = (|a\rangle + |b\rangle)/\sqrt{2}, |\{-\rangle = (|a\rangle - |b\rangle)/\sqrt{2}|$ after the atom-field evolution for time $t$ such that $\tau t = \pi$. The basis rotation can be readily performed by the application of Raman pulse (of area $\pi/2$) coupling the two states to form the GHZ state, in the familiar basis $|\{a\rangle, |b\rangle\}, |\{aaa\ldots\rangle + |b\{b\ldots\rangle\rangle = \sqrt{2}$. Direct implementation in Bose-Einstein condensates (BEC) may also be possible by using the scheme of \cite{11}, which employs the interparticle interaction for generation of arbitrary Dicke States within a BEC. Step 2: Entanglement Transfer. Once a GHZ state of the atoms is prepared, a coupled STIRAP process (See Fig. \ref{fig:2}) can be achieved by controlling the time-variation of the pump field Rabi frequency $\Omega_p(t)$ to generate an entangled state of cavity photons: $|N0::0N\rangle$ that is entangled in the two counter-rotating polarization modes. This polarization mode entanglement can be readily converted into path entanglement with the help of simple optical elements as shown in detail in Fig.\ref{fig:2}

![FIG. 1: Nonlinear Process to convert the generated small NOON state to high NOON state via Phaseonium based cross-Kerr nonlinearity. (a) The scheme of \cite{7} (b) The Bootstrapping procedure. Notice that the difference in (a) and (b) is in the state of the control.](image)

The Step 2, described above, requires a coupled STIRAP operation that allows transfer of entanglement from the GHZ state of the atoms into the photons entangled in polarization-modes. This is the most important result of this letter, which offers a device that we call a NOON gun. In the following, we discuss the physics of this NOON gun in sufficient detail.

The initial state of the atomic cloud is the GHZ state with the component states $|a\rangle$ and $|b\rangle$ being the hyperfine sublevels $m_F = -1$ and $m_F = 1$ of the $F = 1$ hyperfine manifold of an alkali atom respectively. This atomic cloud is then trapped in a cavity such that the cloud size is much smaller than the wavelength of the fundamental mode of the cavity. Also the external pumping field is assumed to couple to all the atoms identically. These restrictions could be easily obtained within the current experimental parameters of the optical cavity, when the number of atoms is confined to about hundred. The interaction of the atomic cloud with the two polarization modes of the cavity and the $\pi$-polarized pump field can be described by the Hamiltonian

$$H = \hbar \sum_i \Omega_p (t) \left( |a\rangle \langle a|_i + |b\rangle \langle b|_i + |a\rangle \langle b|_i + |b\rangle \langle a|_i \right) + g_L (\hat{c}_L |a\rangle \langle g|_i + \hat{c}_R |b\rangle \langle g|_i + \hat{c}_R^\dagger |g\rangle \langle a|_i \right) + g_R (\hat{c}_R |b\rangle \langle g|_i + \hat{c}_R^\dagger |g\rangle \langle b|_i \right) \right) \right)$$

$$H = \hbar \left[ \Omega_p(t) \frac{d_g^\dagger d_a + d_a^\dagger d_g + d_b^\dagger d_b + d_b^\dagger d_b}{2} + g_L (\hat{c}_L d_a^\dagger d_a + \hat{c}_R^\dagger d_b + \hat{c}_R^\dagger d_a^\dagger d_a) + g_R (\hat{c}_R d_g^\dagger d_a + \hat{c}_R d_g^\dagger d_b) \right]$$

where $\hat{c}_{LR}$ and $\hat{c}_{LR}^\dagger$ are the cavity mode photon annihilation and creation operators for the left (L) and right (R) circular modes of polarization and $i$ is the label for the atoms. We arrived at the second line (Eq. (2)) by introducing number representation for the collective atomic states and the corresponding operators $\hat{d}$ and $\hat{d}^\dagger$ labeled by the appropriate atomic level.

It can be readily realized that this system contains two coupled $\Lambda$ systems. It is important to identify quantities conserved under the action of the Hamiltonian in order to understand the form of eigenstates of the system. The conserved quantities are the total number of atoms in the five states, $N = \sum_i d^\dagger_i d_i$ with $i \in \{a, a', b, b', g\}$, and the quantity $M = d_g^\dagger d_a - \hat{c}_L^\dagger \hat{c}_L - \hat{c}_R^\dagger \hat{c}_R$. It should also be noted that the initial state in the given manifold, labeled by $N$ and $M$, shall always remain in that manifold under the action of the Hamiltonian. Moreover, each manifold contains a dark state, which does not contain any atoms in the excited states $|a\rangle$ and $|b\rangle$. We do not give the complete form of the general dark state; however, we note an important observation that is useful for the problem at hand. If the initial state of the cavity fields is chosen such that $\hat{c}_L^\dagger \hat{c}_L = \hat{c}_R^\dagger \hat{c}_R = 0$ and the atomic state is such that all the atoms are in the $\Lambda$-type sys-
tem formed by either \(|\psi\rangle - |\psi'\rangle - |\phi\rangle - |\phi'\rangle\) or \(|\psi\rangle - |\psi'\rangle - |\phi\rangle\) (e.g. the GHZ state) then the interaction Hamiltonian reduces to either
\[
H_{\text{eff}}^{(1)} = \hbar \left[ \Omega_p(t) (d^*_a d_b + d^*_b d_a) + g_R (\hat{c}_R d_a^* d_b + \hat{c}_R^* d_b^* d_a) \right]
\]
or
\[
H_{\text{eff}}^{(2)} = \hbar \left[ \Omega_p(t) (d^*_a d_b + d^*_b d_a) + g_L (\hat{c}_L d_a^* d_b + \hat{c}_L^* d_b^* d_a) \right]
\]
as the effective Hamiltonian governing the system dynamics. This gives two different manifolds of states with the conserved quantities \(N_a = \sum_i \hat{d}_i^* \hat{d}_i\) and \(M^{(1)} = \hat{d}_a^* \hat{d}_b - \hat{c}_L^* \hat{c}_R\) or \(M^{(2)} = \hat{d}_a^* \hat{d}_b - \hat{c}_L^* \hat{c}_R\). It is clear that these manifolds to not couple to each other and the corresponding dark states are given by
\[
|\Psi^{(1)}\rangle = \frac{1}{D} \sum_{j=0}^{N} \left( \frac{\Omega_p(t)/g_L}{\sqrt{(N-j)!j!}} \right) |(N-j)\rangle_a, |0\rangle_b, |j_L\rangle_b, |0_R\rangle_b
\]
\[
|\Psi^{(2)}\rangle = \frac{1}{D'} \sum_{k=0}^{N} \left( \frac{\Omega_p(t)/g_R}{\sqrt{(N-k)!k!}} \right) |0_a, (N-k)\rangle_b, |k_L\rangle_b, |0_R\rangle_b
\]
where \(D\) and \(D'\) are the appropriate normalization constants. These specialized dark states of a three-level \(\Lambda\)-type system for a collective atomic ensemble simultaneously coupled to a quantized and classical field are well studied in Refs. \[12\],[13].

The notation for the complete atom-cavity field states is self-explanatory, where the excited levels \(|\phi\rangle\) and \(|\phi'\rangle\) are not shown as they are not occupied. It is imperative to point out at this place is that the dark states \(|\Psi^{(1)}\rangle\) and \(|\Psi^{(2)}\rangle\) are dynamically decoupled from each other in the sense that if the initial state contains a certain proportion of both states, that proportion shall be left unchanged by the evolution. The evolution or relative dominance of the components of the dark states can be controlled by changing the time evolution of the pump field via the Rabi frequency \(\Omega_p(t)\). Thus, the adiabatic transformations
\[
|N_a, 0_b, 0_g, 0_L, 0_R\rangle \rightarrow |0_a, 0_b, N_g, N_L, 0_R\rangle
\]
and
\[
|0_a, N_b, 0_g, 0_L, 0_R\rangle \rightarrow |0_a, 0_b, N_g, 0_L, N_R\rangle
\]
can be obtained deterministically. The result is such that the two components of the initially prepared atom-cavity state,
\[
|N_a, 0_b, 0_g, 0_L, 0_R\rangle + |0_a, N_b, 0_g, 0_L, 0_R\rangle /\sqrt{2}
\]
evolve independently into the state
\[
|0_a, 0_b, N_g, N_L, 0_R\rangle + |0_a, 0_b, N_g, 0_L, N_R\rangle /\sqrt{2}
\]
just by adiabatic increase of the pump-field intensity such that \(\Omega_p(t) \gg g_L, g_R\) in the long time limit like in the STIRAP processes. The final state of all the atoms is \(|\psi\rangle\) and the field state can be written in an abbreviated manner as
\[
|N_c, 0_r\rangle \equiv |N_c, 0_r\rangle /\sqrt{2}
\]
This polarization entangled state of photons can be readily converted into path entangled NOON state in a chosen polarization mode by outcoupling it and passing through simple optical elements as depicted in Fig.2. The polarization NOON state itself can be used for Heisenberg-limited measurement of polarization angle shifts such as is exploited in magnetometry \[14\].

Physically, the above-mentioned NOON gun is similar in operation to the experimentally demonstrated deterministic single photon source \[15\], and a recent theoretical proposal for Fock-state generation of Ref. \[13\]. In fact, the NOON gun proposed here could be thought of as two coupled Fock-State generators, which when fed with atoms in GHZ state facilitate generation of a NOON state. Essentially, the device performs complete entanglement transfer from the atoms to the photons leaving all the atoms in an identical final state.

This completes the discussion of the Bootstrapping procedure for high-NOON generation. Now we briefly discuss a different implementation of Kerr nonlinearity based on the atom-cavity dispersive interaction \[16\] that can be used in place of the cross-Kerr nonlinearities obtained via quantum coherence effects. The scheme is depicted in Fig.3 where a cavity is introduced in the path of the optical mode \(b\) after the beam splitter BS1. In comparison with Fig.1(a), the new scheme uses a Ramsey Interferometer in place of the MZI-2 as seen in Fig.3. The mathematical transformation of the Quantum Fredkin Gate is given by
\[
\hat{U}_{\text{QFG}} = \exp(i \chi \hat{c}_L^* \hat{c}_R) \exp(i \chi \hat{c}_L \hat{c}_R) \exp(i \chi \hat{c}_R^* \hat{c}_R) \exp(i \chi \hat{c}_L \hat{c}_R^*),
\]
where \(\hat{f}_0 = (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b})/2\) and \(\hat{f}_2 = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2i\) are the Schwinger angular momentum representations of the photonic operators and \(\chi = \pi\) is the cross-Kerr nonlinearity required for
To summarize, we have devised a bootstrapping approach to NOON-state generation for photons based on Quantum Fredkin gate via atom coherence effect based cross-Kerr nonlinearities. In the process we have proposed a device that can produce NOON states of up to one hundred photons on demand in a deterministic manner. Furthermore, we have devised a scheme for NOON-state generation based on Ramsey interferometry and a cavity-enhanced Kerr nonlinearity. It is assumed that the Fock states of arbitrarily large number of photons are available as inputs; which could be generated by applying the proposal of Ref. [13] when applied to cold Rydberg atoms trapped in microwave cavities. Our strong hope is that the ideas presented here shall simulate a growth of experimental activity in generation of entangled photon states with larger and larger number of photons.

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FIG. 3: Quantum Fredkin gate based on nonlinearity obtainable via cavity QED and Ramsey Interferometry