Social Dilemmas and Cooperation in Complex Networks

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Abstract
In this paper we extend the investigation of cooperation in some classical evolutionary games on populations where the network of interactions among individuals is of the scale-free type. We show that the update rule, the payoff computation and, to some extent the timing of the operations, have a marked influence on the transient dynamics and on the amount of cooperation that can be established at equilibrium. We also study the dynamical behavior of the populations and their evolutionary stability.

1 Introduction and Previous Work

The object of game theory is the analysis of situations where the different social actors have conflicting requirements and individual decisions will have a mutual influence on each other[1]. In this framework, and due to their importance as simplified models of many common important socio-economic situations, the Prisoner’s Dilemma (PD) and the Snowdrift (SD) games have received much attention in the literature. According to game theory, the PD and the SD are paradigmatic examples of games in which cooperative attitude should vanish in the PD, and should be limited to a given fraction in the SD. This is also the case when large populations of individuals play the game pairwise in a random manner and anonymously, as prescribed by evolutionary game theory [2]. In spite of this, numerical simulations of the PD have convincingly shown that, when the population of players possesses a spatial structure, a certain amount of cooperation can emerge and remain stable. Nowak and May [3] were the first to empirically show this using a population structured as a square lattice where each site is a player. Standard evolutionary game theory is based on an infinite (or very large) population model, and on the random pairing of two players at each time step. This amounts to a mean-field spatially homogeneous model. The square grid is also spatially homogeneous but the absence of random mixing enables the formation of clusters of cooperators, which allows for more frequent positive encounters between cooperators than would be possible in the mean-field case. More recently, it has become apparent that many real networks are neither regular nor random graphs; instead, they have short diameters, like random graphs, but much higher clustering coefficients than the latter, i.e. they have more local
structure. These networks are collectively called small-world networks (see \cite{4} for a recent review). Many technological, social, and biological networks are now known to be of this kind. Social networks, in addition, show recognizable community structure \cite{5,6}. Since evolutionary PD or SD games are metaphors for conflicting social interactions, the research attention has recently shifted from random graphs and regular lattices towards better models of social interaction structures \cite{7,8,9,10}.

Recently, Santos and Pacheco \cite{9} presented a numerical study of the evolution of cooperation on (static) scale-free (SF) networks for the PD and the SD games. Their main result was that, in contrast with what one observes in mixing populations or on regular lattices, much higher levels of cooperation are sustainable on this kind of graphs, both for the PD as well as the SD. These results are obviously interesting and encouraging for cooperation but they prompt a number of questions. First of all, Barabási–Albert or correlationless configuration SF graphs \cite{11} that were used in \cite{9} are not faithful representations of most typical social networks. In fact, although social interaction networks where the degree distribution can be well described by a power-law have been found \cite{12,13}, several recent studies show that social networks in general do not have a pure power-law degree distribution function, as they often show signs of exponential decay of the tail of the distribution \cite{14,15}. In addition, they usually have more clustering than pure scale-free graphs \cite{4}. Nevertheless, model SF networks are a useful bounding case to study as they are closer to typical social networks than other more artificial kind of graphs, such as Watts–Strogatz small worlds \cite{16}. A second aspect of social networks that is not captured by fixed graph structures is that they are not static; rather, the number of vertices and the links between them continuously evolve as social actors come and go, and relationships are created or abandoned. Dynamical features such as these have been introduced in evolutionary games, among others, in \cite{17,18,19,20,21}. However, in this paper we only focus on the static aspects of the interaction networks. In other words, we make the hypothesis that the network is at equilibrium and that network dynamics are either absent, or their time scale is longer (slower) with respect to the strategy-change dynamics. This proves to be a useful approach, especially for social acquaintance networks.

In the following we present a brief introduction to the games studied. This is followed by a discussion of the population model and of individual’s payoff calculation scheme for the players in a complex network. Next we describe the numerical simulations and their results, including a study of evolutionary stability. We finally present our conclusions.

2 Two Social Dilemmas

Let us first recall a few elementary notions on the PD and the SD. These are two-person, symmetric games in which each player has two possible strategies: cooperate (C) or defect (D). In strategic form, also known as normal form, these games have the payoff bi-matrix of table \cite{1}. In this matrix, R stands for the reward the two players receive if they both cooperate, P is the punishment for bilateral defection, and T is the temptation, i.e. the payoff that a player receives if it defects, while the other cooperates. In this latter case, the cooperator gets the sucker’s payoff S. For the PD, the payoff values are
ordered numerically in the following way: $T > R > P > S$, while in the SD game $T > R > S > P$. Defection is always the best rational individual choice in the PD – (D,D) is the unique Nash equilibrium and also an evolutionary stable strategy (ESS). Mutual cooperation would be preferable but it is a strongly dominated strategy. Thus the dilemma is caused by the “selfishness” of the actors.

In the SD, when both players defect they each get the lowest payoff; (C,D) and (D,C) are Nash equilibria of the game in pure strategies, and there is a third equilibrium in mixed strategies where strategy D is played with probability $\frac{1}{2\beta - 1}$, and strategy C with probability $1 - \frac{1}{2\beta - 1}$, where $\beta$ is another name for the temptation $T$, used in biological circles. The dilemma in this game is caused by “greed”, i.e. players have a strong incentive to “bully” their opponent by playing D, which is harmful for both parties if the outcome produced is (D,D).

3 Numerical Simulations

The two games were simulated in [9] on Barabási-Albert (BA) [11] and configuration model [4] scale-free networks of size $10^4$ over $10^4$ time steps, using a discrete analogue of replicator dynamics equations [2, 22]. The customary rescaling of the payoff values was used such that there is only one independent parameter. For the PD, setting $R = 1$, $P = S = 0$, leaves $T = b > 1$ to be the only parameter (temptation). For the SD, $T$ is set equal to $\beta > 1$, $R = \beta - 1/2$, $S = \beta - 1$, and $P = 0$, which makes the cost-to-benefit ratio of mutual cooperation $r = 1/(2\beta - 1)$ the only parameter. For the sake of comparison, our simulations were done under the same conditions as in [9] ($10^4$ players and $10^4$ time steps).

However, replicator dynamics is not the only possibility for updating the agents’ strategies in discrete, finite populations of players using hard-wired strategies. Moreover, in small non degree-homogeneous populations, the mathematical requirements behind the replicator dynamics, strictly speaking, are not satisfied [23]. Thus, we extended the investigation by simulating an imitate the best evolution rule according to which an individual $i$ will adopt the strategy of the player with the highest payoff among its neighbors and itself. If a tie occurs, the winner is chosen uniformly at random between the best. This rule is deterministic and was the original rule used in [3].

Concerning the calculation of an individual’s payoff, there are several possibilities. A player’s payoff may be defined as the sum (accumulated payoff) of all pair interactions with its nearest neighbors, which is the form used for instance in [9]. Another possibility consists in using average payoff, which is the accumulated payoff divided by the number of interactions. Accumulated and average payoff give the same results when considering degree-homogenous networks such as lattices. Accumulated payoff

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Table 1: Payoff matrix for a standard two-person, two-strategies game (see text).
seems more logical in degree-heterogeneous networks such as scale-free graphs since it reflects the very fact that players may have different numbers of neighbors in the network. Average payoff, on the other hand, smooths out the possible differences although it might be justified in terms of number of interactions that a player may sustain in a given time. For instance, an individual with many connections is likely to interact less often with each of its neighbors than another that has a lower number of connections. Also, if there is a cost to maintain a relationship, average payoff will roughly capture this fact, while it will be hidden if one uses accumulated payoff. For the sake of comparing the two extreme views, here we use both accumulated and average payoff.

Under discrete replicator dynamics rule with accumulated payoff, and using synchronous update, Santos and Pacheco [9] found that, when compared to regular lattices, SF networks lead to high levels of cooperation for all values of the parameters $b$ (for PD) and $r$ (for SD). These results have been reproduced by us and are shown in the upper half of figure 1. Cooperation is also much higher in SF graphs than what has been obtained for Watts–Strogatz small-world graphs [7, 10]. When using the “imitation of the best” strategy-switching rule with synchronous update and accumulated payoff the results are similar, as one can see in the lower part of figure 1 although there is a marked fall in the high-$b$ and high-$r$ region with respect to replicator dynamics. However, when one lingers on the standard deviations (represented as error bars in the figure), one sees that the results for the imitate the best rule are noisy, with quite large fluctuations. Deviations are smaller for the replicator dynamics, see figure 1. The reason for the instability and the large fluctuations can be traced to the step function nature of the update rule, as can be seen in figure 3(a), in which 40 individual PD runs are plotted, all with $b = 1.8$. In all runs cooperation falls at the beginning, the cooperators then often recover but not always, as there are several runs (about 1/5 for the data used here) in which cooperation never recovers. On the other hand, when using replicator dynamics, there is still a systematic drop of cooperation at the beginning (figure 3(c)), nevertheless it tends to rise again in the long run, although this may happen very late in the simulation (see figure 3(b)). To better observe this phenomenon, we have doubled the number of time steps ($2 \times 10^4$).

We thus see that the results on BA SF graphs depend on the update rule, although the level of cooperation is still higher than what is found on regular, Watts–Strogatz, and random graphs [22, 10]. However, we wish to point out that if we use an asynchronous update policy with the “imitate the best” rule instead of the usual synchronous one, the result is a higher level of cooperation with far less fluctuations than the synchronous case (compare lower parts of figures 1 and 2). One might reason that the combination of synchronous update and of “imitate the best” is fully deterministic, which implies that particular chains of events, such as cascades of defection, will be amplified. Introducing stochasticity through asynchrony in the update sequence strongly mitigates the likelihood of such series of events. On the other hand, when using replicator dynamics, the lack of stochasticity in synchronous update is somehow compensated for by the probabilistic strategy change rule, which could explain the similarity of the results in this latter case (compare the upper parts of figures 1 and

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1We use the standard uniform random choice (with replacement) of players in the population, which is a discrete approximation of a Poisson process.
To illustrate the influence of timing when “imitate the best” is the rule used for strategy update, suppose that a defector occupies the most highly connected node in the graph and that it is surrounded by cooperators exclusively. Then, at the next time step in synchronous update, all those cooperators will turn into defectors. From there, a wave of defection could quickly propagate through the network, leading to a state whereby cooperation cannot be recovered. On the other hand, when players are updated in random order, only a fraction of the neighbors will imitate the defector, at the same time lowering the payoff of the central defector, and thus making it less attractive to be imitated in future encounters. This kind of process limits the propagation of defection and allows cooperation to establish itself and be stable. This highlights some shortcomings of synchronous dynamics, which is unrealistic and may give rise to spurious effects [24]. Our conclusion is that, although there is often no significant difference between synchronous and asynchronous update in evolutionary games, as it is the case here under replicator dynamics, the latter is to be preferred for reasons of generality and reliability. However, for the sake of comparison with previous results, in the rest of the paper we use synchronous update.

Now we turn our attention to the assumption that a player’s utility is the sum, i.e. the accumulated payoff of all pair interactions with its nearest neighbors. Although this appears to be a logical step to follow, we shall show that it may cause
Figure 2: Fraction of cooperators on SF BA networks of size $10^4$ and average degree $\bar{k} = 4$ with accumulated payoff and asynchronous dynamics. Mean values over 50 runs. Upper figures: replicator dynamics; lower figures: imitation of the best.

both conceptual and technical problems. Obviously, one would assume that if an individual has more links to cooperators, and that the payoffs are positive quantities, she should earn more than another player with fewer cooperating neighbors. However, this begs the question of how the network got there in the first place. BA SF graphs are incrementally built by using linear preferential attachment [11]. In this model there is no cost associated to the formation of a new link. However, although this model may be adequate for citation networks or, to some extent, the Web, it is well known that this cannot be the case in most other instances. Thus, other models have been proposed that take into account cost and other factors in network formation [4]. In our case, it is as if the population would be “injected” on an already full-grown, topology-favorable network, while the rules of the game and other considerations necessarily should play a role in the network formation and dynamics. The same remarks also hold for the “configuration” SF graphs, although these networks are built starting from the degree distribution and a fixed number of nodes, rather than incrementally. Furthermore, a technical problem arises when combining replicator dynamics with accumulated payoff. In infinite mixing populations, classical evolutionary game theory states that replicator dynamics is invariant under positive affine transformations of payoffs with merely a possible change of time scale [2]. This invariance still holds in finite degree-homogenous populations. However, when different individuals start having different degrees, things are not quite the same. Let $\Pi_i$ denote
Figure 3: PD time series with $b = 1.8$; (a) imitation of the best; (b) replicator dynamics; (c) replicator dynamics (first 70 steps).

a player $i$’s aggregated payoff. Furthermore, let $\phi(\Pi_j - \Pi_i) = (\Pi_j - \Pi_i)/(Dk_>)$ be the probability function according to which $i$ adopts neighbor $j$’s strategy, with $D = max\{T, R, P, S\} - min\{T, R, P, S\}$ and $k_> = max\{k_i, k_j\}$, where $k_x$ represents the degree of player $x$ [9]. If we now apply a positive affine transformation of the payoff matrix, this leads to the new aggregated payoff $\Pi'_i = \alpha\Pi_i + \beta k_i$ and hence $\phi(\Pi'_j - \Pi'_i) = (\alpha\Pi_j + \beta k_j - \alpha\Pi_i - \beta k_i)/(\alpha Dk_>) = \phi(\Pi_j - \Pi_i) + (k_j - k_i)/(\alpha Dk_>)$. One can clearly see that using accumulated payoff does not lead to an invariance of the replicator dynamics under shifts of the payoff matrix. As an illustration of the violation of this invariance, figure 4 shows cooperation curves for the PD when applying such payoff transformations.

This has several implications such as limiting the results obtained in [9] strictly to the studied values of $b$ and $r$, and to an impossibility to rescale the payoff matrix. In a more recent study [25] Santos et al. investigated the same games in a wider parameter space, but still using accumulated payoff, which again makes the results non-invariant with respect to a positive affine transformation. Therefore, we repeated the numerical simulations with average payoff, i.e. the aggregated payoff obtained by one player divided by the number of links the player has to nearest neighbors, which, along with the shortcomings described above, has the advantage of leaving the replicator dynamics
Figure 4: Fraction of cooperation for the PD game using replicator dynamics and accumulated payoff. A translation of the payoff matrix can produce a fall in cooperation (shift of $-1$) as well as unpredictable behaviors (shift of $+5$) with some runs containing high levels of cooperation and others ending up with massive defection. Standard deviations are not plotted here to improve readability.

Figure 5: Fraction of cooperators on SF BA networks of size $10^4$ with average degree $\bar{k} = 4$ using average payoff and synchronous dynamics. Mean values over 50 runs. Upper figures: replicator dynamics; lower figures: imitation of the best.
invariant under positive affine transformations.

In figure 5 we report results for the PD and SD games using average payoff with synchronous updating dynamics, and the same parameter set as in [9]. Looking at the figures, and comparing them with the results of [9] (replicated here for \( k = 4 \) in figure 1), one immediately sees that the cooperation level reached after the transient equilibration period is much lower, and comparable with the results found for regular and random graphs. This is reasonable, given that now it is as if each individual had the same average number of neighbors as far as its payoff is concerned.

To reach a better understanding of the difference between accumulated and average payoff, we interpolated between the two extreme cases according to the formula

\[
\Pi_i = \frac{1}{kd} \sum_j \pi_{i,j},
\]

where \( d \in [0, 1] \), \( \Pi_i \) is the net payoff of player \( i \), and \( \pi_{i,j} \) is the payoff player \( i \) obtains when interacting with neighbor \( j \). One can see that, when \( d = 0 \) we recover the accumulated payoff value, while \( d = 1 \) corresponds to the average payoff case. Figure 6 clearly shows that, as \( d \) varies from 0 to 1, and thus the ratio varies from 1 to \( 1/k \), cooperation levels steadily decrease for all values of the temptation on the y-axis. So, the way in which individual payoff is computed has a large influence on cooperation levels that can be reached, in the average, on a given network topology.

Figure 6: Cooperation level as a function of the parameter \( d \) of equation 1 in the PD for temptation values between 1 and 2. Cooperation prevails in light areas; Darker areas mean more defection. Results are the average of 50 runs.

4 Evolutionary Stability

Evolutionary stability, i.e. the resistance to invasion by mutant strategies, is an important issue when dealing with evolutionary games [2]. The effect of switching the
strategy of the hub with largest connectivity in a totally cooperating population has been studied in [26]. Here we use a different approach to perturb the population after it has reached a quasi-stable state by switching the strategy of a few players having the strategy of the greater number. This was done for values of $b \in \{1.2, 1.5, 1.8\}$ and $r \in \{0.2, 0.5, 0.8\}$. We then give the system 6000 time steps to attempt to reattain its initial stable state. For reasons of space, we only plot the results obtained for $b = 1.8$ and $r = 0.5$ (see figure 7). Given the scale-free nature of the interaction network, introducing a small amount of random noise does not have any effect on the population stability. On the other hand, when cooperator hubs switch strategy (one to five in our study), avalanches of defection can form and propagate through the population. Under replicator dynamics and when using accumulated payoff, about $1/6$ of the PD runs do not recover the state previously attained at time step $10^4$. This fraction rises to $1/3$ for the SD game. With the imitation of the best rule, $1/10$ of the PD and SD runs fail to recover from the perturbations. In contrast to accumulated payoff, average payoff does not allow perturbations to generate any noticeable effect, i.e. the system remains quite stable.

Figure 7: System stability when using accumulated payoff. For each parameter set, 100 runs have been perturbed, but only a few individual runs are plotted here to expose the behaviors encountered. Upper figures: replicator dynamics; lower figures: imitation of the best. Left-hand figures: $b = 1.8$; right-hand figures: $r = 0.5$. 
5 Conclusions

In conclusion, we have deepened and extended the study presented in [9] clarifying the role of the updating rule and the type of payoff attributed to players. We have shown that the games are not invariant under linear affine transformations when using accumulated payoff, while average payoff does not have this problem, although it may artificially reduce the impact of scale-free degree networks. We have also seen that asynchronous update dynamics, being more likely in a system of independently interacting agents, by eliminating artificial effects due to the nature of synchronous update, may give rise to steadier quasi-equilibrium states. Moreover, we have studied several dynamical aspects of the evolution of the populations such as their transients before attaining the steady-state, and their evolutionary stability, showing that scale-free networks of interactions provide a quite stable environment for the emergence of cooperation when using accumulated payoff, except when hubs are targeted by the mutations, in which case a sizable number of runs do not recover the original state, at least within the simulation times allowed in our numerical experiments.

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References


