On the mathematical structure and hidden symmetries of the Born-Infeld field equations

Diego Julio Cirilo-Lombardo
Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, 141980, Dubna, Russian Federation.
diego@thsun1.jinr.ru ; diego77jcl@yahoo.com

1 Introduction

The most significant nonlinear theory of electrodynamics is mainly the Born-Infeld (BI) theory. The Lagrangian density describing the BI theory (in arbitrary space-time dimensions) is

$$\mathcal{L}_{BI} = \sqrt{-g} L_{BI} = \frac{b^2}{4\pi} \left\{ \sqrt{-g} - \sqrt{-\det(g_{\mu\nu} + b^{-1} F_{\mu\nu})} \right\} \tag{1}$$
where $b$ is a fundamental parameter of the theory with field dimension $s$. In four space-time dimensions the determinant in (1) may be expanded out to give

$$L_{BI} = \frac{b^2}{4\pi} \left\{ 1 - \sqrt{1 + \frac{1}{2} b^{-2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} b^{-4} \left( F_{\mu\nu} F^{\mu\nu} \right)^2} \right\}$$

(2)

which coincides with the usual Maxwell Lagrangian in the weak field limit.

Similarly, if we consider the second rank tensor $\mathbb{F}^{\mu\nu}$ defined by

$$\mathbb{F}^{\mu\nu} = -\frac{1}{2} \frac{\partial L_{BI}}{\partial F_{\mu\nu}} = \frac{F^{\mu\nu} - \frac{1}{4} b^{-2} \left( F_{\rho\sigma} F^{\rho\sigma} \right) \tilde{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{16} b^{-4} \left( F_{\rho\sigma} \tilde{F}^{\rho\sigma} \right)^2}}$$

(3)

(so that $\mathbb{F}^{\mu\nu} \approx F^{\mu\nu}$ for weak fields), this second kind of antisymmetrical tensor satisfies the electromagnetic equations of motion

$$\nabla_\mu \mathbb{F}^{\mu\nu} = 0$$

(4)

which are highly nonlinear in $F_{\mu\nu}$. Another interesting object to analyze is the energy-momentum tensor that can be written as

$$T_{\mu\nu} = \frac{1}{4\pi} \left\{ F_{\mu\lambda} F_{\nu\lambda} + b^2 \left[ \mathbb{R} - 1 - \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} \right] g_{\mu\nu} \right\}$$

$$\mathbb{R} \equiv \sqrt{1 + \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{16} b^{-4} \left( F_{\rho\sigma} \tilde{F}^{\rho\sigma} \right)^2}$$

(5)

However, besides these more or less obvious statements, Born and Infeld observed in their original work [1] that the tensor $F$ has a relation to $\mathbb{F}$ similar to that in Maxwell theory of macroscopical bodies between the dielectric displacement and magnetic induction, but in the BI case this relation is a discrete electric-magnetic duality invariance [5] that is associated with underlying $SO(2)$ symmetry. In ref. [1], the relations that put in evidence the symmetries of these transformations that are characteristic of the BI field equations only, are

$$\mathbb{F}^{\mu\nu} = \frac{F^{\mu\nu} - G \tilde{F}^{\mu\nu}}{\sqrt{1 + S - G^2}}$$

(6)

\(^1\)In open superstring theory (2-dimensions), for example, loop calculations lead to this Lagrangian with $b^{-1} = 2\pi \alpha'$ ($\alpha' \equiv$ inverse of the string tension)
\[ F^{\mu \nu} = \frac{F^{\mu \nu} + Q \tilde{F}^{\mu \nu}}{\sqrt{1 + P - Q^2}}, \quad Q \equiv G \]  

(7)

where \( G, Q, S \) and \( P \) are the electromagnetic invariants constructed of the two types of fields \( F \) and \( \mathcal{F} \) which will be expressed explicitly in the next section. Although it is by no means obvious, it may be verified that equations (3), (6) and (7) are invariant under electric-magnetic rotations of duality \( F \leftrightarrow \tilde{F} \), but notice that the BI Lagrangian (1) is not. This fact was firstly pointed out in the general publications about the electromagnetic duality rotations of Gaillard and Zumino [4] and more recently and specifically for the BI case, in the papers of Gibbons and Rasheed [5,6].

The main task in this work is to complete in any sense the analysis given in refs. [4,5,6] for the BI theory showing explicitly the quaternionic structure of the field equations. The starting point to complete such an analysis is based on a previous paper of the author [7] where it was explicitly shown that the transformations (6,7) are produced by a quaternionic operator acting over vectors in which the components are the corresponding electromagnetic fields

\[
\begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix}^{\mu \nu} = \frac{1}{\mathbb{R}} (\sigma_0 - i\sigma_2 \mathbb{P}) \begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix}^{\mu \nu}
\]

(8)

\[
\begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix}^{\mu \nu} = \frac{1}{1 + \mathbb{P}^2} (\sigma_0 + i\sigma_2 \mathbb{P}) \begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix}^{\mu \nu}
\]

(9)

where \( \tilde{F}^{kl} \equiv \frac{\partial L_{BI}}{\partial F^{kl}} \); \( \mathbb{R} \equiv \sqrt{1 + \frac{1}{2}b^{-2}F_{\rho \sigma} F^{\rho \sigma} - \frac{1}{16}b^{-4} (F_{\rho \sigma} \tilde{F}^{\rho \sigma})^2} \) and \( \mathbb{P} = -\frac{1}{4} F^{\mu \nu} \tilde{F}_{\mu \nu} \) (\( b \equiv \) absolute field of the BI-theory) and the complex conjugation indicated by the horizontal-bar over the operators. The Pauli matrices are defined as (Landau-Lifshitz 1968)

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \quad \sigma_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

The norms of the operators \( A \) and \( B \) are

\[ \overline{A}A = A\overline{A} = \frac{1 + \mathbb{P}^2}{\mathbb{R}^2} \]
\[ \overline{B}B = B\overline{B} = \frac{\mathbb{R}^2}{1 + \mathbb{F}^2} \]

where from expressions (8,9) we have

\[ A\overline{B} = \overline{A}B = 1 \]

The plan of this paper is as follows: in Section 2, the Quaternionic structure of the BI field equations is manifestly presented and the mathematical structure is carefully analyzed and extended. In Section 3, we describe the phase space determined by the symmetries of the BI field equations from the Hamiltonian point of view. In Section 4, the constitutive-like relations of the BI theory are studied by comparing them with the ordinary Maxwell electrodynamics in Riemannian space with an arbitrary metric and the expression for the Fresnel equation is explicitly given for the BI case. Finally, the remarks and conclusions are given in Section 5.

Our convention is as in ref.[2] with signatures of the metric, Riemann and Einstein tensors (-+++); the internal indeces (gauge group) are denoted by \( a, b, c,... \), space-time indeces by Greek letters \( \mu, \nu, \rho... \) and the tetrad indeces by capital Latin letters \( A, B, C... \).

2 The quaternionic structure

Now we will show in an explicit and compact form how the transformations (6,7) can be realized by means of a quaternionic structure. We start with the following definitions for the invariants of the electromagnetic field \( S \equiv \frac{1}{2\mathbb{F}} F_{\rho\sigma} F^{\rho\sigma}, \quad G = \frac{1}{2\mathbb{F}} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad R \equiv \sqrt{1 + S - G^2} \), and the following signature for the metric tensor is adopted \( g_{\mu\nu} = (- - + +) \) Starting from expressions (6,7) with the new definitions for the invariants we have

\[
\begin{pmatrix}
[F] \\
[\overline{F}]
\end{pmatrix}
^{\mu\nu} = \frac{1}{R} \left( \sigma_0 - i \sigma_2 G \right)
\begin{pmatrix}
[F] \\
[\overline{F}]
\end{pmatrix}
^{\mu\nu}
\]

\( (10) \)

It is interesting to notice that, due to the fact that the following identity holds \( F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbb{F}_{\mu\nu} \overline{\mathbb{F}}^{\mu\nu} \), the quaternion \( \mathbb{Q} \) is invariant from the topological point of view. It is a very important property because the mapping between the different sets of fields, \( F \) and \( \mathbb{F} \), respectively, preserves the topological
charge unaltered. This means that the topological charge is a fixed point of the $Q$ transformation. Defining the “spinors”

$$\Psi = \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}, \quad \Psi = (\sigma_3 \Psi)^\dagger,$$

and

$$\Phi = \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}, \quad \Phi = (\sigma_3 \Phi)^\dagger,$$

the square root $R$ in (10) is simplified to the following expression:

$$\sqrt{1 + S - G^2} = \sqrt{1 + \frac{1}{4} (\Psi Q \Psi)}$$

and relation (10) takes the compact form

$$\Phi = \frac{Q \Psi}{\sqrt{1 + \frac{1}{4} (\Psi Q \Psi)}} \quad (11)$$

As we could see in the introduction [1], in the same manner it is possible to invert the above equation putting all as a function of the spinor $\Psi$. In order to do this, it is sufficient to consider:

$$P \equiv \frac{1}{2\pi} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}, \quad Q = G = \frac{1}{2\pi} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

and the following property $F_{\rho\sigma} F^{\rho\sigma} = -\tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma}$. The square root in this inverted transformation is ($\overline{Q} \equiv (\sigma_0 + i\sigma_2 G)$)

$$\sqrt{1 + P - Q^2} = \sqrt{1 - \frac{1}{4} (\Phi \overline{Q} \Phi)}$$

and the inverse transformation becomes

$$\Psi = \frac{\overline{Q} \Phi}{\sqrt{1 - \frac{1}{4} (\Phi \overline{Q} \Phi)}} \quad (12)$$

We stop here to see in a little more detail the mathematical structure of the operators $Q$. From (10) we can see that the $Q$ form part of a commutative ring of complex operators $Q \equiv \{\alpha + i\beta \mathbb{I} / \alpha, \beta \in \mathbb{C}\}$, equipped with addition.
and multiplication laws induced by those in $\mathbb{C}$, such as addition and multiplication on $\mathbb{Q}$ are given by the usual matrix addition and multiplication, with $I$ having the following form:\(^2\)

\[
I = \pm \begin{pmatrix}
0 & 1_d/2 \\
-1_d/2 & 0
\end{pmatrix}
\]

It is easily seen that $\mathbb{Q}$ is a commutative ring with zero divisors

\[Q_0^0 \equiv \{ \lambda (1_d \pm i_2 I_d) , \lambda \in \mathbb{C} \},\]

$Q_0^0$, $Q_0^+$ are the only multiplicative ideals in $\mathbb{Q}$, for instance, are maximal ideals. Thus, the only fields that we can construct from $\mathbb{Q}$ are

\[
\frac{\mathbb{Q}}{Q_0^0} \cong \mathbb{Q}_0^0 \cong \mathbb{C}
\]

In the general case (e.g.: $\alpha, \beta \in \mathbb{C}$), the map $|\cdot|^2 : \mathbb{Q} \to \mathbb{R}/|\mathbb{Q}|^2 \equiv \mathbb{Q}^\square = \alpha^2 + \beta^2$ can be seen as a semi-modulus on the ring $\mathbb{Q}$

\[\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^0 \cup \mathbb{Q}^- ,\]

according to the sign of the modulus of $\mathbb{Q}$. It is important to note that in contrast with the analysis of ref.[9], for the BI case $\alpha, \beta \in \mathbb{R}$ ($\alpha, \beta$, are the identity and the pseudoscalar invariant of the electromagnetic field respectively) the commutative ring described by $\mathbb{Q}$ has no pseudo-complex structure.

Another interesting thing about this commutative ring of complex operators is that it permits us to define for $d = 2$ the following exponential mapping

\[e^{(\alpha \sigma_0 - i_2 \beta \sigma_2)} = e^\alpha (\cos \beta - i_2 \sin \beta)\]

where the mathematical structure described in a abstract way before is clearly seen.

The important thing is that the correct analysis of the algebraic and divisor ring structure of the BI-field equations is a crucial point to go towards a truly noncommutative BI theory. The generalization of transformations

\(^2\)here $d$ is the dimension.
(10) is performed due to the fact that the operators can be realized over the non-commutative field of full-quaternions in the following manner:

\[ \frac{1}{R} \left[ \sigma_0 \delta' - iG (\sigma_1 \alpha' + \sigma_2 \beta' + \sigma_3 \gamma') \right] \]

We assume the coefficients \( \alpha', \beta', \gamma' \): reals and \( \delta' \): complex, in principle, taking the operator the following form

\[ \frac{1}{R} \left[ \begin{pmatrix} \delta' & 0 \\ 0 & *\delta' \end{pmatrix} - iG \begin{pmatrix} \gamma' & \alpha' - i\beta' \\ \alpha + i\beta & -\gamma' \end{pmatrix} \right] \]

where the star means complex conjugation, and the quantity \( G \) will have a different meaning that is in the initial expression (10), obviously. The question that immediately arises is: Is it possible to impose conditions on the coefficients \( \alpha', \beta', \gamma' \) and \( \delta' \) in the above expression in order to obtain a full-quaternionic non-commutative operator from the equations of motion of a determinant-geometrical action? The answer is affirmative if and only if \( \gamma' = 0 \) and \( \delta' = \alpha' - i\beta' \). With these particular values of the coefficients the square root of the determinant in the BI action (where the equations of motion determining the mapping coming from) is

\[ \sqrt{-\det(g_{\mu\nu} + b^{-1}\chi F_{\mu\nu})} \]

where \( \chi^4 = i(\alpha' - i\beta') \) and \( G = \frac{\chi^2}{2\delta^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \), following the same conventions from the beginning. A carefully study of the possible physical meaning will be carried out elsewhere in [11].

3 Hamiltonian point of view

We can show that the SO(2) structure of the BI theory is more easily seen in the following operator form [7]:

\[ \frac{1}{R} \left( \sigma_0 - i\sigma_2 \tilde{F} \right) L = L \]

\[ \frac{R}{(1 + \tilde{F}^2)} \left( \sigma_0 + i\sigma_2 \tilde{F} \right) L = L \]
\[ \mathbb{P} \equiv \frac{P}{b} \]

where we defined the following quaternionic operators:

\[ L = F - i\sigma_2 \tilde{F} \]
\[ \mathbb{L} = \mathbb{F} - i\sigma_2 \tilde{\mathbb{F}} \]

the pseudoescalar of the electromagnetic tensor \( F^{\mu\nu} \)

\[ \mathbb{P} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \]

where \( \sigma_0, \sigma_2 \) are the well known Pauli matrices that we define previously.

Now, with the definitions given before, we pass to the description of the phase space from the Hamiltonian point of view in a similar form as in ref.[5].

The 6-dimensional space \( V = \Lambda^2 (\mathbb{R}^4) \to 2\text{-forms} \in \mathbb{R}^4 \), has coordinates \( F_{\mu\nu} \) and carries a Lorentz invariant metric with signature \((+++---)\) defined by

\[ k_H (F, F) \equiv L\tilde{L} = 2F\tilde{F} \]

The dual space \( V^* \) of \( V \) consists of skew-symmetric second rank contravariant tensors \( F^{\mu\nu} \). The phase space \( P = V \oplus V^* \) carries a natural quaternionic symplectic structure given by

\[ d\mathbb{L} \wedge d\tilde{L} = d\mathbb{F} \wedge dF - d\tilde{\mathbb{F}} \wedge d\tilde{F} \]

Notice that now, from the mathematical description of the phase space, the \( SO (2) \) symmetry is, in fact, embedded in a large quaternionic structure.

### 4 Maxwell equations in the Riemannian space-time and the Born-Infeld theory

We now want to give some curious aspects about the relation between the BI field equations and the Maxwell equations in the Riemannian space-time.

From ref.[2] we know that when a gravitational field exists (i.e.:curved space-time), it is possible to write the Maxwell equations in vacuum similarly that in a hypotetic medium\(^3\)

\(^3\)Here \( g_\alpha = -\frac{2m}{g_{00}}, \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{2m g_{0\alpha} g_{0\beta}}{g_{00}} \) and \( h = g_{00} \) as in ref.[2]
\[ D = \frac{E}{\sqrt{h}} + [B \times g], \quad H = \frac{B}{\sqrt{h}} - [E \times g] \]

(i.e. for a *gyrotropic* medium [3]). Analogously to the Born-Infeld case, we can put these constitutive relations in the following form:

\[
\begin{pmatrix} D \\ H \end{pmatrix}^\alpha \equiv \left[ \frac{\sigma_0}{\sqrt{h}} + i\sigma_2 \epsilon_{\beta\gamma} g^\beta \right]^\alpha \begin{pmatrix} E \\ B \end{pmatrix}^\gamma
\] (13)

Notice a remarkable analogy with the similar expression (10). This means that the BI theory can be formulated as an effective metric theory, as it was shown in references [8]. For the BI case the constitutive-like relations give \( D \) and \( H \) in terms of \( E \) and \( B \)[5]

\[
\begin{align*}
D &= \frac{E + b^{-2} (E \cdot B) B}{\sqrt{1 + b^{-2} (B^2 - E^2) - b^{-4} (E \cdot B)^2}} \\
H &= \frac{B - b^{-2} (E \cdot B) E}{\sqrt{1 + b^{-2} (B^2 - E^2) - b^{-4} (E \cdot B)^2}}
\end{align*}
\] (14)

From the introduction we know that these equations can be solved to give \( E \) and \( H \) in terms of \( E \) and \( B \):

\[
\begin{align*}
E &= \frac{(1 + b^{-2}B^2) D + b^{-2} (D \cdot B) B}{\sqrt{(1 + b^{-2}B^2) (1 + b^{-2}D^2) - b^{-4} (D \cdot B)^2}} \\
B &= \frac{(1 + b^{-2}D^2) B + b^{-2} (D \cdot B) D}{\sqrt{(1 + b^{-2}B^2) (1 + b^{-2}D^2) - b^{-4} (D \cdot B)^2}}
\end{align*}
\]

that make the explicit comparison between (13) and (14) easy when the fields \( D \) and \( H \) are the same in both the cases: BI fields in flat space-time and linear field in curved space-time

\[
\left. \frac{E_\alpha}{\sqrt{h}} \right|_{BI} = \left. \frac{E_\alpha}{\sqrt{h}} \right|_{f}
\]

\(^4\)here \( \epsilon \) is the full-antisymmetric tensor, as usual
\[
\left(\gamma^{\beta\gamma} B_\beta E_\gamma\right) \left. \frac{B_\alpha}{g_{\alpha\beta}} \right|_{BI} = \sqrt{\gamma} \epsilon_{\alpha\beta\gamma} g^{0\alpha} B^\gamma |_f
\]

where the subindices \( BI \) and \( f \) indicate the fields in the BI theory (flat space-time) and the Maxwell fields in any frame (curved), respectively.

Following the same procedure as in ref. [3] for the Maxwell case without any background (gravitational and/or electromagnetic), the Fresnel equation in the Born-Infeld case (in a Lorentz frame) takes the following form:

\[
-n^2 \left( C_{xx} n_x^2 + C_{yy} n_y^2 + C_{zz} n_z^2 \right) + n_x^2 C_{xx} (C_{yy} + C_{zz}) +
+n_y^2 C_{yy} (C_{xx} + C_{zz}) + n_z^2 C_{zz} (C_{xx} + C_{yy}) - C_{xx} C_{yy} C_{zz} = 0 \quad (15)
\]

where \( n_i \) are the coordinates of the surface of propagation (wave number) and

\[
C_{ij} \equiv \frac{(\delta_{ij} + (E \cdot B) E_i B_j)}{1 + b^{-2} (B^2 - E^2) - b^{-4} (E \cdot B)^2} \quad (16)
\]

Notice that expression (15) has the same form as in reference [3] but with \( \epsilon_{ij} \) replaced by \( C_{ij} \) given by (16). Notice also that in the presence of any electromagnetic background the particular form of the Fresnel equation (15) can take a more general form depending on the components for \( C_{ij} \) with \( c \neq j \) (i.e. ref. [3]). It is also interesting as a theoretical tool to test the nonlinearity of the BI field as the deviation of the Maxwell theory. This fact takes particular importance in astrophysical phenomena [10].

5 Concluding remarks

In this work, the Born-Infeld field equations were written in the most compact form by means of quaternionic operators constructed according to the symmetries of the theory.

We also showed that the \( \mathbb{Q} \) operators defined here form part of a commutative ring of complex operators and the \( SO(2) \) symmetry of the BI field equations is in a manner embedded into a larger quaternionic structure. This extension can be realized by transforming the commutative ring of complex operators to a non-commutative ring. Our results agree with the observation of Gibbons and Rasheed in [5,6] that there exists discrete symmetry in the structure of the field equations that is unique in the case of the Born and Infeld nonlinear electrodynamics: this fact is easily seen in our work because these discrete symmetries generated by the \( \mathbb{Q} \) operators are invertible.
The quaternionic structure of the phase space was explicitly derived and described from the Hamiltonian point of view, showing at the same time that the results on the structure of the phase space of ref.[5] are naturally included in this large quaternionic symmetry.

Finally, the analogy between the BI theory and the Maxwell (linear) electrodynamics in a curved space-time was explicitly shown and the equation for the Fresnel equation in the nonlinear BI case without background was explicitly given and proposed as a theoretical tool to test this particularly interesting nonlinear electrodynamics of M. Born and L. Infeld.

6 Acknowledgements

I am very thankful to Professors Boris M. Barbashov and Alexander Dorokhov for their advisement and, in particular, to Professor J. A. Helayel-Neto for his interest to put in this research. I am very grateful to the JINR-Directorate, in particular, the Bogoliubov Laboratory of Theoretical Physics for their hospitality and support.

7 References