Quasi-hermitian Quantum Mechanics in Phase Space

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January 16, 2007

Abstract

We investigate quasi-hermitian quantum mechanics in phase space using standard deformation quantization methods: Groenewold star products and Wigner transforms. We focus on imaginary Liouville theory as a representative example where exact results are easily obtained. We emphasize spatially periodic solutions, compute various distribution functions and phase-space metrics, and explore the relationships between them.

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1 Introduction

Superficially non-hermitian Hamiltonian quantum systems are of considerable current interest, especially in the context of PT symmetric models \[3,13\]. For such systems the Hilbert space structure is at first sight very different than that for hermitian Hamiltonian systems inasmuch as the dual wave functions are not just the complex conjugates of the wave functions, or equivalently, the Hilbert space metric is not the usual one. While it is possible to keep most of the compact Dirac notation in analyzing such systems (see Appendix E), in the main body of this paper we will work with explicit functions and avoid abstract notation. Our goal is to expose the underlying mechanisms (as in \[8,9\]) rather than to hide them.

Our discussion is focused on a system with potential \( \exp(2ix) \). This model, as well as its field theory extension, is of interest for applications to table-top physical systems \[2,4\] and to deeper problems in string theory \[19,20\]. We will not discuss those applications here, but rather we will simply develop the phase-space formalism for the point particle model. We believe this formalism will be helpful in understanding the applications cited, as well as others. Other recent work along these same lines can be found in \[16,17\].

2 Imaginary Liouville quantum mechanics

Consider “imaginary” or “periodic” Liouville quantum mechanics as governed by the apparently non-hermitian Hamiltonian

\[
H = p^2 + m^2 e^{2ix}
\]  

Without essential loss of generality, we take \( m = 1 \) in most of the following. Obviously, this is a “PT symmetric” model. But more to the point, this is actually a “quasi-hermitian” theory \[18\] with a real energy spectrum, as explained in \[8\] and as we shall clarify further here. We will analyze this system in phase-space using the methods of deformation quantization.

2.1 Eigenfunctions

But first, let us briefly review the position representation Schrödinger eigenvalue problem for this system (see e.g. \[8\]). With \( x \) on the real line and with the condition that the wave functions remain bounded, the corresponding Schrödinger equation has energy eigenvalues given by all real \( E \geq 0 \). The eigenfunctions are just Bessel functions, \( J_{\pm \sqrt{E}}(e^{i\pi}) \). These are doubly degenerate when \( \sqrt{E} \neq n \in \mathbb{N} \), but they merge into a single, nondegenerate eigenfunction when \( E = n^2 \). For these nondegenerate cases the eigenfunctions are
2π-periodic in $x$, and the solutions are the analytic Bessel functions $J_n$ \[22\].

$$n^2 J_n (e^{ix}) = \left(-\frac{d^2}{dx^2} + e^{2ix}\right) J_n (e^{ix})$$

\[2\]

$$J_n (e^{ix}) = \frac{1}{2^n} e^{inx} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} e^{2ikx}, \quad n = 0, 1, 2, \cdots$$

\[3\]

We will consider only such periodic solutions here, since in our opinion this is the most interesting situation.

We note that these periodic eigenfunctions are superpositions of only \textit{right-moving} plane waves. In fact, in this periodic situation the discrete energy spectrum is precisely the same as would be found for particles moving freely on a circle \textit{but} restricted to non-negative momentum.

### 2.2 Dual polynomials

The periodic Bessel functions and their complex conjugates do \textit{not} form an orthonormal set on the circle. To obtain an orthonormal set of functions it is necessary to combine $\{J_n (z)\}$ with an associated set of \textit{polynomials} in $z^{-1}$, $\{A_n (z)\}$, the so-called Neumann polynomials. These are dual to $\{J_n (z)\}$ on any contour enclosing the origin $z = 0$, in the following sense:

$$\frac{1}{2\pi i} \oint dz \frac{A_j (z) J_k (z)}{z} = \delta_{j,k}$$

\[4\]

The Neumann polynomials on the circle are given explicitly by

$$A_0 (e^{ix}) = 1, \quad A_1 (e^{ix}) = 2e^{-ix}, \quad A_n (e^{ix}) = 2^n n e^{-inx} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-k-1)! \ e^{2ikx}}{4^k}$$

\[5\]

$$I_n (e^{ix}) = \left(-\frac{d^2}{dx^2} + e^{2ix} - n^2\right) A_n (e^{ix})$$

\[6\]

As indicated, the $\{A_n\}$ obey \textit{inhomogeneous} modifications of Bessel’s equation, where the inhomogeneity is either $I_n (z) \propto z^2$ for even $n$ or $I_n (z) \propto z$ for odd $n$, according to

$$I_n (e^{ix}) = \begin{cases} \varepsilon_n e^{2ix} & \text{for even } n \geq 0 \\ 2ne^{ix} & \text{for odd } n > 0 \end{cases}$$

\[7\]

While \[6\] is inhomogeneous, nevertheless the usual proof of orthogonality between pairs of non-degenerate $H$ eigenfunctions and their duals goes through because the inhomogeneities are orthogonal to the eigenfunctions.

$$\int_0^{2\pi} dx \ I_j (e^{ix}) J_k (e^{ix}) = 0, \quad \text{for all } j, k \in \mathbb{N}$$

\[8\]

\[1\]For convenience we have modified the usual notation of the associated polynomials as given in [15, 22], namely $O_n$, and have defined $A_n (z) = \varepsilon_n \ z \ O_n (z)$ where $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n \neq 0$. 

\[5\]
3 Phase space distributions

We next compute Wigner transforms of various function bilinears for the $2\pi$-periodic Bessel/Neumann system. For a system so-defined, on a circle, momentum is quantized. Thus we would expect that the $(x,p)$ phase space is not the usual $\mathbb{R}^2$ nor even $S^1 \times \mathbb{R}$, but rather that it is reduced to $S^1 \times \mathbb{Z}$. In fact, the periodic energy eigenfunctions of the imaginary Liouville Hamiltonian consist of superpositions of positive momentum plane waves, so we would also expect not to need the $\mathbb{Z}_{<0}$ momentum sector at all. Well, both expectations are almost true. But not quite. We shall see below to what extent these expectations are born out.

3.1 Eigenfunction WFs

As a first step, we remind the reader about the structure of real (diagonal) Wigner functions (WFs) made from $2\pi$-periodic plane waves. They are just Kronecker deltas, with $p \in \mathbb{Z}$ as expected.

$$e_n (x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi_n (x-y) \overline{\phi_n (x+y)} \, e^{2ipy} \, dy$$

$$= \delta_{n,p}$$

upon choosing $\phi_n (x) = \exp (inx)$. By analogy, for the Liouville eigenfunctions $\psi_n$ the WFs are again manifestly real, and again have support for $p \in \mathbb{Z}$ as expected, as given by

$$f_n (x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \psi_n (x-y) \overline{\psi_n (x+y)} \, e^{2ipy} \, dy$$

$$= \frac{(-1)^{p-n}}{4p} \sum_{k=0}^{p-n} \frac{e^{2ix(n-p+2k)}}{k! (n+k)! (p-k)! (p-k-n)!}$$

where the sum results from taking $\psi_n (x) = J_n (e^{ix})$ as given by the series in (3). Note for the Liouville case, as opposed to the free particle case, the support in $p$ is infinite: This particular $f_n (x,p)$ is non-zero for all $p \geq n \geq 0$. Another way to write these WFs makes use of the associated Legendre functions:

$$f_n (x,p) = \frac{(-1)^n}{p! (p-n)!} \left( \frac{i \sin 2x}{2} \right)^p \text{LegendreP} (p, -n, i \cot 2x)$$

In particular, for any point in the reduced phase space, $(x,p) \in S^1 \times \mathbb{Z}$, we have

$$f_0 (x,p) = 1 \times \delta_{p,0} - \frac{1}{2} (2 \cos 2x) \times \delta_{p,1} + \frac{1}{2^5} (2 + \cos 4x) \times \delta_{p,2} + \cdots$$

$$f_1 (x,p) = \frac{1}{2^2} \times \delta_{p,1} - \frac{1}{2^4} (2 \cos 2x) \times \delta_{p,2} + \frac{1}{2^8} (3 + \cos 4x) \times \delta_{p,3} + \cdots$$

$$f_2 (x,p) = \frac{1}{2^6} \times \delta_{p,2} - \frac{1}{2^{10}} (2 \cos 2x) \times \delta_{p,3} + \frac{1}{2^{12}} (4 + 3 \cos 4x) \times \delta_{p,4} + \cdots$$

$$f_3 (x,p) = \frac{1}{2^{10}} \times \delta_{p,3} - \frac{1}{2^{12}} (2 \cos 2x) \times \delta_{p,4} + \frac{1}{2^{18}} (5 + 4 \cos 4x) \times \delta_{p,5} + \cdots$$

\footnote{A form which facilitates a continuation to non-integer $p$, should anyone wish to do that.}
But now we have more functions at our disposal, namely the Neumann polynomials, so we may build another set of WFs for comparison to those in [10] and [12].

3.2 Dual WFs

For dual functions $\chi_n$ the WFs are also manifestly real, with $p \in \mathbb{Z}$, as given by

$$\tilde{f}_n(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \chi_n(x-y) \chi_n(x+y) e^{2iyp} dy$$

$$= 4p n^2 \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{(n-k-1)! (n-l-1)!}{k! l!} e^{2ix(l-k)} \delta_{n-p,k+l}$$

where the sums result from taking $\chi_n(x) = A_n(e^{ix})$ as given by the series in [10]. Note the support in $p$ is now finite: This particular $\tilde{f}_n(x,p)$ is non-zero for $0 \leq n-2 \lfloor n/2 \rfloor \leq p \leq n$. That is to say, $0 \leq p \leq n$ for $n$ even, and $1 \leq p \leq n$ for $n$ odd.

Resolving the constraint set by the Kronecker delta in (16) eliminates one sum and further restricts the range of the other to yield

$$\tilde{f}_n(x,p) = 4p n^2 \sum_{k=\max(0,n-p-\lfloor n/2 \rfloor)}^{\min(\lfloor n/2 \rfloor, n-p)} \frac{(n-k-1)! (k+p-1)!}{k! (n-p-k)!} e^{2i(n-p-2k)x}$$

In particular, again for any phase-space point $(x,p) \in S^1 \times \mathbb{Z}$,

$$\tilde{f}_0(x,p) = \delta_{p,0}$$

$$\tilde{f}_1(x,p) = 4\delta_{p,1}$$

$$\tilde{f}_2(x,p) = 4 \times \delta_{p,0} + 32 \cos(2x) \times \delta_{p,1} + 64 \times \delta_{p,2}$$

$$\tilde{f}_3(x,p) = 36 \times \delta_{p,1} + 576 \cos(2x) \times \delta_{p,2} + 2304 \times \delta_{p,3}$$

Thus, at any given momentum level, we find the same set of functions of $x$ (i.e. $\cos(2kx)$) no matter whether we consider $\{f_n\}$ or $\{\tilde{f}_n\}$.

There is a basic orthogonality relation for WFs and dual WFs.

$$\frac{1}{2\pi} \int_{x,p} f_k(x,p) \tilde{f}_n(x,p) = \delta_{k,n}$$

This follows in a straightforward way from the bilinear structure of the Wigner transform and from the orthogonality of the wave functions and their duals. There is also a corresponding pseudo-local relation on

---

3. To dispel any confusion about our conventions for $\tilde{f}_n$, the complex conjugation in [15] is different from that in [9] and [11] just because we chose in [9, 10] to define the duals such that $\delta_{n,k} = \frac{1}{2\pi} \int_0^{2\pi} \chi_n(x) \psi_k(x) dx$ without any explicit conjugations.

4. The limits on the sum may also be written as $\min(\lfloor n/2 \rfloor, n-p)$ and $n-p - \min(\lfloor n/2 \rfloor, n-p)$. 
the phase space that involves the Groenewold star product (see (61) and (65) below). The form of these results can be seen most easily through the use of formal density operator methods, as in Appendix E.

Perhaps it is useful to present the specific examples of \( f_n \) and \( \tilde{f}_n \), for \( n = 0, 1, 2, 3 \) and for \( p = 0, 1, 2, 3 \), in Table form. This facilitates checking (19) for these few cases, and illuminates the orthogonality mechanism.

<table>
<thead>
<tr>
<th>WFs &amp; Dual WFs (non-zero values)</th>
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<tr>
<td>( p = 0 )</td>
</tr>
<tr>
<td>( f_0, \tilde{f}_0 )</td>
</tr>
<tr>
<td>( f_1, \tilde{f}_1 )</td>
</tr>
<tr>
<td>( f_2, \tilde{f}_2 )</td>
</tr>
<tr>
<td>( f_3, \tilde{f}_3 )</td>
</tr>
<tr>
<td>( \vdots )</td>
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4 Wigner transform of the bilocal metric

It is explained in [8] – as well as in the classic literature on the subject – how a scalar product for a biorthogonal system such as \( \{ A_k, J_n \} \) can always be written as an integral over a doubled configuration space involving a “bilocal metric” \( K(x, y) \).

\[
(\phi, \psi) = \int \int \phi(x) K(x, y) \psi(y) \, dx \, dy
\]  

4.1 Bilocal ↔ phase space

When a scalar product is so expressed as a bilocal bilinear form then it is naturally and very easily re-expressed in phase space (which we suppose to be \( \mathbb{R}^2 \) in this paragraph) through the use of a Wigner transform [11].

\[
f_{\psi \phi}(x, p) = \frac{1}{2\pi} \int \psi \left( x - \frac{1}{2} y \right) \phi \left( x + \frac{1}{2} y \right) e^{iyp} \, dy
\]
We have chosen the normalization here so that for \( p \) on the real line

\[
\psi(x) \phi(x) = \int_{-\infty}^{\infty} f_{\psi\phi}(x, p) \, dp
\]  

(22)

More generally, Fourier inverting (21) gives the point-split product

\[
\phi(x) \psi(y) = \int_{-\infty}^{\infty} e^{i(y-x)p} f_{\psi\phi}(x + \frac{y}{2}, p) \, dp
\]  

(23)

Thus the scalar product (20) can be re-written as

\[
(\phi, \psi) = \iint \, dx \, dp \, R(x, p) \, f_{\psi\phi}(x, p)
\]  

(24)

where the phase-space metric is the Wigner transform of the bilocal metric.

\[
R(x, p) = \int e^{iyp} K \left( x - \frac{1}{2} y, x + \frac{1}{2} y \right) \, dy
\]  

(25)

and inversely

\[
K(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-y)p} R \left( \frac{x + y}{2}, p \right) \, dp
\]  

(26)

In a more abstract notation (as in Appendix E) the form of (24) is

\[
(\phi, \psi) = \text{Tr} \left( \psi \langle \phi \rangle \right) = \text{Tr} \left( \langle \phi \rangle \psi \right)
\]

4.2 Liouville dual metric

The preceding results are quite general. To be more specific, for \( 2\pi \)-periodic dual functions of imaginary Liouville quantum mechanics, the scalar product was shown in [8] to be

\[
\chi_j(x) J(x, y) \chi_k(y) = \delta_{j,k}
\]

where

\[
J(x, y) = J_0 \left( e^{-ix} - e^{iy} \right) = J_0 \left( e^{-ix} \right) J_0 \left( e^{iy} \right) + 2 \sum_{n=1}^{\infty} J_n \left( e^{-ix} \right) J_n \left( e^{iy} \right)
\]  

(27)

Again, just a Bessel function. Or, re-expressed in a form which is immediately useful in the following,

\[
J(x, y) = J_0 \left( -2ie^{i(y-x)/2} \sin \left( \frac{x + y}{2} \right) \right)
\]  

(28)

Up to a normalization the corresponding metric in phase space is given by the Wigner transform of this bilocal.

A bit of care is needed since the Wigner transforms \( \tilde{f}_n(x, p) \) on which this metric will act are actually defined so that a dual function \( \chi \) plays the role of \( \phi \) and a conjugate dual function \( \overline{\chi} \) plays the role of \( \psi \) in the above (compare (21) to (15)). Thus, acting on \( \tilde{f}_n(x, p) \) the metric would be the Wigner transform of \( J \) as above, only with first and second arguments interchanged. We also adjust the normalization here (and

\^5To take the free particle limit, the parameter \( m \) in (11) must first be restored. See Appendix A.
again later, in (33)) to take into account our conventions and the fact that we are dealing with $2\pi$-periodic functions (see Appendix C). In view of all this, we finally obtain a dual phase-space metric given by
\[
\tilde{R}(x,p) = \frac{1}{2\pi} \int_0^{2\pi} J(x+w, x-w) e^{2iwp} dw = \frac{1}{2\pi} \int_0^{2\pi} J_0 (-2ie^{-iw}\sin x) e^{2iwp} dw
\]
\[
= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(\sin x)^{2k}}{(k!)^2} \int_0^{2\pi} e^{2iw(p-k)} dw = \sum_{k=0}^{\infty} \frac{(\sin x)^{2k}}{(k!)^2} \delta_{p,k}
\]
Hence the simple final answer.
\[
\tilde{R}(x,p) = \frac{(\sin^2 x)^p}{(p!)^2} \quad \text{for integer } p \geq 0, \text{ but vanishes for integer } p < 0
\]
An equivalent operator expression can be obtained by the method of Weyl transforms. (See Appendix F.)

Another way to characterize $\tilde{R}(x,p)$ is to note that it satisfies the differential-difference equation
\[
p\partial_x \tilde{R}(x,p) = \sin (2x) \tilde{R}(x,p-1)
\]
even when $p = 0$, since $\partial_x \tilde{R}(x,p = 0) = 0$ as well as $\tilde{R}(x,p = -1) = 0$. (This should be compared to (141) given below. Note that $\tilde{R}$ actually corresponds to $R^{-1}$ in that later discussion.)

In fact, there is an obvious continuation of (30) to all real $p$, or even to complex $x$ and $p$. Namely
\[
\tilde{R}(x,p) = \frac{(\sin^2 x)^p}{(\Gamma(p+1))^2}
\]
with its manifest zeroes and singularities (poles and cuts). This continuation also transparently satisfies (31).

Taking into account all our conventions, we may now express the normalizations of pure states in terms of the dual WFs and the dual phase-space metric as
\[
\varepsilon_n = \frac{1}{2\pi} \int_{x,p} \tilde{R}(x,p) \tilde{f}_n(x,p) = \frac{1}{2\pi} \sum_{p=0}^{n} \int_0^{2\pi} \frac{(\sin^2 x)^p}{(p!)^2} \tilde{f}_n(x,p) dx
\]
where as usual $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n > 0$. It is tedious but straightforward to use (17) to check this normalization and confirm that the dual metric does its job. More importantly, (33) is consistent with (19) for the simple reason that
\[
\tilde{R}(x,p) = \sum_{k=0}^{\infty} \varepsilon_k \tilde{f}_k(x,p)
\]
This in turn follows from the expansion of the bilocal metric in terms of Bessel bilinears, in (27).

5 Homogeneous versus inhomogeneous ★genvalue equations

Were the dual functions just the complex conjugates of the wave functions, the two types of WFs that we have defined would be identical, $\tilde{f}_n(x,p) = f_n(x,p)|_{\psi = \chi}$, but this is obviously not true for the case at hand.
5.1 WFs as eigenfunctions

As an important distinguishing feature, for the first of these WFs we have

\[ H \star f_n = n^2 f_n = f_n \star \Pi \]  

(35)

where the associative Groenewold star product operation \(^{12}\) is \((\hbar \equiv 1)\)

\[ \star \equiv \exp \left( \frac{i}{2} \partial_x \partial_p - \frac{i}{2} \partial_p \partial_x \right) \]  

(36)

Thus the wave function WFs are energy ★genfunctions.

The ★genvalue equation \(^{33}\) for the WFs is not only a distinguishing feature but can actually be used to define the \(f_n\) as real functions, without any prior knowledge of the underlying wave functions. Taking imaginary and real parts of \(^{35}\) we obtain an equation like \(^{31}\), namely

\[ p \partial_x f_n (x, p) = \sin (2x) f_n (x, p - 1) \]  

(37)

as well as

\[ \left( p^2 - \frac{1}{4} \partial_x^2 \right) f_n (x, p) + \cos (2x) f_n (x, p - 1) = n^2 f_n (x, p) \]  

(38)

Using \(^{37}\) twice, the second derivative in \(^{38}\) becomes

\[ \partial_x^2 f_n (x, p) = \frac{2 \cos (2x)}{p} f_n (x, p - 1) + \frac{\sin^2 (2x)}{p (p - 1)} f_n (x, p - 2) \]  

(39)

Hence \(^{38}\) reduces to a second-order difference equation in the momentum.

\[ (p^2 - n^2) f_n (x,p) + \frac{2p - 1}{2p} \cos (2x) f_n (x, p - 1) - \frac{\sin^2 (2x)}{4p (p - 1)} f_n (x, p - 2) = 0 \]  

(40)

We may solve this second-order equation by forward recursion under the condition \(^{6}\) that \(f_n (x, p < 0) = 0\).

The resulting WFs have support only for non-negative integer \(p\). We find that \(f_n (x, p < n) = 0, f_n (x, p = n)\) is arbitrary, and all \(f_n (x, p > n)\) are uniquely determined by \(^{40}\) in terms of our choice for \(f_n (x, p = n)\). For example, for the ground state \(n = 0\), the choice \(f_0 (x, 0) = 1\) immediately reproduces the terms in \(^{14a}\).

Similarly the choice (with no \(x\) dependence)

\[ f_n (x, p = n) = \frac{1}{4^n (n!)^2} \]  

(41)

reproduces the series \(^{12}\). The choice for \(f_n (x, p = n)\) must be independent of \(x\) so that \(f_n (x, p < n) = 0\) is consistent with \(^{37}\). Thus the WFs are determined by the ★genvalue equation.

\(^{6}\)Allowing support for negative integer \(p\) leads to singularities at \(p = 0\) and \(p = 1\). Therefore we rule out this possibility.
5.2 Entwining the dual metric

It follows immediately from (34) and (35) that

\[ H \ast \tilde{R}(x,p) = \tilde{R}(x,p) \ast \overline{H} \]  

(42)

which amounts to just (31). (This should be compared to (135) given below.) The equivalent operator statement is obvious, and follows directly from the Weyl correspondence. (See Appendix F.) As direct verification of (42), we compute

\[ H \ast \tilde{R}(x,p) = \frac{p}{2 \sin^2 x} \tilde{R}(x,p) = \tilde{R}(x,p) \ast \overline{H} \]  

(43)

The relation (42) is actually a special case of the “two star equation” given in [7], Eqn(73). In the language of that paper, (42) is the ultra-local version for which

\[ T(x,p;X,P) = \delta(x-X) \delta(p-P) \tilde{R}(x,p) \]

and one Hamiltonian function (\(H\)) is the complex conjugate of the other (\(\overline{H}\)).

5.3 Inhomogeneities for dual WFs

For the dual WFs (15), by direct calculation using standard methods (as in an Appendix B) we find

\[ \tilde{f}_n \ast H = \left( \left( p + \frac{1}{2} i \partial_x \right)^2 + e^{2ix} e^{\partial_p} \right) \frac{1}{2\pi} \int_0^{2\pi} \chi_n(x-y) \chi_n(x+y) e^{2iyp} dy \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} e^{2iyp} \chi_n(x-y) \left( \frac{1}{4} (\partial_y + \partial_x)^2 + e^{2i(x+y)} \right) \chi_n(x+y) dy \]  

(44)

Then from (6) we have

\[ \tilde{f}_n \ast H = n^2 \tilde{f}_n + \frac{1}{2\pi} \int_0^{2\pi} dy e^{2iyp} \chi_n(x-y) \begin{cases} 2ne^{i(x+y)} & \text{for } n \in \mathbb{N}_{\text{odd}} \\ \varepsilon_n e^{2i(x+y)} & \text{for } n \in \mathbb{N}_{\text{even}} \end{cases} \]  

(45)

Taking \(\chi_n(x) = A_n(e^{ix})\) and using (5), this gives

\[ \tilde{f}_n \ast H = n^2 \tilde{f}_n + 2n \sum_{k=0}^{n/2} \frac{(n-k-1)!}{4^k k!} e^{i(n-2k)x} \frac{1}{2\pi} \int_0^{2\pi} dy e^{iy(2p+2k-n)} \begin{cases} 2ne^{i(x+y)} \\ \varepsilon_n e^{2i(x+y)} \end{cases} \]  

\[ = n^2 \tilde{f}_n + 2n \sum_{k=0}^{n/2} \frac{(n-k-1)!}{4^k k!} e^{i(n-2k)x} \begin{cases} 2ne^{i(x+y)} \\ \varepsilon_n e^{2i(x+y)} \end{cases} \]  

(46)

That is to say, in complete analogy with the dual wave functions, the dual WFs obey inhomogeneous eigenvalue equations.

In particular, for the dual ground state WF

\[ \tilde{f}_0 \ast H = e^{2ix} \delta_{p,-1}, \quad \overline{H} \ast \tilde{f}_0 = e^{-2ix} \delta_{p,-1} \]  

(47)
and more generally
\[
\tilde{f}_n \ast H = n^2 \tilde{f}_n + \begin{cases} 
\frac{n^2 (\frac{|n/2|+p)!}{(n/2+p)!} 4^{1+p} e^{2i(1+p)x}} & \text{for odd } n, \text{ if } \lfloor n/2 \rfloor \geq p \geq 0, \text{ otherwise } 0 \\
2n \frac{(\frac{|n/2|+p)!}{(n/2-p)!} 4^{1+p} e^{2i(2+p)x}} & \text{for even } n > 0, \text{ if } \lfloor n/2 \rfloor \geq 1 + p \geq 0, \text{ otherwise } 0 
\end{cases}
\]
(48)
\[
\overline{H} \ast \tilde{f}_n = n^2 \tilde{f}_n + \begin{cases} 
\frac{n^2 (\frac{|n/2|+p)!}{(n/2-p)!} 4^{1+p} e^{-2i(1+p)x}} & \text{for odd } n, \text{ if } \lfloor n/2 \rfloor \geq p \geq 0, \text{ otherwise } 0 \\
2n \frac{(\frac{|n/2|+p)!}{(n/2-p-1)!} 4^{1+p} e^{-2i(2+p)x}} & \text{for even } n > 0, \text{ if } \lfloor n/2 \rfloor \geq 1 + p \geq 0, \text{ otherwise } 0 
\end{cases}
\]
(49)

As an alternative derivation, this can be checked by acting directly on the dual WFs as explicit sums, (17).

These equations for the dual WFs do not determine them uniquely, when they are only required to be finite real solutions to the implied differential-difference equations (again with \( \tilde{f}_n (x, p = n) \) independent of \( x \)) since we could always add to any \( \tilde{f}_n \) a solution to the homogeneous equation, namely an \( f_n \). However, if we require the stronger conditions that we seek solutions with finite momentum support, such that \( \tilde{f}_n (x, p < 0) = 0 \) and \( \tilde{f}_n (x, p > n) = 0 \), then the solutions are uniquely determined, up to normalization. In fact, the form of the inhomogeneities is also fixed by these requirements. This is similar to what happens in the analysis of the dual wave functions, which may be constructed without knowledge of the wave functions if we require that they be finite polynomials in \( \exp (-ix) \). In that analysis the form of the inhomogeneities is also fixed (see [8] and [9]).

In (47), and in (48) and (49) for other even \( n \), we see that the inhomogeneity has support for \( p = 1 \), even though \( \tilde{f}_n \) has support only for non-negative \( p \). The star product with the Hamiltonian has spread out the distribution, in a typical quantum fashion, just slightly into the realm of negative \( p \). Thus our original expectation that we should be able to ignore all integer \( p < 0 \) was not quite correct.

### 5.4 \( \langle H \rangle \) from dual WFs

For imaginary Liouville QM the difference between (48) and (49) is
\[
\tilde{f}_n \ast H - \overline{H} \ast \tilde{f}_n = \begin{cases} 
2i n^2 (\frac{|n/2|+p)!}{(n/2+p)!} 4^{1+p} \sin (2x (1+p)) & \text{for odd } n, \text{ if } \lfloor n/2 \rfloor \geq p \geq 0 \\
4i n (\frac{|n/2|+p)!}{(n/2-p)!} 4^{1+p} \sin (2x (2+p)) & \text{for even } n > 0, \text{ if } \lfloor n/2 \rfloor \geq 1 + p \geq 0 
\end{cases}
\]
(50)

The peculiar structure of the RHS combines with that of the metric (30) to establish specifically for the Liouville model that the expectation of the Hamiltonian is real within the phase-space framework.

\[
0 = \frac{1}{2\pi} \oint_{x,p} \overline{R} \left( f_n \ast H - \overline{H} \ast \tilde{f}_n \right)
\]
(51)
This holds just because \(0 = \int_0^{2\pi} (\sin^2 x)^p \sin(2x(1+p)) \, dx\) as well as \(0 = \int_0^{2\pi} (\sin^2 x)^p \sin(2x(2+p)) \, dx\), for integer \(p \geq 0\), and because the vanishing of the metric for negative \(p\) conveniently eliminates the \(p = -1\) possibility for even \(n\).

Indeed, for all \(n\)

\[
n^2 = \frac{\sum_{x,p} \tilde{R}(x,p) \left( \tilde{f}_n(x,p) \star H \right)}{\sum_{x,p} \tilde{R}(x,p) \tilde{f}_n(x,p)} = \frac{\sum_{x,p} \tilde{R}(x,p) \left( \overline{\tilde{f}_n(x,p)} \right)}{\sum_{x,p} \tilde{R}(x,p) \tilde{f}_n(x,p)}
\]

(52)

despite the inhomogeneities in (48) and (49), because \(0 = \int_0^{2\pi} (\sin^2 x)^p \exp(2ix(1+p)) \, dx\) as well as \(0 = \int_0^{2\pi} (\sin^2 x)^p \exp(2ix(2+p)) \, dx\) for integer \(p \geq 0\). Again the vanishing of the metric for negative \(p\) eliminates the possibility of anguish at \(p = -1\).

However it must be said that (51) can also be established more generally by the method of combining (42) with the “Lone Star Lemma” [23] which allows us the option of inserting or removing a single \(\star\) from the phase space summand/integrand, hence to write

\[
\frac{1}{2\pi} \sum_{x,p} \tilde{R}(x,p) \left( \tilde{f}_n \star H \right) = \frac{1}{2\pi} \sum_{x,p} \tilde{R}(x,p) \star \tilde{f}_n \star H = \frac{1}{2\pi} \sum_{x,p} H \star \tilde{R}(x,p) \star \tilde{f}_n = \frac{1}{2\pi} \sum_{x,p} \tilde{R}(x,p) \left( \overline{\tilde{f}_n} \star H \right)
\]

(53)

6 A sesquilinear star product and bracket

There is a compelling formal way to express the entwining relation (42) between the dual metric and the Hamiltonian which suggests it would be appropriate to abbreviate quasi-hermitian to q-hermitian in the title of this paper. Certainly, the structure of (42) brings to mind the deformations of commutators and braiding relations such as occur in q-algebras and quantum groups [6]. To pursue that statement, let us rewrite the entwining relation as

\[
H \star \tilde{R}(x,p) = \tilde{R}(x,p) \star H \mathbb{K}
\]

(54)

where \(\mathbb{K}\) is the anti-linear operation of complex conjugation. Since \(\tilde{R}\) is in fact a real function, this may also be written as

\[
H \star \mathbb{K}\tilde{R}(x,p) = \tilde{R}(x,p) \star \mathbb{K}H
\]

(55)

By defining a “sesquilinear star product”

\[
\tilde{\otimes} \equiv \star \mathbb{K}
\]

(56)

\footnote{Otherwise the RHS of, say, (42), would be troublesome when multiplied by \((\sin x)^{2p}\big|_{p=-1}\) and integrated over \(x\), although not impossible to handle (cf. Cauchy’s principal value prescription).}
and a corresponding modification of the Moyal bracket \[14\], the entwining relation becomes simply:

\[
0 = \left[ H, \tilde{R} \right] \equiv H \circ \tilde{R} - \tilde{R} \circ H
\]

(57)

A form of sesquilinearity is evident, although left↔right ordering dependent, in the properties

\[
C \circ (\alpha A + \beta B) = (C \circ A) \alpha + (C \circ B) \beta = \bar{\alpha} (C \circ A) + \bar{\beta} (C \circ B)
\]

(58)

\[
(\alpha A + \beta B) \circ C = \alpha (A \circ C) + \beta (B \circ C) = (A \circ C) \bar{\alpha} + (B \circ C) \bar{\beta}
\]

(59)

for constants \( \alpha \) and \( \beta \). Also note that \( * \mathbb{K} \neq \mathbb{K} * \), rather

\[
\mathbb{K} = \mathbb{K} * \mathbb{K} = \exp \left( -\frac{i}{2} \frac{\partial}{\partial x} \frac{\partial}{\partial p} + \frac{i}{2} \frac{\partial}{\partial p} \frac{\partial}{\partial x} \right)
\]

(60)

This “star-bar” is again an associative but non-commutative product, like \[30\], and just amounts to flipping the sign of the deformation parameter, \( \hbar \) (\( \equiv 1 \) throughout this paper).

### 7 Other phase space distributions

We may also take star products of the various WFs. This leads in a routine way to a larger class of distributions on the phase space, although the details for the present circumstances are rather novel not only because the functions are periodic in \( x \) but also because \( \chi_n \neq \bar{\psi}_n \). This means that in general there is a third class of “hybrid” WFs involving mixed bilinears in the wave functions and their duals.

#### 7.1 Hybrid WFs

By direct calculation, with \( p \in \mathbb{Z} \), we find

\[
f_k (x, p) \ast \tilde{f}_n (x, p) = g_n (x, p) \delta_{k,n}
\]

(61)

\[
g_n (x, p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \psi_n (x - y) \chi_n (x + y) e^{2iyp} dy
\]

(62)

\[
g_n (x, p) = \frac{(-1)^{p-n}}{4^{p-n} (n-1)!} e^{2ipx} \sum_{k=0}^{[n/2]} \frac{(n-k-1)! (-1)^k}{4^{2k} k! (p+k)! (p+k-n)!} e^{4ikx}
\]

(63)

\footnote{For the less symmetric Voros product \( \triangleright \equiv \exp \left( i \frac{\partial}{\partial x} \frac{\partial}{\partial p} \right) \) (see the Appendix in \[21\]) an analogous construction is not quite so elegant, since in that case the metric would not be a real function.}
the latter from the series (3) and (5). So \( g_n(x,p) \neq 0 \) if and only if \( p \geq n - \lfloor n/2 \rfloor \), and thus has infinite support in \( p \) similar to \( f_n \). In particular, again for any phase-space point \((x,p) \in \mathbb{S}^1 \times \mathbb{Z}\),

\[
g_0(x,p) = \frac{(-1)^p}{4^p (p!)^2} e^{2ipx} \quad \text{for} \quad p \geq 0
\]

(64a)

\[
g_1(x,p) = \frac{(-1)^{p-1}}{4^{p-1} p! (p-1)!} e^{2i(p-1)x} \quad \text{for} \quad p \geq 1
\]

(64b)

\[
g_2(x,p) = \frac{(-1)^p}{4^p (p+1)! (p-1)!} \left( 16 \left( p^2 - 1 \right) e^{2i(p-2)x} - e^{2ipx} \right) \quad \text{for} \quad p \geq 1
\]

(64c)

etc. Without further calculation we also find

\[
\tilde{f}_k(x,p) \star f_n(x,p) = g_n(x,p) \delta_{k,n}
\]

(65)

\[
g_n(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \chi_n(x-y) \psi_n(x+y) e^{2iyp} dy
\]

(66)

as consequences of the previous relations, upon making use of \( \tilde{f}_n \star f_n = f_n \star \tilde{f}_n \). Summing and integrating over \( p \) and \( x \) the RHS of either (61) or (65) gives back (19). A sum over integer \( p \) suffices in this case. These hybrids also obey a form of the completeness relation that follows from the corresponding statement for wave functions and their duals.

\[
\frac{u}{w \minus z} = \sum_{n=0}^{\infty} A_n (w) \ J_n (z)
\]

(67)

However, the Wigner transform requires some regulation to give a precise meaning to \( \sum_{k=0}^{\infty} g_k(x,p) \).

Finally, we find the remaining simple star products

\[
g_k(x,p) \star g_n(x,p) = g_n(x,p) \delta_{k,n}
\]

(68)

\[
g_k(x,p) \star f_n(x,p) = f_n(x,p) \delta_{k,n}
\]

(69)

\[
\tilde{f}_k(x,p) \star g_n(x,p) = \tilde{f}_n(x,p) \delta_{k,n}
\]

(70)

The first of these is of the usual esthetic form that holds for non-hybrid WFs in familiar hermitian quantum mechanical systems (e.g. the free particle WFs satisfy \( e_k(x,p) \star e_n(x,p) = e_n(x,p) \delta_{k,n} \)). However, unlike in that non-hybrid hermitian situation, the hybrid \( g_n \) is not real. A direct consequence of the second of these star products is that all the \( g_n \) are right-\( \star \)-mapped by the dual metric into \( f_n \).

\[
g_n(x,p) \star \hat{R}(x,p) = \varepsilon_n f_n(x,p)
\]

(71)

\[9\] However, in other situations it may be necessary to sum over 
semi-integer \( p \), i.e. all \( p \) such that \( 2p \in \mathbb{Z} \). While \( e_n, f_n, \tilde{f}_n \), and \( g_n \) only have support for integer \( p \), there are “non-diagonal” variants of these distributions, namely \( e_{k,n}, f_{k,n}, \tilde{f}_{k,n} \), and \( g_{k,n} \) for \( k \neq n \) (see Appendix D), which have support for semi-integer \( p \) when \( k - n \) is an odd integer. In dealing with such nondiagonal cases, some care is required to respect the positivity of the metric.
This follows immediately from \[34\].

Once again, the form of the various star products of WFs can most easily be seen through the use of density operator methods, as in Appendix E. However, for devotees of phase-space methods, we make the following technical points. The results are obtained directly from the structure of the Wigner transforms \[11\], \[15\], and \[62\], and the orthogonality of the wave functions and their duals. For example, consider \[61\]. Applying the star product to each of the integral representations \[11\] and \[15\], and carrying out the requisite variable shifts on the integrands, the resulting coupled integrals

\[
\int_0^{2\pi} dy_1 \int_0^{2\pi} dy_2 \cdots
\]

can be split into uncoupled integrals

\[
\int_0^{4\pi} dy_1 + \int_0^{2\pi} dy_2 \cdots \times \int_0^{2\pi} dy_1 - \int_0^{2\pi} dy_2 \cdots
\]

upon making use of the periodicity of the integrands, hence the final result can be obtained. (Further details are in Appendix C.)

Moving on, we compute \(H \star g_k\) and \(g_k \star H\). The first of these is easily seen to be

\[ H \star g_n = n^2 g_n \quad (72) \]

but the other has an inhomogeneity.

\[ g_n \star H = n^2 g_n + \begin{cases} 
2nh_{n,-1} & \text{for } n \in \mathbb{N}_{\text{odd}} \\
\varepsilon_n h_{n,-2} & \text{for } n \in \mathbb{N}_{\text{even}}
\end{cases} \quad (73) \]

where the RHS involves non-diagonal versions of hybrid WFs, of the type indicated (see Appendix D and set \(\phi_l(x) = \exp ilx\)). Explicitly

\[
h_{n,l}(x,p) = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(x-y) e^{-il(x+y)} e^{2iyp} dy = \frac{1}{2^n} e^{2i(p-l)x} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k! (k+n)!} \delta_{2p-2l-2k}
\]

\[
= e^{2i(p-l)x} \frac{(-1)^{p-\frac{l+n}{2}}}{2^{2p-l} (2n)! (p-\frac{l+n}{2})! (p-\frac{l-n}{2})!} \quad (74)
\]

for integer \((p-\frac{1}{2}(l+n)) \geq 0\), but zero otherwise, upon using the series \[49\]. We note that the right action of \(H\) has spread the \(g_n\) distributions in momentum, through the effects of the inhomogeneities, and in particular has produced a contribution at \(p = -1\) from \(g_0\).

In similar language, we could have abbreviated the explicit results in \[48\] and \[49\] as

\[
\overline{H} \star \overline{f}_n = n^2 \overline{f}_n + \begin{cases} 
2n \overline{h}_{-1,n} & \text{for } n \in \mathbb{N}_{\text{odd}} \\
\varepsilon_n \overline{h}_{-2,n} & \text{for } n \in \mathbb{N}_{\text{even}}
\end{cases} \quad (75)
\]

\[
\overline{f}_n \star H = n^2 \overline{f}_n + \begin{cases} 
2n \overline{h}_{-1,n} & \text{for } n \in \mathbb{N}_{\text{odd}} \\
\varepsilon_n \overline{h}_{-2,n} & \text{for } n \in \mathbb{N}_{\text{even}}
\end{cases} \quad (76)
\]
where again the RHSs involve non-diagonal versions of hybrid WFs, of the type indicated (again see Appendix D and set $\phi_l(x) = \exp ilx$). Explicitly, upon using the series (5),

$$\tilde{h}_{l,n}(x,p) = \frac{1}{2\pi} \int_0^{2\pi} e^{il(x-y)} \chi_n(x+y) e^{2iyp} dy = 2^n n e^{2i(l-p)x} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n - k - 1)!}{4^k k!} \delta_{2p,l+n-2k}$$

for integer $(\frac{1}{2} (l + n) - p) \geq 0$ and $(\frac{1}{2} (l + n) - p) \leq \lfloor n/2 \rfloor$, but zero otherwise.

It is significant that the RHSs of (73), (75), and (76) only have support for $p \in \mathbb{Z}$, and so conform to our original expectation about momentum quantization. But in general, (74) and (77) do not vanish when $2p \in \mathbb{Z}$, where contributions at semi-integer $p$ can occur when $l + n$ is an odd integer. So, upon considering these more general, nondiagonal WFs, the phase space must be expanded to include all points $(x,p) \in S^1 \times \mathbb{Z}/2$.

### 7.2 More hybrid WFs

As anticipated in our abbreviated expressions for the various inhomogeneities, (74) and (77), we may continue the process of hybridizing WFs by constructing Wigner transforms of other pairs of functions. If one function is an imaginary Liouville eigenfunction while the other is a free particle solution, we are led to define the hybrid

$$h_n(x,p) = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(x-y) \phi_n(x+y) e^{2iyp} dy$$

where $\psi_n$ are the Bessels, and $\phi_n$ are free solutions, $2\pi$-periodic, and orthonormal in the usual sense.

$$-\frac{d^2}{dx^2} \phi_n(x) = n^2 \phi_n(x), \quad \frac{1}{2\pi} \int_0^{2\pi} \overline{\phi_k(x)} \phi_n(x) dx = \delta_{k,n}$$

Then by standard techniques we have

$$H \ast h_n(x,p) = n^2 h_n(x,p) = h_n(x,p) \ast p^2$$

$$\overline{h_n(x,p)} \ast \overline{H} = n^2 \overline{h_n(x,p)} = p^2 \ast \overline{h_n(x,p)}$$

as well as

$$h_k(x,p) \ast \overline{h_n(x,p)} = f_n(x,p) \delta_{k,n}$$

and so on. However, since the Bessels on the circle are not orthonormal, we do not have similar simple results for $\overline{h_n(x,p)} \ast h_k(x,p)$, although this does not really seem to matter in practice.
These $\star$ product results can be established without explicit forms for either the $h_n$ or the $f_n$, but it is perhaps useful to have such expressions in hand. Unlike the situation for the $f_n$, as given in (12), there is more freedom in the construction of the $h_n$ since we have a choice between taking $\phi_n(x)$ to be $e^{inx}$ or $e^{-inx}$, for $n \geq 0$, or some linear combination of the two. Explicit results depend on how we make this choice. Perhaps the simplest choice, conceptually, is to take only right- or left-moving plane waves for the $\phi_n$. For example, when $\phi_n$ is right-moving, then $\phi_n(x) = e^{-inx}$ for $n \geq 0$ is left-moving, and

$$h^R_n(x, p) = \frac{1}{2\pi} \int_0^{2\pi} J_n \left( e^{i(x-y)} \right) e^{-in(x+y)} e^{2ipy} dy = \frac{2^n (-1)^{p-n}}{4^p p! (p-n)!} e^{2i(p-n)x} \quad \text{for} \quad p \geq n$$

(83)

but vanishes for $p < n$. On the other hand, if we take only left-moving plane waves for the $\phi_n$, then $\phi_n(x) = e^{inx}$ for $n \geq 0$ is right-moving, and

$$h^L_n(x, p) = \frac{1}{2\pi} \int_0^{2\pi} J_n \left( e^{i(x-y)} \right) e^{in(x+y)} e^{2ipy} dy = \frac{(-1)^p}{4^p 2^n p! (p+n)!} e^{2i(p+n)x} \quad \text{for} \quad p \geq 0$$

(84)

but vanishes for $p < 0$.

In conjunction with the $h_n(x, p)$, we also define the duals

$$\tilde{h}_n(x, p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi_n(x-y) \chi_n(x+y) e^{2ipy} dy$$

(85)

where $\chi_n$ are the duals to the $\psi_n$. Various star products follow immediately.

$$\tilde{h}_n(x, p) \star H = n^2 \tilde{h}_n(x, p) + \frac{1}{2\pi} \int_0^{2\pi} \phi_n(x-y) e^{2ipy} \begin{cases} \varepsilon_n e^{2i(x+y)} & \text{for even } n \geq 0 \\ 2n e^{i(x+y)} & \text{for odd } n > 0 \end{cases}$$

(86)

$$p^2 \star \tilde{h}_n(x, p) = n^2 \tilde{h}_n(x, p)$$

(87)

$$f_k(x, p) \star \tilde{h}_n(x, p) = \tilde{h}_n(x, p) \delta_{k,n}$$

(88)

$$f_k(x, p) \star h_n(x, p) = h_n(x, p) \delta_{k,n}$$

(89)

$$h_k(x, p) \star \tilde{h}_n(x, p) = g_n(x, p) \delta_{k,n}$$

(90)

$$\tilde{h}_k(x, p) \star \tilde{h}_n(x, p) = f_n(x, p) \delta_{k,n}$$

(91)

$$\tilde{h}_k(x, p) \star h_n(x, p) = e_n(x, p) \delta_{k,n}$$

(92)
where the last of these brings us back to the plane wave WFs we introduced initially, in \( [19] \). Also, all the 
\( h_n \) and \( \tilde{h}_n \) are respectively left- and right-\( \star \)-mapped by the dual metric into the \( h_n \) and their conjugates.

\[
\tilde{R}(x,p) \star h_n(x,p) = \varepsilon_n h_n(x,p) \\
\tilde{h}_n(x,p) \star \tilde{R}(x,p) = \varepsilon_n h_n(x,p)
\]  
(93)  
(94)

This follows immediately from \([53], [58]\), and \([53]\).

More explicit results again depend on how we choose the free particle solutions. Choosing right-moving plane waves, with \( n \geq 0 \) as above, we have

\[
\tilde{h}_n^R(x,p) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\eta(x-y)} \chi_n(x+y) e^{2iyp} \, dy = 2^{2p-n}n! e^{2i(n-p)x} \frac{(p-1)!}{(n-p)!}
\]  
(95)

for \( 0 \leq n-p \leq |n/2| \), but zero otherwise. Choosing left-moving plane waves, with \( n \geq 0 \), we have

\[
\tilde{h}_n^L(x,p) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\eta(x-y)} \chi_n(x+y) e^{2iyp} \, dy = 2^{2p+n}n! e^{-2i(n+p)} \frac{(n+p-1)!}{(-p)!}
\]  
(96)

for \( 0 \leq -p \leq |n/2| \), but zero otherwise. For these specific choices we compute the right action of \( H \) to be

\[
\tilde{h}_n^R(x,p) \star H = n^2 \tilde{h}_n^R(x,p) + \begin{cases} 
\varepsilon_n e^{i(n+1)x} \delta_{2p,n-1} & \text{for even } n \geq 0 \\
2ne^{i(n+1)x} \delta_{2p,n-1} & \text{for odd } n > 0
\end{cases}
\]  
(97)

\[
\tilde{h}_n^L(x,p) \star H = n^2 \tilde{h}_n^L(x,p) + \begin{cases} 
\varepsilon_n e^{i(-n+1)x} \delta_{2p,n-1} & \text{for even } n \geq 0 \\
2ne^{i(-n+1)x} \delta_{2p,n-1} & \text{for odd } n > 0
\end{cases}
\]  
(98)

The inhomogeneities here can be identified with nondiagonal free particle WFs, \( e^{R}_{n,-2} \& e^{R}_{n,-1} \), and \( e^{L}_{n,2} \& e^{L}_{n,1} \). It is significant that the inhomogeneities have support only for integer \( p \)\(^{10}\). But note that this includes, e.g., \( p = -1 \) when \( n = 0 \) even though only \( \tilde{h}_0^{R,L}(x,p = 0) \neq 0 \). Once again the star product of \( H \) with the \( \tilde{h}_n(x,p) \) has spread the distributions on the phase space to give contributions outside their initial momentum support through the effects of the inhomogeneities. (For \( \tilde{h}_{n>0}^{L,L}(x,p) \) cases, the initial support was only for negative momentum, and the star action of \( H \) also gives only negative momentum contributions, but spread out nonetheless.)

\(^{10}\)Although in general the nondiagonal free WFs have support at semi-integer \( p \) when \( l+n \) is odd: \( e^{R}_{n,l}(x,p) = e^{i(n-l)x} \delta_{2p,n+l} \) and \( e^{L}_{n,l}(x,p) = e^{i(-n+l)x} \delta_{2p,n-l} \). So once again, considering these more general, nondiagonal WFs, the phase space must be expanded to include all points \( (x,p) \in S^1 \times \mathbb{Z}/2 \).
8 Direct solutions of the dual metric equation

Basic solutions to (42), or equivalently (31), are obtained by separation of variables. We find two classes of solutions. The first of these is non-singular for all real \( p \), although there are zeroes for negative integer \( p \).

\[
\tilde{R}(x,p;s) = \frac{1}{s^p \Gamma (1 + p)} \exp \left( -\frac{1}{2} s \cos 2x \right)
\]

(99)

For real \( s \) this is real and positive definite on the positive momentum half-line. The other class of solutions has poles for all positive integer \( p \).

\[
\tilde{R}_{\text{other}}(x,p;s) = \frac{\Gamma (-p)}{s^p} \exp \left( \frac{1}{2} s \cos 2x \right)
\]

(100)

For later use we also compute the left- and right-actions of \( H \) and \( \overline{H} \) on \( \tilde{R}(x,p;s) \).

\[
H \ast \tilde{R}(x,p;s) = \left( p^2 - \frac{1}{8} s^2 + \left( p - \frac{1}{2} \right) s \cos 2x + \frac{1}{8} s^2 \cos 4x \right) \tilde{R}(x,p;s) = \tilde{R}(x,p;s) \ast \overline{H}
\]

(101)

Linear combinations of (99) and/or (100) are also solutions. This permits us to build a “composite” metric from members of the first class by using a contour integral representation. For \( t > 0 \)

\[
\tilde{R}(x,p;s,t) \equiv \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \tilde{R}(x,p;st) e^{\tau} d\tau
\]

(102)

The contour begins at \(-\infty\), with \( \arg \tau = -\pi \), proceeds below the real \( \tau \) axis towards the origin, loops in the positive, counterclockwise sense around the origin (hence the \( (0+) \) notation), and then continues above the real \( \tau \) axis back to \(-\infty\), with \( \arg \tau = +\pi \). By construction, \( \tilde{R}(x,p;s,t) \) actually depends only on the ratio \( t/s \). Evaluation of the contour integral gives

\[
\tilde{R}(x,p;s,t) = \left( \frac{t}{s} - \frac{1}{2} \cos 2x \right)^p \frac{1}{(\Gamma (1 + p))^2}
\]

(103)

where we have made use of

\[
\frac{1}{\Gamma (1 + p)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \tau^{-p-1} e^\tau d\tau
\]

(104)

From \( \tilde{R}(x,p;s,t) \) we therefore recover our original dual metric by setting \( s = 2t \).

\[
\tilde{R}(x,p;2t,t) = \left( \frac{\sin^2 x}{(\Gamma (p + 1))^2} \right)^p = \tilde{R}(x,p)
\]

(105)

9 The \( \ast \) root of the dual metric

9.1 \( \tilde{S} \) as a direct solution of an entwining equation

We look for an equivalence between the Liouville \( H = p^2 + e^{2ix} \) and the free particle \( \overline{H} = p^2 \) as given by solutions of

\[
\tilde{S}(x,p)^{-1} \ast H \ast \tilde{S}(x,p) = p^2
\]

(106)
or, barring complete invertibility, as solutions of the entwining equation (cf. \[105\] below)

\[ H \ast \tilde{S}(x,p) = \tilde{S}(x,p) \ast p^2 \]  

(107)

This is again a special case of the ultra-local two star equation as given in \[7\]. For the Liouville–free-particle case, this amounts to an equation similar to that for \( \tilde{R} \), but inherently complex.

\[ 2ip\partial_x \tilde{S}(x,p) = e^{2ix} e^{-\partial_p} \tilde{S}(x,p) = e^{2ix} \tilde{S}(x,p-1) \]  

(108)

Once again solutions are easily found through the use of a product ansatz. For any value of a parameter \( s \), we find two immediate solutions:

\[ \tilde{S}(x,p; s) = \frac{1}{s^p\Gamma(1+p)} \exp\left(-\frac{1}{4}s\exp(2ix)\right) \]  

(109)

\[ \tilde{S}_{\text{other}}(x,p; s) = \frac{1}{s^p\Gamma(-p)} \exp\left(\frac{1}{4}s\exp(2ix)\right) \]  

(110)

The first of these is a “good” solution for \( p \in (-1, \infty) \), say, while the second is good for \( p \in (-\infty, 0) \), thereby providing an overlapping pair of solutions that cover the entire real \( p \) axis, but not smoothly or continuously. These solutions could always be multiplied by periodic functions of \( p \), \( \exp(2i\pi np) \) for \( n \in \mathbb{Z} \), but for integer-valued \( p \) (which is our primary interest) this has no effect.

The two solutions for \( \tilde{S} \) are brought closer in appearance by shifting \( s \to s\exp(\pm i\pi/2) \) and using the reflection relation for the \( \Gamma \):

\[ \Gamma(-p) = \frac{-\pi}{\Gamma(1+p)\sin\pi p} \]  

Thus we may take as our two basic solutions the alternate forms

\[ \tilde{S}(x,p; s\exp(+i\pi/2)) = e^{-i\pi p/2} \frac{1}{s^p\Gamma(1+p)} \exp\left(-\frac{i}{4}s\exp(2ix)\right) \]  

(111)

\[ \frac{i}{\pi} \tilde{S}_{\text{other}}(x,p; s\exp(-i\pi/2)) = e^{i\pi(1+p)/2} \left(\frac{1}{\sin\pi p}\right) \frac{1}{s^p\Gamma(1+p)} \exp\left(-\frac{i}{4}s\exp(2ix)\right) \]  

(112)

\[ = e^{i\pi(1+2p)/2} \left(\frac{1}{\sin\pi p}\right) \tilde{S}(x,p; s\exp(+i\pi/2)) \]  

(113)

which now coincide at the point \( p = -1/2 \), since we have also adjusted the normalization and phase of the 2nd solution. However, their derivatives with respect to \( p \) still do not match at \( p = -1/2 \), due to

\[ \frac{\partial}{\partial p} \left(\frac{-e^{i\pi(1+2p)/2}}{\sin(\pi p)}\right) \bigg|_{p=-1/2} = i\pi. \]

Taking all this into account, and exploiting the linearity of the equation, \( \widetilde{\text{III}} \), a more general solution would be

\[ \tilde{S}_{\text{general}}(x,p) = \sum_n \int ds \left( c_n(s) e^{-i\pi p/4} + d_n(s) e^{i\pi(1+p)/4} \frac{-1}{\sin\pi p} \right) \times e^{2\pi i np} \frac{1}{s^p\Gamma(1+p)} \exp\left(-\frac{i}{4}s\exp(2ix)\right) \]  

(114)

For integer-valued \( p \) the sum over \( n \) may be omitted.
9.2 The dual metric as an absolute ★ square

Each such solution for \( \tilde{S} \) leads to a candidate real metric, given by

\[
\tilde{R} = \tilde{S} \star \tilde{S}
\]  

To verify this, we note that the entwining equation for \( \tilde{S} \), and its conjugate \( \tilde{S} \),

\[
H \star \tilde{S} = \tilde{S} (x, p) \star p^2, \quad p^2 \star \tilde{S} = \tilde{S} \star H
\]

may be combined with the associativity of the star product to obtain

\[
H \star \tilde{S} \star \tilde{S} = \tilde{S} (x, p) \star p^2 \star \tilde{S} = \tilde{S} \star \tilde{S} \star H
\]

Thus the form in (115) yields a solution to (42). We need to work out a representative \( \tilde{S} \star \tilde{S} \) star product to understand the relation to the previous solutions for \( \tilde{R} \).

We do this for the first form of the basic \( \tilde{S} \) solutions, (109), using the standard integral representation for \( 1/\Gamma \). We find a result that coincides with one of the composite dual metrics for \( \tilde{R} \).

\[
\tilde{S} (x, p; s) \star \tilde{S} (x, p; s) = \left( 1 + \frac{1}{16} s^4 - \frac{1}{2} s^2 \cos 2x \right)^p \frac{1}{s^{2p} (1 + p)^2} = \tilde{R} \left( x, p; s^2, 1 + \frac{1}{16} s^4 \right)
\]

By choosing \( s = \pm 2 \), we again obtain the original dual metric.

\[
\tilde{S} (x, p; \pm 2) \star \tilde{S} (x, p; \pm 2) = \tilde{R} (x, p; 4, 2) = \frac{(\sin^2 x)^p}{(\Gamma (p + 1))^2} = \tilde{R} (x, p)
\]

This provides a greater appreciation of the information contained in \( \tilde{S} \), and motivates us to consider an alternative construction of such ★ roots of \( \tilde{R} \).

9.3 \( \tilde{S} \) as a sum of hybrid WFs

The basic ideas here are essentially the same as used in the construction of \( \tilde{R} \) as a sum of \( f_n \), only the bilinears appearing in the Wigner transforms involve two different types of functions: One wave function is an imaginary Liouville eigenfunction while the other is a free particle solution, precisely the \( h_n (x, p) \) defined earlier in (78).

We form the sums

\[
\tilde{S} (x, p) = \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} h_n (x, p), \quad \overline{S} (x, p) = \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} \overline{h_n (x, p)}
\]
Then from (S0) and (S1) we immediately obtain

\[
H \star \tilde{S}(x, p) = \tilde{S}(x, p) \ast p^2, \quad p^2 \star \tilde{S}(x, p) = \tilde{S}(x, p) \ast \overline{H}
\]

and, upon using (S2), we also obtain \( \tilde{R} \) as the absolute star-square of \( \tilde{S} \).

\[
\tilde{S}(x, p) \ast \overline{\tilde{S}(x, p)} = \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} h_n(x, p) \ast \sum_{k=0}^{\infty} \sqrt{\varepsilon_k} h_k(x, p) = \sum_{n=0}^{\infty} \varepsilon_n f_n(x, p) = \tilde{R}(x, p)
\]

However, since the Bessels on the circle are not orthonormal, we do not have a similar relation for \( \tilde{S}(x, p) \ast \overline{\tilde{S}(x, p)} \). While these results can be established without explicit forms for the \( h_n(x, p) \) and for \( \tilde{S}(x, p) \), it is perhaps useful to have such expressions in hand.

Like the individual hybrid WFs, explicit results for the sum depend on how we choose the free particle solutions. In particular, we may take the hybrid WFs built from right-moving and left-moving plane waves in (S3) and (S4). This gives rise to the phase-space equivalent of the “chiral kernel” in [S].

\[
\tilde{S}_R(x, p) = \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} h_n^R(x, p) = \frac{(-1)^p}{4p!} e^{2ipx} \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} \left( \frac{-1}{p!} \right)^n 2^n e^{-2inx} = \left(1 - \sqrt{2} \right) \frac{(-1)^p}{2p^2} e^{2ipx} + \sqrt{\varepsilon} \frac{1}{2p^2} e^{\frac{1}{2}e^{2ix}} \Gamma \left( p + 1, -\frac{1}{2}e^{2ix} \right)
\]

where the incomplete \( \Gamma \) function has made an appearance (e.g. see [I]).

\[
\sum_{k=0}^{p} \frac{1}{k!} z^k = e^z \int_0^1 \frac{t^p}{(1-t)^{p+1}} dt = \frac{\Gamma(p+1, z)}{\Gamma(p+1)}
\]

We leave it as an exercise for the reader to use the properties of the incomplete \( \Gamma \) to check directly that \( \tilde{S}_R(x, p) \ast \overline{\tilde{S}_R(x, p)} = \tilde{R}(x, p) \).

Similarly, we have

\[
\tilde{S}_L(x, p) = \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} h_n^L(x, p) = \frac{(-1)^p}{4p!} e^{2ipx} \sum_{n=0}^{\infty} \sqrt{\varepsilon_n} \frac{1}{2^n (p+n)!} e^{2inx} = \frac{(-1)^p}{4p^2} e^{2ipx} + \sqrt{\varepsilon} \frac{(-1)^p}{2p^2} e^{\frac{1}{2}e^{2ix}} \left( \Gamma(p+1) - \Gamma(p+1, -\frac{1}{2}e^{2ix}) \right)
\]

Comparison with the results in [S], §4, Equn’s (67) and (69), shows that \( \tilde{S}_{R,L}(x, p) \) are essentially Fourier transforms with respect to one variable of a particular combination of Lommel’s functions of two variables.

Who knew?

Alternatively, we could take the free particle solutions to be non-chiral: 1 for \( n = 0 \) but for \( n > 0 \), \( (e^{inx} - (-1)^n e^{inx}) / \sqrt{2} \). That is

\[
\phi_n(x) = \frac{\sqrt{\varepsilon_n}}{2} (e^{inx} - (-1)^n e^{inx})
\]

(128)
which are properly normalized, as in (79). In this case we would be led to the Wigner transform of the well-known generating function.

\[
e^{iz \sin x} = \sum_{n=-\infty}^{\infty} J_n(z) e^{inx} = J_0(z) + \sum_{n=1}^{\infty} J_n(z) (e^{inx} + (-1)^n e^{-inx})
\]

(129)

Thus we obtain a remarkably simple result for the non-chiral kernel.

\[
\tilde{S}_{NC}(x,p) \equiv \frac{1}{2\pi} \int_{0}^{2\pi} e^{ie^{i(x-y)} \sin(x+y)} e^{2ipy} dy = \left(\frac{-1}{2}\right)^p \exp\left(\frac{1}{2}e^{2ix}\right)
\]

(131)

which we recognize as a particular case of our basic solutions, (109).

\[
\tilde{S}_{NC}(x,p) = \tilde{S}(x,p; s = -2)
\]

(132)

The selected non-chiral free particle solutions have WFs with symmetric momentum support.

\[
e_n(x,p) = \frac{\varepsilon_n}{4} (\delta_{p,n} + \delta_{p,-n} + 2 (-1)^n \cos(2nx) \times \delta_{p,n})
\]

(133)

The corresponding hybrids are linear combinations of those in (83) and (84), as given by

\[
h_{NC}^L(x,p) = \frac{\sqrt{\varepsilon_n}}{2} \left( h_n^L(x,p) + (-1)^n h_n^R(x,p) \right)
\]

\[
h_{NC}^R(x,p) = \frac{\sqrt{\varepsilon_n}}{2} \left\{ \begin{array}{ll}
\frac{(-1)^p}{4^2 p!(p+n)!} e^{2i(p+n)x} & \text{for } n > p \\
\frac{(-1)^p}{4^2 p!(p+n)!} e^{2i(p-n)x} + (-1)^n \frac{2^n (-1)^{p-n}}{4^p p!(p-n)!} e^{2i(p-n)x} & \text{for } p \geq n
\end{array} \right.
\]

(134)

and vanish for \(p < 0\). All this leads us once again to the expected relation (122).

10 Meanwhile, back at the metric

We might also wish to systematically determine all metrics, rather than dual metrics, through which the Liouville Hamiltonian is rendered hermitian. This is perhaps a more conventional problem to attack in the framework of quasi-hermitian theories [18, 16, 17]. That is to say, we seek all solutions to

\[
H^* R = R^* H
\]

(135)

for real functions \(R(x,p) = R(x,p)\). (This should be compared to (42) given above.) In this context it is somewhat repetitious but perhaps instructive to go through a few details omitted in the previous two sections, including some forays into calculational cul-de-sacs.
10.1 $R$ as a formal sum of the dual WFs (Not!)

In parallel to the previous construction of the dual metric as a sum of WFs, \( [34] \), we might try

\[
Q(x,p) = \sum_{k=0}^{\infty} \varepsilon_k \tilde{f}_k(x,p)
\]

which gives, at least formally, the expected phase-space orthogonality relation

\[
\varepsilon_n = \frac{1}{2\pi} \int_{x,p} Q(x,p) f_n(x,p)
\]

However, \( Q \) does not satisfy the homogeneous equation \([35]\) due to the inhomogeneities resulting when \( H \) and \( \mathbb{T} \) act on the individual \( \tilde{f}_k \). Moreover, the best we can do with the sum \([36]\), so far, is to interpret it as an asymptotic series related to the Wigner transform of an integral whose asymptotic expansion is a formal generating function for the dual wave functions. Rather than pursue this here, we turn to direct solutions of the differential-difference equation corresponding to \([35]\).

10.2 Solving directly for $R$

We find the following left- and right-sided star products, and their difference.

\[
\mathbb{H} \star R(x,p) = (p^2 + e^{-2ix}) \star R(x,p) = \left( \left( p - \frac{i}{2} \partial_x \right)^2 + e^{-2ix} \partial_p \right) R(x,p)
\]

\[
R(x,p) \star H = R(x,p) \star (p^2 + e^{2ix}) = \left( \left( p + \frac{i}{2} \partial_x \right)^2 + e^{2ix} \partial_p \right) R(x,p)
\]

\[
R(x,p) \star H - \mathbb{H} \star R(x,p) = 2ip \partial_x R(x,p) + 2i \sin(2x) R(x,p + 1)
\]

So then, solving \([35]\) amounts to solving the linear differential-difference equation\(^{11}\)

\[
-p \partial_x R(x,p) = \sin(2x) R(x,p + 1)
\]

This should be compared to \([31]\) given above, which it becomes upon letting \( p \to -p \). (Note that \( R \) actually corresponds to \( \tilde{R}^{-1} \) in that earlier discussion.) This is immediately solved upon assuming a product form, \( R(x,p) = q(x) r(p) \). There are two distinct sets of solutions corresponding to positive and negative constants of separation, \( \pm s \).

The first one-parameter \( (s) \) set of solutions is

\[
R(x,p,s) = s^p \Gamma(p) \exp \left( \frac{1}{2} s \cos(2x) \right)
\]

\(^{11}\)This first-order equation becomes a second-order equation if the Voros product is used. That is, if we demand \( \mathbb{H} \triangleleft R = R \triangleleft H \) then we have to solve: \( \left( \partial_x^2 - 2ip \partial_x - \exp(2ix) \right) R(x,p) + \exp(-2ix) R(x,p + 2) = 0 \). Moreover, the solutions of this second-order equation are not real. So we prefer to use \( \star \) and not \( \triangleleft \).
up to an overall multiplicative constant. Identification with the previous dual metric basic solution is given by $R_{\text{other}}(x, -p; s) = R(x, p; s)$. Also, the duplication formula $\Gamma(p) = \frac{2^p \sqrt{\pi}}{\Gamma(\frac{1}{2} p)} \Gamma\left(\frac{1}{2} + \frac{1}{2} p\right)$ gives

$$R(x, p; s) = \frac{(2s)^p}{\sqrt{4\pi}} \Gamma\left(\frac{1}{2} p\right) \Gamma\left(\frac{1}{2} + \frac{1}{2} p\right) \exp\left(\frac{1}{2} s \cos(2x)\right)$$

(143)

for whatever that’s worth\(^{12}\) while the reflection relation $\Gamma(p) = \frac{\pi}{\Gamma(1-p) \sin \pi p}$ gives

$$R(x, p; s) = \frac{\pi s^p}{\Gamma(1-p) \sin \pi p} \exp\left(\frac{1}{2} s \cos(2x)\right)$$

(144)

For real $s$, the solutions (142) are hermitian and positive-definite functions of real variables $x, p$ so long as $p > 0$. However, $R(x, p; s)$ is not bounded either on the negative $p$ half-line, or on the positive $p$ half-line for any real $s$ (cf. Stirling’s approximation). Otherwise, $(-1)^{k+1} R(x, p)$ is hermitian and positive-definite for $-k - 1 < p < -k$, where $k = 0, 1, 2, \cdots$.

Now we know from previous considerations that for $n > 0$ the periodic WFs $f_n$ have support only for positive $p$, so (142) would seem to be a preferred set of solutions, except for the ground state: $f_0$ has support at $p = 0$. This issue must still be addressed.

Another set of solutions is given by the form

$$R_{\text{other}}(x, p; s) = \frac{s^p}{\Gamma(1-p)} \exp\left(-\frac{1}{2} s \cos(2x)\right)$$

(145)

Identification with the previous dual metric basic solution is given by $\tilde{R}(x, -p; s) = R_{\text{other}}(x, p; s)$. The duplication $\frac{1}{\Gamma(1-p)} = \frac{2^p \sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{1}{2} p) \Gamma(1 - \frac{1}{2} p)}$ and reflection relations $\frac{1}{\Gamma(1-p)} = \frac{\Gamma(p) \sin \pi p}{\pi}$ now give

$$R_{\text{other}}(x, p; s) = \frac{(2s)^p \sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{1}{2} p\right) \Gamma\left(1 - \frac{1}{2} p\right)} \exp\left(\frac{1}{2} s \cos(2x)\right)$$

(146)

$$R_{\text{other}}(x, p; s) = \frac{s^p \Gamma(p) \sin \pi p}{\pi} \exp\left(-\frac{1}{2} s \cos(2x)\right)$$

(147)

again for whatever that’s worth. For real $s$, these other solutions are hermitian and positive-definite functions of real variables $x, p$ so long as $p < 1$. Otherwise, $(-1)^k R_{\text{other}}(x, p; s)$ is hermitian and positive-definite for $k < p < k + 1$, where $k = 1, 2, \cdots$. Also, $R_{\text{other}}$ is now bounded for all real $p$.

An interesting problem now is to find the real square-root of either $R(x, p; s)$, or $R_{\text{other}}(x, p; s)$. This is not just $s^{p/2} \sqrt{\Gamma(p)} e^{\frac{1}{2} s \cos(2x)}$, say, since the square-root must be taken in a $*$ sense. That is, we seek a real $S$ such that

$$R(x, p; s) = S(x, p; s) \ast S(x, p; s)$$

(148)

\(^{12}\)These solutions immediately call to mind the Mellin transforms of Bessel functions, $J_\nu(p) = \frac{\Gamma\left(\frac{1}{2} \nu\right)}{2^{\nu} \Gamma\left(\frac{1}{2} + \frac{1}{2} \nu\right)} J_\nu(p)$, which are solutions to the Liouville energy eigenvalue problem in momentum space, $(p^2 + \frac{1}{2} \nu^2) J_\nu(p) = \nu^2 J_\nu(p)$.\]
Alternatively, since $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$, this would be easy since
\[
\exp(a \cos(2x) + bp) = \exp \left( \frac{a}{2 \cosh(b/2)} \cos(2x) + \frac{b}{2} p \right) \ast \exp \left( \frac{a}{2 \cosh(b/2)} \cos(2x) + \frac{b}{2} p \right)
\]
But the presence of the $\Gamma$ function makes the problem a little more challenging.

Recall for $p > 0$, $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$ while for all $p$ there is again the contour integral representation
\[1 \over \Gamma(1-p) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{p-1}e^{t}dt\]
The contour here is the same as in $[102]$. Thus
\[
R_{\text{other}}(x, p; s) = \int_{-\infty}^{(0+)} t^{p-1} \exp \left( t + ps - \frac{1}{2} e^{2x} \cos(2x) \right) dt
\]
\[
= \int_{-\infty}^{(0+)} \exp \left( e^{ln t} + p \ln t + ps - \frac{1}{2} e^{2x} \cos(2x) \right) d\ln t
\]
Star composition of two such integrands is now possible, as in $[149]$, although we then have to deal with a double integral. But taking the square-root in this approach is not transparent.

From another perspective, the problem involves computation of either
\[
\Gamma(p) \ast f(x) = \int_0^\infty t^{p-1} \ast f(x) e^{t}dt = \int_0^\infty \exp(p \ln t) \ast f(x) e^{t}d\ln t
\]
\[
= \int_0^\infty \exp(p \ln t) f \left( x - \frac{1}{2} i \ln t \right) e^{t}d\ln t = \int_{-\infty}^{(0+)} t^{p-1}e^{t}f \left( x - \frac{1}{2} i \ln t \right) dt
\]
or
\[
\frac{1}{\Gamma(1-p)} \ast f(x) = \int_{-\infty}^{(0+)} t^{p-1} \ast f(x) e^{t}dt = \int_{-\infty}^{(0+)} \exp(p \ln t) \ast f(x) e^{t}d\ln t
\]
\[
= \int_{-\infty}^{(0+)} \exp(p \ln t) f \left( x - \frac{1}{2} i \ln t \right) e^{t}d\ln t = \int_{-\infty}^{(0+)} t^{p-1}e^{t}f \left( x - \frac{1}{2} i \ln t \right) dt
\]
Alternatively, since $\Gamma(p) \ast e^{ikx} = e^{ikx} \Gamma(p + \frac{1}{2} k)$, then if $f(x) = \int e^{ikx}F(k) dk$ the two computations become
\[
\Gamma(p) \ast f(x) = \int \Gamma(p) \ast e^{ikx}F(k) dk = \int e^{ikx} \Gamma(p + \frac{1}{2} k) F(k) dk
\]
\[
\frac{1}{\Gamma(1-p)} \ast f(x) = \int \frac{1}{\Gamma(1-p)} \ast e^{ikx}F(k) dk = \int e^{ikx} \frac{F(k)}{\Gamma(1-p - \frac{1}{2} k)} dk
\]
We also note that
\[
\exp(a \cos 2x) = \sum_{n=-\infty}^{\infty} I_n(a) e^{2inx}, \quad I_n(a) = \frac{1}{\pi} \int_0^\pi e^{a \cos \theta} \cos(n \theta) d\theta
\]
with $I_n(a) = I_{-n}(a)$ and $I_n(-a) = (-1)^n I_n(a)$. Therefore we may write, as least formally, the ordinary product in $R$ as a star product
\[
\Gamma(p) \ast \exp(a \cos 2x) = \sum_{n=-\infty}^{\infty} I_n(a) \Gamma(p) e^{2inx} = \sum_{n=-\infty}^{\infty} I_n(a) \frac{\Gamma(p)}{\Gamma(p + n)} \Gamma(p) \ast e^{2inx} = \Gamma(p) \ast \sum_{n=-\infty}^{\infty} \frac{I_n(a)}{(p)_n} e^{2inx}
\]
where the Pochhammer symbol is \((p)_n = (p)(p+1)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p)\). Note the zeroes in the summand for integer \(p > 0\) when \(n \leq -p\). Unfortunately, this is not of much use without a closed form for the sum. This would appear to be one of the aforementioned cul-de-sacs, so let us try a different route, parallel to that used previously to construct \(\widetilde{S}\).

### 10.3 Solving for \(\sqrt{R}\)

We look for an equivalence between the Liouville \(H = p^2 + e^{2ix}\) and the free particle \(\mathbb{H} = p^2\) as given by

\[ \mathbb{H} = S \ast H \ast S^{-1} \]

That is to say, in phase-space we wish to solve the linear equation (cf. (107) above)

\[ p^2 \ast S = S \ast H \]

This is again a special case of the ultra-local two star equation as given in [7]. For the Liouville–free-particle case, this again amounts to an equation similar to that for \(R\), but inherently complex.

\[ -2ip\partial_x S(x,p) = e^{2ix}e^{\partial_p}S(x,p) = e^{2ix}S(x,p+1) \]

which becomes the previous (108) upon \(p \to -p\). Once again solutions are easily found through the use of a product ansatz. For any value of the parameter \(s\) we have two immediate solutions:

\[ S(x,p;s) = sp\Gamma(p)\exp\left(\frac{1}{4}s\exp(2ix)\right) \]

\[ S_{\text{other}}(x,p;s) = \frac{sp}{\Gamma(1-p)}\exp\left(-\frac{1}{4}s\exp(2ix)\right) \]

The first of these is a good solution for \(p \in (0,\infty)\), say, while the second is good for \(p \in (-\infty,1)\), thereby providing a solution for the entire real \(p\) axis, albeit a discontinuous one. The relation to the previous dual solutions is obviously

\[ S(x,p;s) = \widetilde{S}_{\text{other}}(x,-p;s) , \quad S_{\text{other}}(x,p;s) = \widetilde{S}(x,-p;s) \]

As in the dual situation, these solutions could always be multiplied by periodic functions of \(p\), \(\exp(2i\pi np)\) for \(n \in \mathbb{Z}\), but again, for integer-valued \(p\) this has no effect.

The two solutions for \(S\) are brought closer in appearance by shifting \(s \to se^{\pm i\pi/2}\) and using the reflection
relation for the $\Gamma$. Thus we may take as our two basic solutions the alternate forms

$$S(x,p; s \exp (+i\pi/2)) = e^{i\pi p/2} s \exp (2ix)$$ (164)

$$\frac{i}{\pi} S_{\text{other}}(x,p; s \exp (-i\pi/2)) = e^{i\pi(1-p)/2} s \exp (2ix)$$ (165)

$$= e^{i\pi(1-2p)/2} \sin (\pi p) \Gamma (p) \exp (i 4 s \exp (2ix))$$ (166)

which now coincide at the point $p = 1/2$, since we have also rescaled the 2nd solution. However, their derivatives with respect to $p$ still do not match at $p = 1/2$, due to

$$\left. \frac{\partial}{\partial p} (e^{i\pi(1-2p)/2} \sin (\pi p)) \right|_{p=1/2} = -i\pi.$$

Taking all this into account, and exploiting the linearity of the equation, a more general solution would be

$$S_{\text{general}}(x,p) = \sum_n \int ds \left( c_n(s) e^{i\pi p/2} + d_n(s) e^{i\pi(1-p)/2} \sin (\pi p) \right) \times e^{2\pi i n p} s \exp (2ix) \Gamma (p) \exp (i 4 s \exp (2ix))$$ (167)

Once again, for integer-valued $p$ the sum over $n$ may be omitted.

Since these solutions for $S$ are complex, a suitable real metric would now be given by

$$R = \bar{S} * S$$ (168)

and as before we have $\bar{H} * R = R * H$ consistent with $S * H * S^{-1} = H = \bar{H} = \bar{S}^{-1} * \bar{H} * \bar{S}$. For completeness, we extend this to a simple theorem that can be used to construct other composite solutions for the metric through multiple star products:

**[Lemma]** Any solution of (159), i.e. (160), gives a real solution of (135), i.e. (141), namely $R \equiv \bar{S} * S$. Moreover, appropriately ordered odd star products of any solutions to (159), (160), and the complementary equation (42) are also solutions to (141) and (160). For example, $\bar{H} * R_1 * \bar{R}_2 * R_3 = R_1 * \bar{R}_2 * R_3 * H$. Similarly, $H * S_1 * \bar{R}_2 * R_3 = S_1 * \bar{R}_2 * R_3 * H$.

We need to work out a star product at least for a particular case just to understand the relation to the previous solutions for $R$. So we do this for the first form of the solutions, (161), using the standard integral representation for $\Gamma (p)$. We find a composite metric

$$R = \bar{S} (x,p; s) * S (x,p; s)$$

$$= \left( 1 - \frac{1}{4} s^2 e^{2ix} \right)^{-p} \left( 1 - \frac{1}{4} s^2 e^{-2ix} \right)^{-p} s^{2p} \Gamma^2 (p)$$

$$= \left( 1 + \frac{1}{16} s^4 - \frac{1}{2} s^2 \cos 2x \right)^{-p} s^{2p} \Gamma^2 (p)$$ (169)

assuming $s$ is real and provided $\text{Re} \left( s^2 e^{\pm 2ix} \right) < 4$ to avoid the singularities in the $\left( 1 - \frac{1}{4} s^2 e^{\pm 2ix} \right)^{-p}$ factors. For example, if $e^{2x} < 4$, then all real $x$ satisfy this condition, and the composite metric is real and positive
definite for real \( p > 0 \). The double pole at \( p = 0 \) is somewhat troubling, and again presents an issue for the ground state WF. Nevertheless, we easily check that this composite \( R \) satisfies the metric equation (141).

Similarly, for the other solution (162), we find

\[
R = \sum_{\text{other}} (x, p) * S_{\text{other}} (x, p)
\]

\[
= \left( 1 - \frac{1}{4} s^2 e^{2ix} \right)^{-p} \left( 1 - \frac{1}{4} s^2 e^{-2ix} \right)^{-p} \frac{s^{2p}}{\Gamma^2 (1 - p)}
\]

\[
= \left( 1 + \frac{1}{16} s^4 - \frac{1}{2} s^2 \cos 2x \right)^{-p} s^{2p} \frac{\Gamma^2}{\Gamma^2 (1 - p)}
\]

(170)

again for real \( s \) and provided \( \text{Re} (e^{2x} e^{\pm 2ix}) < 4 \). Again we easily verify that this composite \( R \) satisfies the metric equation (141). We also note that these two composite metrics are more simply related than the previous forms of the metric or even the pairs of solutions for \( S \). Namely, we go from one composite \( R \) to the other just by the interchange \( \Gamma^2 (p) \leftrightarrow 1/\Gamma^2 (1 - p) \).

We should also express the composite solution in terms of (sums) of the previous solutions for \( R \), if that is possible. Indeed, just as in the dual metric situation, it is possible. We find

\[
\left( 1 + \frac{1}{16} s^4 - \frac{1}{2} s^2 \cos 2x \right)^{-p} s^{2p} \Gamma^2 (p) = \int_0^\infty \frac{d\tau}{\tau} e^{-(1 + \frac{1}{16} s^4) \tau} R (x, p; s^2 \tau)
\]

(171)

where \( R (x, p; s^2 \tau) \) is as defined in (142). Also

\[
\left( 1 + \frac{1}{16} s^4 - \frac{1}{2} s^2 \cos 2x \right)^{-p} s^{2p} \Gamma^2 (1 - p) = \int_{-\infty}^{(0+)} \frac{d\tau}{\tau} e^{(1 + \frac{1}{16} s^4) \tau} R_{\text{other}} (x, p; s^2 \tau)
\]

(172)

where \( R_{\text{other}} (x, p; s^2 \tau) \) is as defined in (145).

10.4 \( \sqrt{R} \) as a sum of hybrid WFs (Not!)

In parallel to the previous construction of \( \tilde{S} \) as a sum of the hybrid WFs \( h_n \), as in (120), we might try to construct roots of \( R \) as formal sums of the dual hybrids \( \tilde{h}_n \).

\[
T (x, p) = \sum_{n=0}^{\infty} \sqrt{\epsilon_n} \tilde{h}_n (x, p) \quad T (x, p) = \sum_{n=0}^{\infty} \sqrt{\epsilon_n} \tilde{h}_n (x, p)
\]

(173)

However, these are not solutions to (159) or its conjugate due to the inhomogeneities resulting when \( H \) and \( \tilde{H} \) act on the individual \( \tilde{h}_n \) and \( h_n \). Even so, these and similar sums do yield some interesting star products, at least formally. For example,

\[
\overline{T (x, p)} \ast T (x, p) = \sum_{k=0}^{\infty} \varepsilon_k \tilde{f}_k (x, p) = Q (x, p)
\]

(174)

Rather than pursue this here, we consider other direct solutions of the differential-difference equation corresponding to (159).
10.5 Additional solutions for $S$

Re-instating the coupling constant $m$ via $\exp (ix) \to m \exp (ix)$, as in (1), a straightforward series solution in powers of $m$ gives another form for solutions to (159).

$$S (m; x, p) = \Gamma (p) \left( \frac{2}{m} \right)^p e^{-ipx} I_p (me^{ix})$$

(175)

$$S_{\text{other}} (m; x, p) = \frac{1}{\Gamma (1 - p)} \left( \frac{2}{m} \right)^p (-i)^p e^{-ipx} I_p (ime^{ix})$$

(176)

the latter obtained from the first form by $x \to x + \pi/2$ and $\Gamma (p) \to 1/\Gamma (1 - p)$. We also note that (176) may be written as

$$S_{\text{other}} (m; x, p) = \frac{1}{\Gamma (1 - p)} \left( \frac{2}{m} \right)^p e^{-ipx} J_p (ime^{ix})$$

(177)

Here we have made use of the modified Bessel function $I_\nu (z)$ with properties

$$I_\nu (z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k! \Gamma (\nu + k + 1)}$$

(178a)

$$J_\nu (z) = (-i)^\nu I_\nu (iz) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k! \Gamma (\nu + k + 1)}$$

(178b)

$$z \frac{d}{dz} I_\nu (z) = zI_{\nu \pm 1} (z) \pm \nu I_\nu (z)$$

(178c)

$$I_n (z) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{z \cos \theta} e^{\pm i \nu \theta} d\theta$$

for $n \in \mathbb{N}$

(178d)

$$\exp (z \cos \theta) = I_0 (z) + 2 \sum_{k=1}^{\infty} I_k (z) \cos (k\theta)$$

(178e)

Either the first or the third of these properties leads to the most direct verification that (175) and (176) satisfy the equation (159).

We also recall the elegant contour integral representations of Schläflî and Sonine which are valid for all $\nu$ and $z$.

$$I_\nu (z) = \left( \frac{z}{2} \right)^\nu \frac{1}{2\pi i} \int_{-\infty}^{(0+)} w^{-\nu - 1} \exp \left( w + \frac{z^2}{4w} \right) dw$$

(179)

$$J_\nu (z) = \left( \frac{z}{2} \right)^\nu \frac{1}{2\pi i} \int_{-\infty}^{(0+)} w^{-\nu - 1} \exp \left( w - \frac{z^2}{4w} \right) dw$$

(180)

The contour is the same as that used in (102). Thus we identify (175) and (177) as linear combinations of the basic solutions (161) and (162) (after re-instating the coupling constant $m$ via $x \to x - i \ln m$), expressed as integrals over $w = 1/s$.

$$S (m; x, p) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} S \left( x - i \ln m, p; s = \frac{1}{w} \right) e^{s \frac{dw}{w}}$$

(181)

$$S_{\text{other}} (m; x, p) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} S_{\text{other}} \left( x - i \ln m, p; s = \frac{1}{w} \right) e^{s \frac{dw}{w}}$$

(182)
In the same manner, we can construct additional solutions to \( (107) \), with \( m \) re-instated, to obtain \( \tilde{S}(m; x, p) = S_{other}(m; x, -p) \) and \( \tilde{S}_{other}(m; x, p) = S(m; x, -p) \). We leave as an exercise the calculation of the corresponding composite metric, \( R(m; x, p) = S(m; x, p) \ast S(m; x, p) \).

### 11 Metric dependence of expectation values

It is important to realize that expectation values are metric-dependent, in general. As a first illustration of this, we consider expectations of WFs for energy eigenstates using the basic solutions \( (142) \) for the phase-space metric.

\[
N_n(s) \equiv \frac{1}{2\pi} \int_{x, p} R(x, p; s) f_n(x, p) = \frac{1}{2\pi} \sum_{p} s^p \Gamma(p) \int_{0}^{2\pi} f_n(x, p) \exp\left(\frac{1}{2} s \cos 2x\right) dx
\]

For the ground state case, \( n = 0 \), there is a problem here. The sum over \( p \) must include \( p = 0 \) and this diverges due to the \( \Gamma(p) \) in \( R(x, p; s) \). The individual basic metric solutions are not useful in this one case. It would appear to be necessary to take a linear combination of the basic metrics – an integral over \( s \), perhaps – to tame this singularity. We leave this as an unsolved problem.

For all \( n > 0 \), the individual \( R(x, p; s) \) give finite, reasonably well-behaved \( N_n(s) \). We have

\[
N_n(s) = \frac{1}{2\pi} \int_{x, p} R(x, p; s) f_n(x, p) = \frac{1}{2\pi} \sum_{p=n}^{\infty} s^p \Gamma(p) \int_{0}^{2\pi} f_n(x, p) \exp\left(\frac{1}{2} s \cos 2x\right) dx
\]

From the series \( (12) \) and the integral representation of the modified Bessels, \( (178d) \), these basic norms are

\[
N_n(s) = \sum_{p=n}^{\infty} \frac{s^p \Gamma(p)}{4^p} \sum_{k=0}^{p-n} \frac{1}{k! (n+k)! (p-k)! (p-k-n)!} \times I_{n-p+2k}\left(\frac{s}{2}\right)
\]

Alternatively,

\[
N_n(s) = \sum_{p=n}^{\infty} \frac{s^p}{p (p-n)!} \frac{1}{2\pi} \int_{0}^{2\pi} \left(\frac{i \sin 2x}{2}\right)^p \text{LegendreP}(p, -n, i \cot 2x) \exp\left(\frac{1}{2} s \cos 2x\right) dx
\]

\[
= \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k! (n+k)!} \left(\frac{1}{4}\right)^{n+k} \frac{1}{2\pi} \int_{0}^{2\pi} e^{2ikx} \text{KummerM}(n+k, 1+n, \frac{s}{4} e^{-2ix}) \exp\left(\frac{1}{2} s \cos 2x\right) dx
\]

The infinite sums here are convergent, but we have not found a closed-form. Nevertheless, the norms are clearly \( s \)- and hence metric-dependent. If we switch our attention to the dual WFs, we encounter only finite sums of Bessel functions.
11.1 Diagonal dual WFs

As further illustration of the metric dependence of phase-space expectations, we compute expectations of
dual WFs for energy eigenstates using the basic solutions (9 9) for the phase-space dual metric.

\[ \widetilde{N}_n (s) \equiv \frac{1}{2\pi} \int_{x,p} \widetilde{R} (x, p; s) \widetilde{f}_n (x, p) = \frac{1}{2\pi} \sum_{p=0}^{n} \frac{1}{s^p (1 + p)} \int_{0}^{2\pi} \widehat{f}_n (x, p) \exp \left( -\frac{1}{2} s \cos 2x \right) \, dx \]  

(187)

From the series (17) and the integral representation (178d) these dual norms are

\[ \widetilde{N}_n (s) = \sum_{p=0}^{n} \frac{1}{s^p \Gamma (1 + p)} \sum_{k=\max (0, n-p-\lfloor n/2 \rfloor)}^{\min \{ n/2, n-p \}} \frac{(n-k-1)! (k+p-1)!}{k! (n-p-k)!} \times (-1)^{2k+p-n} \frac{1}{2} I_{2k+p-n} \left( \frac{s}{2} \right) \]  

(188)

Note that \( \widetilde{N}_n (s) = \widetilde{N}_n (-s) \). For example:

\[ \widetilde{N}_0 (s) = I_0 \left( \frac{s}{2} \right) \]  

(189)

\[ \widetilde{N}_1 (s) = 4 s I_0 \left( \frac{s}{2} \right) - 32 s I_1 \left( \frac{s}{2} \right) \]  

(190)

\[ \widetilde{N}_2 (s) = 4 s^2 \left( s^2 + 8 \right) I_0 \left( \frac{s}{2} \right) - \frac{32 s}{s} I_1 \left( \frac{s}{2} \right) \]  

(191)

\[ \widetilde{N}_3 (s) = 12 s I_2 \left( \frac{s}{2} \right) + \frac{4 s^2}{s^2} \left( 8 - s^2 \right) I_0 \left( \frac{s}{2} \right) \]  

(192)

These results should be compared to the much simpler norms (33) obtained upon computing phase-space
averages using (30). The basic norms are indeed positive for all \( s \), but \( s \) dependent, as is evident upon
graphing the chosen examples.

11.2 Non-diagonal dual WFs

We also compute, for \( k \neq n \),

\[ \widetilde{N}_{k,n} (s) \equiv \frac{1}{2\pi} \int_{x,p} \widetilde{R} (x, p; s) \widetilde{f}_{k,n} (x, p) = \overline{\mathcal{N}_{n,k} (s)} \]  

(193)

to see if they vanish. As a consequence of the inhomogeneities in the dual energy eigenvalue equations, it
is not obvious that they should vanish for \( k \neq n \). In fact, they do not.

To verify this statement and to compute \( \widetilde{N}_{k,n} (s) \), for general \( k \neq n \), we first determine the non-diagonal
dual WFs.

\[ \widetilde{f}_{k,n} (x, p) \equiv \frac{1}{2\pi} \int_{0}^{2\pi} \chi_k (x-y) \chi_n (x+y) e^{2iyp} \, dy \]  

\[ = 4^p k n e^{ix(k-n)} \sum_{j=0}^{\lfloor k/2 \rfloor} \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{(k-j-1)! (n-l-1)!}{j! l!} e^{2ix(l-j)} \delta_{k+n-2p,2j+2l} \]  

(194)

34
The sums result from using the series in (3). Given the range of the sums, we must have $0 \leq 2p \leq k + n$ for a non-zero result. So, as for the $k = n$ case, the support in $p$ is finite. Also, for a contribution to the sums, we must have $k + n = 2p \mod 2$. Thus $\tilde{f}_{k,n} (x,p)$ can be non-zero for semi-integer $p$, when $k + n$ is an odd integer. Let us simplify the double sum in (194). First, suppose $k + n$ is even. Then $\delta_{k+n-2p,2j+2l}$ under the double sum implies $p$ must be integer and $\leq \frac{k+n}{2}$. The upper limits of the sums further imply $k/2 - \lfloor k/2 \rfloor + n/2 - \lfloor n/2 \rfloor \leq p$. That is to say, for $k + n$ even, we must have $p$ integer and $0 \leq (k + n)/2 - \lfloor k/2 \rfloor - \lfloor n/2 \rfloor \leq p \leq \frac{k+n}{2}$. On the other hand, for $k + n$ odd, we must have $p$ semi-integer and $\frac{1}{2} \leq p \leq \frac{k+n}{2}$. Incorporating these conditions and using the Kronecker delta to eliminate one sum, for both even and odd $k + n$ cases, we find

$$\tilde{f}_{k,n} (x,p) = 4^p k \kappa e^{i(k-n)x} \sum_{j = \max(0, n + \lfloor \frac{k+n}{2} \rfloor \lceil \frac{p}{2} \rceil \lfloor \frac{p}{2} \rfloor)}^{\min(\lfloor \frac{k+n}{2} \rfloor - \lceil \frac{p}{2} \rceil)} \frac{(k - j - 1)! (\lfloor p \rfloor + j + \lfloor \frac{k+n}{2} \rfloor - 1)!}{j! (n + \lfloor \frac{k-n}{2} \rfloor - \lfloor \frac{p}{2} \rfloor - j)!} e^{2i(n + \lfloor \frac{k+n}{2} \rfloor - |p| - 2j)x}$$

(195)

Note that this coincides with (17) when $k = n$. Also note we do not need the floor function in the $\lfloor \frac{k+n}{2} \rfloor$ factors when $k + n$ is even, but we do need it when $k + n$ is odd. Similarly, we do not need the floor function in $\lfloor p \rfloor$ when $k + n$ is even, but do need it when $k + n$ is odd. In particular

$$\tilde{f}_{0,1} (x,p) = 2e^{-ix} \delta_{p,1/2}$$

(196a)

$$\tilde{f}_{0,2} (x,p) = 2\delta_{p,0} + 8e^{-2ix} \delta_{p,1}$$

(196b)

$$\tilde{f}_{1,2} (x,p) = 4e^{ix} \delta_{p,1/2} + 16e^{-ix} \delta_{p,3/2}$$

(196c)

$$\tilde{f}_{1,3} (x,p) = 12\delta_{p,1} + 96e^{-2ix} \delta_{p,2}$$

(196d)

$$\tilde{f}_{2,3} (x,p) = 12e^{-ix} \delta_{p,1/2} + (48e^{ix} + 96e^{-3ix}) \delta_{p,3/2} + 384e^{-ix} \delta_{p,5/2}$$

(196e)

$$\tilde{f}_{2,4} (x,p) = 4\delta_{p,0} + (16e^{2ix} + 64e^{-2ix}) \delta_{p,1} + (256 + 768e^{-4ix}) \delta_{p,2} + 3072e^{-2ix} \delta_{p,3}$$

(196f)

We now finish checking nondiagonal orthogonality for the basic metrics, i.e. whether $\tilde{N}_{k,n} (s) = 0$ for $k \neq n$. This is not true, in general.

First, suppose $k + n$ is even, and consider the simplest case. The average over position is

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_{0,2} (x,p) \bar{R} (x,p;s) \, dx = \frac{1}{sp\Gamma (1 + p)} \frac{1}{2\pi} \int_0^{2\pi} (2\delta_{p,0} + 8e^{-2ix} \delta_{p,1}) \exp \left(-\frac{s}{2} \cos 2x\right) \, dx$$

$$= \frac{1}{sp\Gamma (1 + p)} \left(2I_0 \left(\frac{s}{2}\right) \delta_{p,0} - 8I_1 \left(\frac{s}{2}\right) \delta_{p,1}\right)$$

(197)

This is real for real $s$, so

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_{0,2} (x,p) \bar{R} (x,p;s) \, dx = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_{2,0} (x,p) \bar{R} (x,p;s) \, dx$$

(198)
Summing over $p$ now gives

\[
\tilde{N}_{0,2} (s) = \tilde{N}_{2,0} (s) = 2I_0 \left( \frac{s}{2} \right) - \frac{8}{s} I_1 \left( \frac{s}{2} \right) = 2I_2 \left( \frac{s}{2} \right)
\]  

(199)

This does not vanish for $s \neq 0$. Similarly for other even $k + n$ cases:

\[
\tilde{N}_{k,n} (s) = \tilde{N}_{n,k} (s) = \frac{4^p k! n!}{s p} \sum_{j = \max(0, \frac{k+n}{2} - \frac{p}{2} - \frac{s}{4})}^{\min(\frac{s}{2}, \frac{k+n}{2} - p)} \frac{\left( k - j - 1 \right)! \left( p + j - \frac{k+n}{2} - 1 \right)!}{j! \left( \frac{k+n}{2} - p - j \right)! p!} I_{k-p-2j} \left( -\frac{s}{2} \right)
\]  

(200)

We leave odd $k + n$ as an exercise for the interested reader.

In contrast to (200), all the non-diagonal dual WFs vanish when summed/integrated over phase space using the dual metric of (30).

\[
\frac{1}{2\pi} \int_{x,p} \tilde{R}(x,p) \tilde{f}_{k,n} (x,p) = 0 \quad \text{for} \quad k \neq n
\]  

(201)

as follows most easily from (34). For the case when $k + n$ is even, this involves cancellations among the various terms for different $p$ after integrating over $x$. For example,

\[
\frac{1}{2\pi} \int_0^{2\pi} \sum_p f_{0,2} (x,p) \tilde{R}(x,p) \, dx = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_{0,2} (x,p) \frac{\sin^2 x}{(p!)^2} \, dx = \frac{1}{2\pi} \int_0^{2\pi} 2dx + \frac{1}{2\pi} \int_0^{2\pi} 8e^{-2ix} \sin^2 x \, dx = 2 - 2 = 0
\]  

(202)

However, when $k + n$ is odd, and therefore $p$ is semi-integer, we interpret the metric as integer powers of $|\sin x|$. Then the individual terms contributed by $\tilde{f}_{k,n} (x,p)$ to the sum over $p$ each vanish separately upon integration over $x$. In this regard we note that $\frac{\sin x}{} (p+1))$ is still a solution of (31), for $p > 0$, given that $(\sin x)^{2p} \delta (x) = 0$. (That is, all test functions are required to be non-singular at $x = 0$.)

Thus it is evident that requiring the phase-space averages of $\tilde{f}_{k,n}$ to vanish for $k \neq n$ imposes additional conditions on the dual metric, beyond those specified in (31). However, these additional conditions are not sufficient to force the dual metric to be proportional to $\tilde{R}(x,p)$ in (30). Any dual metric of the form given in (34) but with arbitrary coefficients, i.e. $\tilde{R}(x,p) = \sum_{k=0}^{\infty} c_k f_k (x,p)$, will give a vanishing phase-space average for nondiagonal dual WFs. This follows immediately from

\[
\frac{1}{2\pi} \int_{x,p} f_j (x,p) \tilde{f}_{k,l} (x,p) = \delta_{j,k} \delta_{j,l}
\]  

(203)

From (200) we can only conclude that $\tilde{R}(x,p;s)$ cannot be so expressed as a linear combination of the diagonal $f_k$. Of course, this can also be established by other means.

As a consequence of $\tilde{N}_{k,n} (s) \neq 0$ the norms of general pure state dual WFs are even more complicated. For example, consider a linear combination of dual functions, $\chi = \alpha A_0 + \beta A_2$, and form the corresponding dual WF.

\[
\tilde{f} = \frac{1}{2\pi} \int_0^{2\pi} \chi (x - y) \chi (x + y) e^{2iyp \, dy} = |\alpha|^2 \tilde{f}_0 + \overline{\alpha} \beta \tilde{f}_{0,2} + \alpha \overline{\beta} \tilde{f}_{2,0} + |\beta|^2 \tilde{f}_2
\]  

(204)
Under the action of $\bar{R}(x,p;s)$ the normalization of this pure state dual WF is

$$\mathcal{N}(s) = \frac{1}{2\pi} \int_0^{2\pi} \bar{R}(x,p;s) \tilde{f}(x,p) \, dx = |\alpha|^2 \mathcal{N}_0(s) + \bar{\alpha} \beta \mathcal{N}_{0,2}(s) + \alpha \bar{\beta} \mathcal{N}_{2,0}(s) + |\beta|^2 \mathcal{N}_2(s)$$

$$= \left( |\alpha|^2 + \frac{4}{s^2} (8 - s^2) |\beta|^2 \right) I_0 \left( \frac{s}{2} \right) + \left( 2\pi \beta + 2\alpha \bar{\beta} + 8 |\beta|^2 \right) I_2 \left( \frac{s}{2} \right)$$

(205)

On the other hand, under $\bar{R}(x,p)$ in (30), the norm of this dual WF would be just $|\alpha|^2 + 2|\beta|^2$.

Ultimately, $\mathcal{N}_{k,n} \neq 0$ for $k \neq n$ originates in the inhomogeneities in the dual eigenvalue equations. By contrast, a similar calculation for the metric, in those cases where it is well-defined ($k \neq 0 \neq n$), would give

$$\mathcal{N}_{k,n}(s) = \frac{1}{2\pi} \int_{x,p} \bar{R}(x,p;s) f_{k,n}(x,p) = 0$$

(206)

for $k \neq n$. This is a simple consequence of the Lone Star Lemma, the homogeneous equations $H \ast f_{k,n}(x,p) = k^2 f_{k,n}(x,p)$ and $f_{k,n}(x,p) \ast \bar{H} = n^2 f_{k,n}(x,p)$, and (135). Hence, for a linear combination of WFs corresponding to the wave function $\psi = \alpha J_k + \beta J_n$, i.e.

$$f = \frac{1}{2\pi} \int_0^{2\pi} \psi(x-y) \bar{\psi}(x+y) e^{2iyp} \, dy = |\alpha|^2 f_k + 2\bar{\alpha}\bar{\beta}f_{n,k} + \alpha\bar{\beta}f_{k,n} + |\beta|^2 f_n$$

(207)

the action of $\bar{R}(x,p;s)$ would produce the normalization

$$\mathcal{N}(s) = \frac{1}{2\pi} \int_0^{2\pi} \bar{R}(x,p;s) \tilde{f}(x,p) \, dx = |\alpha|^2 \mathcal{N}_k(s) + |\beta|^2 \mathcal{N}_n(s)$$

(208)

We have assumed that neither $k$ nor $n$ are zero.

### 11.3 Expectations of $H$

The corresponding expectations of $H$ under the action of $\bar{R}(x,p;s)$ are given by

$$\mathcal{H}_{k,n}(s) = \frac{1}{2\pi} \int_{x,p} \bar{R}(x,p;s) \left( \tilde{f}_{k,n}(x,p) \ast H \right)$$

$$= \frac{1}{2\pi} \int_{x,p} \bar{R}(x,p;s) \left( n^2 \tilde{f}_{k,n} + \begin{cases} 2n \mathcal{h}_{-1,k} & \text{for } n \in \mathbb{N}_{\text{odd}} \\ \mathcal{e}_n \mathcal{h}_{-2,k} & \text{for } n \in \mathbb{N}_{\text{even}} \end{cases} \right)$$

(209)

where diagonal cases will be denoted by $\mathcal{H}_{n,n} \equiv \mathcal{H}_n$. Alternatively, we may use the Lone Star Lemma and (101) to write

$$\mathcal{H}_{k,n}(s) = \frac{1}{2\pi} \int_{x,p} \left( H \ast \bar{R}(x,p;s) \right) \tilde{f}_{k,n}(x,p)$$

$$= \frac{1}{2\pi} \int_{x,p} \left( p^2 - \frac{1}{8}s^2 + \left( p - \frac{1}{2} \right)s \cos 2x + \frac{1}{8}s^2 \cos 4x \right) \bar{R}(x,p;s) \tilde{f}_{k,n}(x,p)$$

(210)

Since $\bar{R}(x,p;s)$ is real for real $s$, and $\tilde{f}_{n,k} = f_{k,n}$, (12) then gives

$$\mathcal{H}_{k,n}(s) = \mathcal{H}_{n,k}(s)$$

(211)
For example, using (196b) we have
\[
\tilde{H}_{0,2}(s) = \frac{1}{2\pi} \int_0^{2\pi} \left( -\frac{1}{8}s^2 - \frac{1}{2}s \cos 2x + \frac{1}{8}s^2 \cos 4x \right) \tilde{R}(x,0; s) \times 2dx
\]
\[
+ \frac{1}{2\pi} \int_0^{2\pi} \left( 1 - \frac{1}{8}s^2 + \frac{1}{2}s \cos 2x + \frac{1}{8}s^2 \cos 4x \right) \tilde{R}(x,1; s) \times 8e^{-2ix} dx
\]
(212)
From (178d) and (178c) we then obtain
\[
\tilde{H}_{0,2}(s) = \left( -\frac{1}{8}s^2 I_0 \left( \frac{s}{2} \right) + \frac{1}{2}s I_1 \left( \frac{s}{2} \right) + \frac{1}{8}s^2 I_2 \left( \frac{s}{2} \right) \right) \times 2
\]
\[
+ \left( 1 - \frac{1}{8}s^2 \right) I_1 \left( \frac{s}{2} \right) + \frac{1}{4}s I_0 \left( \frac{s}{2} \right) + \frac{1}{4}s I_2 \left( \frac{s}{2} \right) - \frac{1}{16}s^2 I_1 \left( \frac{s}{2} \right) - \frac{1}{16}s^2 I_3 \left( \frac{s}{2} \right) \right) \times \frac{8}{s}
\]
(213)
So, for real \( s \), \( \tilde{H}_{0,2}(s) = \tilde{H}_{0,2}(s) = \tilde{H}_{2,0}(s) \). Similarly, the \( n = 0 \) and 2 diagonal cases are given by
\[
\tilde{H}_0(s) = \frac{1}{8}s^2 I_2 \left( \frac{s}{2} \right)
\]
(214)
\[
\tilde{H}_2(s) = 48I_2 \left( \frac{s}{2} \right) + 16\frac{8}{s^2} (8 - s^2) I_0 \left( \frac{s}{2} \right)
\]
(215)
to be compared to \( \tilde{N}_n(s) \) given above. The important point here is that the average of \( H \) for definite energy dual WFs, when computed as the ratio \( \tilde{H}_n(s) / \tilde{N}_n(s) \) using the basic metric \( \tilde{R}(x,p; s) \), is far from a simple expression. It is certainly not \( n^2 \), as it was in (52), when computed using the dual metric of (30).

With these cases in hand, we can also consider the linear combination of dual WFs in (204).
\[
\tilde{H}(s) = \frac{1}{2\pi} \int_{x,p} \tilde{R}(x,p; s) \left( \tilde{f}(x,p) \ast H \right) = |\alpha|^2 \tilde{H}_0(s) + \overline{\alpha} \beta \tilde{H}_{0,2}(s) + \alpha \beta \tilde{H}_{2,0}(s) + |\beta|^2 \tilde{H}_2(s)
\]
(216)
We find
\[
\tilde{H}(s) = \frac{16}{s^2} (8 - s^2) |\beta|^2 I_0 \left( \frac{s}{2} \right) + \left( \frac{1}{8}s^2 |\alpha|^2 + 8 (\overline{\alpha} \beta + \alpha \overline{\beta}) + 48 |\beta|^2 \right) I_2 \left( \frac{s}{2} \right)
\]
(217)
to be compared to \( \tilde{N}(s) \) given above. Again, the important point is that the ratio \( \tilde{H}(s) / \tilde{N}(s) \) is far from a simple expression, and is not \( 4 \times 2 |\beta|^2 / \left( |\alpha|^2 + 2 |\beta|^2 \right) \), as it would have been had we used the dual metric of (30).

Other cases of even \( k + n \) are handled similarly, using (197). We have
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( p^2 - \frac{1}{8}s^2 + \left( p - \frac{1}{2} \right) s \cos 2x + \frac{1}{8}s^2 \cos 4x \right) \tilde{R}(x,p; s) \int_{x,n} (x,p) dx
\]
\[
= \frac{4^p k_1}{s^p \Gamma(1 + p)} \min \left( \left\lfloor \frac{k}{n} \right\rfloor, n + \left\lfloor \frac{k-n}{2} \right\rfloor - |p| \right) \sum_{j = \max(0,n + \left\lfloor \frac{k-n}{2} \right\rfloor - |p| - \left\lfloor \frac{k}{2n} \right\rfloor, 0)} \frac{(k-j-1)! \left( p+j - \frac{k-n}{2} - 1 \right)!}{j! \left( \frac{k-n}{2} - p - j \right)!} \times
\]
\[
\times \frac{1}{2\pi} \int_0^{2\pi} \left( p^2 - \frac{1}{8}s^2 + \left( p - \frac{1}{2} \right) s \cos 2x + \frac{1}{8}s^2 \cos 4x \right) e^{2i(k-p-2j)x} \exp \left( -\frac{1}{2}s \cos 2x \right)
\]
(218)
Then Eqs. (178d) and (178e) permit evaluation of the spatial integral to obtain

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( p^2 - \frac{1}{8} s^2 + \left( p - \frac{1}{2} \right) s \cos 2x + \frac{1}{8} s^2 \cos 4x \right) e^{2i(k-p-2j)x} \exp \left( -\frac{1}{2} s \cos 2x \right) \]

\[= p^2 I_{k-p-2j} \left( -\frac{s}{2} \right) + \frac{s}{2} p \left( I_{k-p-2j+1} \left( -\frac{s}{2} \right) + I_{k-p-2j-1} \left( -\frac{s}{2} \right) \right) \]

\[+ \frac{s}{4} (k-p-2j) \left( I_{k-p-2j+1} \left( -\frac{s}{2} \right) - I_{k-p-2j-1} \left( -\frac{s}{2} \right) \right) \]  

(219)

Hence, for even \( k + n \),

\[
\tilde{\mathcal{H}}_{k,n} (s) = \sum_{p} \frac{A_p k n}{s p!} \sum_{j=\max(0, k+n-|n/2|)}^{\min(|k+n-p|)} \frac{(k-j-1)! (p+j-k-n)!}{j! (k+n-p-j)!} \times
\]

\[
\left( p^2 I_{k-p-2j} \left( -\frac{s}{2} \right) + \frac{s}{2} p \left( I_{k-p-2j+1} \left( -\frac{s}{2} \right) + I_{k-p-2j-1} \left( -\frac{s}{2} \right) \right) \right) \]

\[+ \frac{s}{4} (k-p-2j) \left( I_{k-p-2j+1} \left( -\frac{s}{2} \right) - I_{k-p-2j-1} \left( -\frac{s}{2} \right) \right) \]  

(220)

Again, this is real for real \( s \), so \( \tilde{\mathcal{H}}_{k,n} = \tilde{\mathcal{H}}_{n,k} \). In particular, for \( k = n \) we obtain

\[
\tilde{\mathcal{H}}_n (s) = \sum_{p} \frac{A_p n^2}{s p!} \sum_{j=\max(0, n-|n/2|)}^{\min(|n|-p)} \frac{(n-j-1)! (p+j-1)!}{j! (n-p-j)!} \times
\]

\[
\left( p^2 I_{n-p-2j} \left( -\frac{s}{2} \right) + \frac{s}{2} p \left( I_{n-p-2j+1} \left( -\frac{s}{2} \right) + I_{n-p-2j-1} \left( -\frac{s}{2} \right) \right) \right) \]

\[+ \frac{s}{4} (n-p-2j) \left( I_{n-p-2j+1} \left( -\frac{s}{2} \right) - I_{n-p-2j-1} \left( -\frac{s}{2} \right) \right) \]  

(221)

We leave odd \( k + n \) as an exercise for the reader.

Ultimately, \( \tilde{\mathcal{H}}_{k,n} \neq 0 \) for \( k \neq n \) can be traced back to its origins in the inhomogeneities in the dual eigenvalue equations, as was also true for \( \tilde{\mathcal{N}}_{k,n} \). A similar calculation for the metric, instead of the dual metric, would give

\[
\mathcal{H}_{k,n} (s) \equiv \frac{1}{2\pi} \iint_{x,p} R(x,p;s) H \ast f_{k,n}(x,p) = k^2 n^2 \mathcal{N}_{k,n} (s) = n^2 \mathcal{N}_n (s) \delta_{k,n} \]

(222)

at least in those cases where is it is well-defined. Hence, for definite energy WFs, we would obtain the expected averages: \( \langle H \rangle = \mathcal{H}_n (s) / \mathcal{N}_n (s) = n^2 \). Also, for a linear combination of WFs corresponding to the wave function \( \psi = \alpha J_k + \beta J_n \), as in (207) above, we would have the more intuitive result:

\[
\mathcal{H} (s) = \frac{1}{2\pi} \iint_{x,p} R(x,p;s) H \ast f(x,p) = |\alpha|^2 k^2 \mathcal{N}_k (s) + |\beta|^2 n^2 \mathcal{N}_n (s) \]  

(223)
We have again assumed that neither $k$ nor $n$ vanishes.

All this raises the question of how physics can be extracted free from any metric dependence, in general, especially in situations where calculations are performed using dual functions that obey inhomogeneous equations. Unfortunately, at this time we have no model-independent answer to this question. (However, for a brief discussion of possible (semi)classical behavior in the specific $\exp(2ix)$ model, please see the last subsection in Section IV of [8].)

12 Conclusions

There are several types of functions involved in the analysis of imaginary Liouville quantum mechanics: Energy eigenfunctions, their duals, and free-particle plane waves related to the eigenfunctions by various equivalence maps. By cross-breeding pairs of these functions we have produced a wide variety of hybrid Wigner transforms that exhibit a rich diversity of star product relations. Through the use of sums of these WFs, and star products of those sums, we have also constructed various phase space equivalence transformations and metrics. We have further shown how these phase space kernels can be built from direct, elementary solutions of their controlling star product equations. Finally, we have discussed, albeit briefly, the import of the metrics for the calculation of physical expectation values. Later, we hope to extend the analysis of this paper to include the general class of Hamiltonians of the form $H = (p + \nu)^2 + \sum_{k>0} \mu_k \exp(ikx)$.

Acknowledgements We thank the Institute for Advanced Study for its hospitality and support, and for providing a stimulating environment in which most of this work was completed. We also thank Luca Mezincescu for useful discussions. One of us (TC) acknowledges the beneficial natural setting of the Aspen Center for Physics in which this work was initiated during the summer of 2005. This material is based upon work supported by the National Science Foundation under Grant No’s. 0303550 and 0555603.
References


Appendix A. The free particle limit

This is achieved by taking \( m \to 0 \), but first we have to rescale the Bessels and the Neumann polynomials to have sensible limits as \( m \) vanishes. So we take the biorthonormal set to be \( \{ m^n A_n (me^{ix}) , m^{-n} J_n (me^{ix}) \} \) which becomes \( \{ 2^n! e^{-inx}, \frac{1}{2^n n!} e^{inx} \} \) as \( m \to 0 \). Indeed, the \( 2^n n! \) factors look weird, but we must attribute that convention to Bessel and Neumann.

Now, what happens as \( m \to 0 \) to the metric on the space of dual functions as in [8]? After rescaling the Bessels, it is just

\[
J(x, y) = \sum_{n=0}^{\infty} m^{-n} J_n (me^{ix}) m^{-n} J_n (me^{iy}) \xrightarrow{m \to 0} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-inx} e^{iny} = I_0 \left( e^{i(n-y)/2} \right) \quad (224)
\]

Another Bessel function. This integral kernel is not unity (i.e. a Dirac delta) but a projector.

When integrated with (dual) functions that are constants or negative powers of \( e^{iy} \), and their conjugates, it gives the expected orthonormality.

\[
\frac{1}{(2\pi)^2} \int dx dy \ m^n A_n^* (me^{ix}) J(x, y) m^k A_k (me^{iy}) = \delta_{n,k} \quad (225)
\]

even as \( m \to 0 \). But when integrated with positive powers of \( e^{iy} \) (not constants) it gives zero.

Appendix B. Arbitrary functions acting on WFs through star products

For any two functions \( \psi \) and \( \phi \), we define the Wigner transform \( f_{\psi\phi} \) somewhat unconventionally (we do not conjugate \( \phi \)) as

\[
f_{\psi\phi}(x, p) \equiv \frac{1}{2N\pi} \int \psi(x-y) e^{2ip\phi}(x+y) \ dy \quad (226)
\]

where \( N \) will be chosen for convenience, later. Then for any other function \( G \) on the phase-space, we have the star product

\[
G(x, p) \star f_{\psi\phi}(x, p) = G \left( x, p - \frac{1}{2} i \partial_x \right) \frac{1}{2N\pi} \int e^{-i\partial_y} e^{2ip\phi}(x-y) \psi(x+y) \ dy \\
= \frac{1}{2N\pi} \int e^{2ipG} \left( x-y, p-\frac{1}{2} i \partial_x \right) \psi(x) \phi(x+y) \ dy \quad (227)
\]

Similarly

\[
f_{\psi\phi}(x, p) \star G(x, p) = G \left( x, p + \frac{1}{2} i \partial_x \right) \frac{1}{2N\pi} \int e^{i\partial_y} e^{2ip\phi}(x-y) \phi(x+y) e^y \ dy \\
= \frac{1}{2N\pi} \int e^{2ipG} \left( x+y, p+\frac{1}{2} i \partial_x \right) \psi(x) \phi(x+y) \ dy \quad (228)
\]
For two classes of special cases things simplify considerably here.

In the first class, if \( G(x,p) = V(x) \) then

\[
V(x) \star f_{\psi \phi}(x,p) = \frac{1}{2N\pi} \int V(x-y) \psi(x-y) e^{2ipy} \phi(y) dy
\]

(229)

\[
f_{\psi \phi}(x,p) \star V(x) = \frac{1}{2N\pi} \int \psi(x-y) e^{2ipy} V(y) \phi(y) dy
\]

(230)

Since \( V(x) \) here is arbitrary, there is a simple and direct application of (229) and (230) to write Wigner transforms themselves as star products of wave functions and their Fourier transforms, with an intercalated exponential. Namely

\[
f_{\psi \phi}(x,p) = \psi(x) \star f_{1\phi}(x,p) = \psi(x) \star \frac{1}{2N\pi} \int e^{2ipy} \phi(y) dy
\]

\[
= \frac{1}{2N-1} \psi(x) \star \left( e^{-2ixp \hat{\phi}}(-2p) \right)
\]

\[
= \frac{1}{2N-1} \psi(x) \star e^{-2ixp} \hat{\phi}(-p)
\]

(231)

\[
f_{\psi \phi}(x,p) = f_{\psi 1}(x,p) \star \phi(x) = \frac{1}{2N\pi} \int \psi(x-y) e^{2ipy} dy \star \phi(x)
\]

\[
= \frac{1}{2N-1} \left( e^{2ixp \hat{\psi}}(2p) \right) \star \phi(x)
\]

\[
= \frac{1}{2N-1} \hat{\psi}(p) \star e^{2ixp} \phi(x)
\]

(232)

where the momentum space wave functions are defined by

\[
\hat{\psi}(2p) \equiv \frac{1}{2\pi} \int \psi(w) e^{-2iwp} dw , \quad \hat{\phi}(-2p) \equiv \frac{1}{2\pi} \int \psi(w) e^{2iwp} dw
\]

(233)

Reasoning along these same lines also leads to a compact expression for Wigner transforms in terms of star products of wave functions with an intercalated momentum-space delta \([5]\), either Dirac or Kronecker.

\[
f_{\psi \phi}(x,p) = 2^{1-N} \psi(x) \star \delta_{2p} \star \phi(x)
\]

(234)

where \( \delta_{2p} = \delta(2p) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2isp} ds \) for \( x \in \mathbb{R} \) and \( \delta_{2p,0} = \delta_{2p,0} \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{2isp} ds \) for \( x \in S^1 \).

In the second class, if \( G(x,p) = K(p) \) then integration by parts assuming no surface contributions...
\[ K(p) \ast f_{\psi\phi}(x, p) = \frac{1}{2N\pi} \int e^{2ipy} K \left( p - \frac{1}{2}i\bar{\partial}_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} K \left( -\frac{1}{2}i\partial_y - \frac{1}{2}i\bar{\partial}_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} \phi(x+y) K \left( -\frac{1}{2}i\left(\bar{\partial}_x - \partial_y \right) \right) \psi(x-y) \, dy \]  \hspace{1cm} (235)

\[ f_{\psi\phi}(x, p) \ast K(p) = \frac{1}{2N\pi} \int e^{2ipy} K \left( p + \frac{1}{2}i\bar{\partial}_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} K \left( -\frac{1}{2}i\partial_y + \frac{1}{2}i\partial_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} \psi(x-y) K \left( \frac{1}{2}i\left(\partial_x + \partial_y \right) \right) \phi(x+y) \, dy \]  \hspace{1cm} (236)

More generally, some care is needed to unravel the remaining derivatives in (227) and (228). For general \( G(x, p) \) there will appear an operator matrix element involving the operator obtained from \( G(x, p) \) by the Weyl correspondence.

As an illustration, consider \( G(x, p) = xp \). Then

\[ xp \ast f_{\psi\phi}(x, p) = \frac{1}{2N\pi} \int e^{2ipy} (x-y) \left( p - \frac{1}{2}i\bar{\partial}_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int (x-y) \left( \frac{1}{2}i\partial_y (e^{2ipy}) - \frac{1}{2}e^{2ipy}i\bar{\partial}_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} \left( -\frac{1}{2}i + (x-y) \frac{1}{2}i\left(\partial_y - \partial_x \right) \right) \psi(x-y) \phi(x+y) \, dy \]

upon IPANS. We recognize part of this as the expected operator matrix element \( (x-y) \frac{1}{2}i \left(\partial_y - \partial_x \right) \psi(x-y) = \langle x-y|xp|\psi \rangle \). The other term in the integrand, \(-\frac{1}{2}i\), is the price we must pay because on the LHS we took just the simple product, \( xp \), and not the star product, \( x \ast p = xp + \frac{1}{2}i \). Had we taken the later instead, we would have

\[ x \ast p \ast f_{\psi\phi}(x, p) = \frac{1}{2N\pi} \int e^{2ipy} \left( x-y \right) \frac{1}{2}i\left(\partial_y - \partial_x \right) \psi(x-y) \phi(x+y) \, dy \]

\[ = \frac{1}{2N\pi} \int e^{2ipy} \left( x-y \right) G(x, p) \, dy \]  \hspace{1cm} (237)

Actually, this illustrates the general situation. If the WF is acted on by a “star function” \( G_\ast(x, p) \) of \( x \) and \( p \), by which terminology we mean \( G_\ast(x, p) \) is a sum of multinomials of star products of \( x \) and \( p \), then the result involves an operator matrix element involving an operator function \( G(x, p) \) with the operators arranged in the same order as the various \( x \) and \( p \) factors were arranged in the multinomials of the star function. Indeed, that operator is just the Weyl correspondent of the star function.

\[ G_\ast(x, p) \ast f_{\psi\phi}(x, p) = \frac{1}{2N\pi} \int e^{2ipy} (x-y) G(x, p) \, dy \]  \hspace{1cm} (238)
A similar simple operator correspondence applies to acting on the right with $G_\ast (x, p)$, provided we complex conjugate the second function.

$$f_{\psi \bar{\phi}} (x, p) \ast G_\ast (x, p) = \frac{1}{2N \pi} \int e^{2iyp} \psi (x - y) \langle \phi | G (x, p) | x + y \rangle \, dy$$ (239)

### Appendix C. Conventions and star product compositions of bilinear Wigner transforms

For functions $\psi$ and $\phi$ either on $\mathbb{R}$ or on $S^1$, we define the Wigner transform $f_{\psi \phi}$ as

$$f_{\psi \phi} (x, p) \equiv \frac{1}{2N \pi} \int_{\mathbb{R} \text{ or } S^1} \psi (x - y) \phi (x + y) e^{2iyp} \, dy$$ (240)

where $N$ is chosen to simplify the normalization of two such functions: $N = 0$ for $\mathbb{R}$; $N = 1$ for $S^1$. With these choices

$$\psi (x) \phi (x) = \begin{cases} \int_{-\infty}^{+\infty} f_{\psi \phi} (x, p) \, dp & \text{for } x \in \mathbb{R} \\ \sum_{2p \in \mathbb{Z}} f_{\psi \phi} (x, p) & \text{for } x \in S^1 \end{cases}$$ (241)

Note that for position coordinates on the circle, $x, y \in [0, 2\pi]$, the momentum sum is over all semi-integer $p$ so that the complete periodic Dirac delta is produced, $\sum_{2p \in \mathbb{Z}} e^{2iyp} = 2\pi \delta (y) = 2\pi \delta (y + 2\pi)$, such that $\int_{0}^{2\pi} \delta (y) \, dy = \int_{-\epsilon}^{2\pi - \epsilon} \delta (y) \, dy = 1$ for all $\epsilon$. This is important since in principle the contributing Fourier mode numbers of $\psi$ and $\phi$ could differ by either even or odd integers.

With these choices for $c$ we also have a uniform appearance for the $\ast$ composition of two such functions

$$f_{\psi \phi} (x, p) \ast f_{\eta \chi} (x, p) = (\phi, \eta) \, f_{\psi \chi} (x, p)$$ (242)

for both cases. On the RHS we have used the notation

$$(\phi, \eta) = \frac{1}{2\pi} \int_{\mathbb{R} \text{ or } S^1} \phi (x) \eta (x) \, dx$$ (243)

That is to say this metric is local, $K (x, y) = \frac{1}{2\pi} \delta (x - y)$ in the language of biorthogonal systems as used in the main text, Eqn (20).

For $\mathbb{R}$ the result (242) is obtained as in the Overview of [23]. For $S^1$ the result is obtained with a modicum of novelty as follows. Applying the star product directly to each of the integral representations
for \( f_{\psi \phi} \) and \( f_{\eta \chi} \), and carrying out the requisite variable shifts on the integrands, we obtain

\[
f_{\psi \phi} (x, p) \ast f_{\eta \chi} (x, p) = \frac{1}{2N\pi} \int \psi (x - y_1) \phi(x + y_1) e^{2iy_1p} e^{yi\bar{\eta}} \frac{1}{2N\pi} \int e^{-i\bar{y}_2 y_2} e^{2iy_2p} \eta (x - y_2) \chi (x + y_2) dy_2 \]
\[
= \frac{1}{(2N\pi)^2} \int \int \psi (x - y_1 - y_2) \phi (x + y_1 - y_2) \eta (x + y_1 - y_2) \chi (x + y_1 + y_2) e^{2i(y_1 + y_2)p} dy_1 dy_2 \]
\[
= \frac{1}{(2N\pi)^2} \int \psi (x - y_1 - y_2) \chi (x + y_1 + y_2) e^{2i(y_1 + y_2)p} d (y_1 + y_2) \frac{1}{2} \int \phi (x + y_1 - y_2) \eta (x + y_1 - y_2) d (y_1 - y_2) \]

(244)

Now we only need worry about the regions for the two integrations. When the original \( y_1 \) and \( y_2 \) are integrated over the whole real line, then so are \( y_1 + y_2 \) and \( y_1 - y_2 \), and the result (244) follows. When the original \( y_1 \) and \( y_2 \) are integrated over the circle, it may seem that the \( y_1 - y_2 \) region of integration depends on \( y_1 + y_2 \). However, this is actually not the case for periodic integrands. The original coupled integrals \( \int_0^{2\pi} dy_1 \int_0^{2\pi} dy_2 \cdots \) may be split into uncoupled integrals \( \int_{2\pi}^{2\pi} d (y_1 + y_2) \cdots \times \frac{1}{2} \int_{-2\pi}^{2\pi} d (y_1 - y_2) \cdots \) upon making use of this periodicity.

The original \( 2\pi \times 2\pi \) square representing the region of integration on the \( y_1 \times y_2 \) plane may be partitioned by diagonal segments into three right-triangular regions (A, B, and C in the Figure). For \( 2\pi \)-periodic integrands the result of the double integration is unchanged if the left-most triangle (A) is translated \( 2\pi \) to the right (to become triangle E), and the lower triangle (B) is translated up by \( 2\pi \) (to become triangle D), to form a \( 2\pi \times 4\pi \) rectangle (C+D+E) on the \( (y_1 + y_2) \times (y_1 - y_2) \) integration plane.3

Then periodicity is again exploited to evaluate

\[
\frac{1}{2} \int_{-2\pi}^{2\pi} d (y_1 - y_2) \phi (x + y_1 - y_2) \eta (x + y_1 - y_2) = 2\pi (\phi, \eta) \]

and hence to obtain the final result (242).

This result is used extensively in the next Appendix.

Appendix D. Non-diagonal WFs, conventional and hybrid

Consider any biorthogonal system involving countable pairs of functions and their duals, \( \{ \psi_n, \chi_n \} \), which has an equivalent hermitian system involving corresponding functions, \( \{ \phi_n, \phi_n^\dagger \} \), orthonormal in the usual

3The scale of the final rectangle in the Figure needs to be increased by \( \sqrt{2} \) to correctly display the relative area of the initial and final regions on the \( y_1 \times y_2 \) and \( (y_1 + y_2) \times (y_1 - y_2) \) planes, as is evident from the Jacobian \( |\partial (y_1 + y_2, y_1 - y_2) / \partial (y_1, y_2)| = 2. \)
sense. That is, in the notation of the previous Appendix,

\[(\chi_k, \psi_n) = \delta_{k,n} = (\phi_k, \phi_n)\]  

(245)

In phase space we define the following conventional and hybrid Wigner transforms.

\[e_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi_k(x-y) \phi_n(x+y) e^{2iyp} dy\]  

(246)

\[f_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \psi_k(x-y) \psi_n(x+y) e^{2iyp} dy\]  

(247)

\[\tilde{f}_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \chi_k(x-y) \chi_n(x+y) e^{2iyp} dy\]  

(248)

\[g_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \psi_k(x-y) \chi_n(x+y) e^{2iyp} dy\]  

(249)

\[h_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi_k(x-y) \phi_n(x+y) e^{2iyp} dy\]  

(250)

\[\tilde{h}_{k,n}(x,p) \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi_k(x-y) \chi_n(x+y) e^{2iyp} dy\]  

(251)

Clearly, in the notation of the main text, \(e_n(x,p) \equiv e_{n,n}(x,p)\), etc.

From the result in the previous Appendix, \(\Box\), mixed products are also given immediately by \(\Box\).

we immediately obtain the following star products.

\[g_{j,k} \star h_{l,m} = h_{j,m} \delta_{k,l}\]  

(259)

\[h_{j,k} \star e_{l,m} = h_{j,m} \delta_{k,l}\]  

(260)

\[e_{j,k} \star \tilde{h}_{l,m} = h_{j,m} \delta_{k,l}\]  

(261)

\[h_{j,k} \star \tilde{h}_{l,m} = \tilde{g}_{j,m} \delta_{k,l}\]  

(262)

\[f_{j,k} \star \tilde{f}_{l,m} = f_{j,m} \delta_{k,l}\]  

(252)

\[f_{j,k} \star \tilde{f}_{l,m} = \tilde{f}_{j,m} \delta_{k,l}\]  

(253)

\[\tilde{f}_{j,k} \star f_{l,m} = \tilde{g}_{m,j} \delta_{k,l}\]  

(254)

\[\tilde{f}_{j,k} \star g_{m,l} = \tilde{f}_{j,m} \delta_{k,l}\]  

(255)

\[h_{j,k} \star \tilde{f}_{l,m} = f_{j,m} \delta_{k,l}\]  

(256)

\[\tilde{h}_{j,k} \star f_{l,m} = \tilde{f}_{j,m} \delta_{k,l}\]  

(257)

\[g_{j,k} \star f_{l,m} = g_{j,m} \delta_{k,l}\]  

(258)

\[g_{j,k} \star \tilde{f}_{l,m} = \tilde{g}_{j,m} \delta_{k,l}\]  

(259)

\[g_{j,k} \star \tilde{f}_{l,m} = \tilde{g}_{j,m} \delta_{k,l}\]  

(260)

\[h_{j,k} \star \tilde{f}_{l,m} = f_{j,m} \delta_{k,l}\]  

(261)

There are a few other obvious relations that follow from complex conjugations, using \(\overline{e_{k,n}} = e_{n,k}\), \(\overline{f_{k,n}} = f_{n,k}\), and \(\overline{\tilde{f}_{k,n}} = \tilde{f}_{n,k}\), but in writing these it should be kept in mind that the hybrid conjugates, \(\overline{\tilde{g}}, \overline{\tilde{h}}, \) and \(\overline{\tilde{h}}\), are in general different functions, inde-
pendent of $g$, $h$, and $\tilde{h}$.

Moreover, there are other star products, such as $g_{j,k} \star \tilde{f}_{l,m}$ or $f_{j,k} \star g_{l,m}$, which are not proportional to $\delta_{k,l}$ as simple consequences of the biorthonormality of $\{\psi_k, \chi_n\}$ or of $\{\phi_k, \phi_n\}$. These involve the scalar products $(\chi_k, \chi_l)$, $(\overline{\psi}_k, \overline{\psi}_l)$, $(\phi_k, \psi_n)$, and $(\phi_k, \chi_n)$, which are model dependent and not single Kronecker deltas. Thus their evaluation depends on specific details of the system. Additional model dependent structure exists, such as the form of specific $\psi$s as infinite series of $\chi$s, and vice versa, or the form of specific $\phi$s as infinite series of $\psi$s, and vice versa, which permit re-expressing some star products as infinite sums of others. These re-expressions may amount to relations already known to hold in general, as above, but often they are new results and highly model dependent.

In summary, the full algebra of Wigner transform star products for a biorthogonal system is quite complicated and varies from one model to another. But the subset of products that we have explicitly shown are valid for any combined $\{\{\psi_k, \chi_n\}, \{\phi_k, \phi_n\}\}$ system.

Appendix E. A brief overview of biorthogonal systems and density operators

As noted in the introduction, for general biorthogonal systems the Hilbert space structure is at first sight very different than that for hermitian Hamiltonian systems inasmuch as the dual wave functions are usually not just the complex conjugates of the wave functions. Still, we may keep most of the compact Dirac notation for a state, $|\psi\rangle$, and its dual, $\langle \psi|$, provided we just allow $\langle \psi| x \rangle = \langle x | \psi \rangle = \langle x | \overline{\psi} \rangle$.

As an alternative notation, we will often write for the dual wave function $\langle \psi| x \rangle = \chi(x)$ (no overbar on $\chi$!) thereby reducing the clutter of symbol decorations. So then, orthonormality for a discretely indexed biorthogonal set of states $\{ |\psi_k\rangle \}$ and their duals $\{ \langle \psi_n| \}$ (i.e. a countable basis) reads $\langle \psi_n| \psi_k \rangle = \delta_{n,k} = \int dx \, \chi_n(x) \, \psi_k(x)$.

Let us now go through some purely formal manipulations to get a quick overview of the abstract constructions that are possible for biorthogonal systems. If we are dealing with a discrete biorthonormal basis, $\{ |\psi_k\rangle, \langle \psi_n| \}$, then through the usual conjugation of wave functions we can always “double-up” the states and their duals\footnote{Keep in mind that the total number of independent Hilbert space states ($L^2$ functions) may not have actually changed by this doubling-up procedure, e.g. as is certainly the case if the original wave functions and their duals were real functions. But this doesn’t affect the formal constructions to follow.} to include $\{ |\widetilde{\psi}_n\rangle, \langle \widetilde{\psi}_k| \}$. And hence write the obvious relation $|\widetilde{\psi}_n\rangle = \sum_k |\psi_k\rangle \, \delta_{k,n} = $
where the dual corresponding to $|\psi_n\rangle$ is just
\[ |\psi_n\rangle = R |\psi_n\rangle \] (268)
where the operator $R$ is just the formal hermitian sum
\[ R = \sum_k |\psi_k\rangle \langle\psi_k| = R^\dagger \] (269)
This is a positive operator, certainly, but without further assumptions about the completeness of the $|\psi_k\rangle$, $R$ may well annihilate some states, so it is not necessarily positive definite. We also have the equally obvious relation
\[ \langle\psi_n| = \langle\psi_n| R \] (270)
Thus the biorthonormality of the system may also be written as
\[ \delta_{n,k} = \langle\psi_n| R |\psi_k\rangle \] (271)
The operator $R$ plays the role of a metric in the span of $\{|\psi_n\rangle\}$. Given the requisite convergence of the sums, for any linear combination of the basis states
\[ |\psi\rangle = \sum_k c_k |\psi_k\rangle \] (272)
we have a manifestly positive definite norm
\[ \sum_k |c_k|^2 = \langle\psi| R |\psi\rangle \] (273)
Equivalently, we may write
\[ \sum_k |c_k|^2 = \langle\psi| \psi\rangle \] (274)
where the dual corresponding to $|\psi\rangle$ in the given basis is just
\[ \langle\psi| = \langle\psi| R = \sum_n c_n \langle\psi_n| \] (275)
as well as the equation dual to this
\[ |\tilde{\psi}\rangle = R |\psi\rangle = \sum_k c_k |\tilde{\psi}_k\rangle \] (276)
Now, we emphasize that it is quite possible, and often the case for non-hermitian systems, that $R \neq 1$. Moreover, it is also often the case that $\delta_{k,n} \neq \langle\psi_k|\psi_n\rangle \equiv \int dx \chi_k(x)\overline{\chi_n(x)}$ and $\delta_{k,n} \neq \langle\psi_k|\psi_n\rangle \equiv \int dx \overline{\psi_k(x)}\psi_n(x)$.

For emphasis and clarity, let us restate the previous nine relations in terms of wave functions $\psi(x) \equiv \langle x|\psi\rangle$, their dual functions $\langle\psi|x\rangle \equiv \chi(x)$, and the integral kernel in the position basis
\[ R(x,y) \equiv \langle x|R|y\rangle \] (277)
We have
\[ \overline{\chi_n(x)} = \int R(x,y)\psi_n(y) dy \] (278)
\[ \overline{\psi_n(y)}R(y,x) dy \] (279)
\[ \chi_n(x) = \int \overline{\psi_n(y)}R(y,x) dy \] (280)
\[ \delta_{n,k} = \int \overline{\psi_n(x)}R(x,y)\psi_k(y) dx dy \] (281)
\[ \psi(x) = \sum_k c_k\psi_k(x) \] (282)
\[ \sum_k |c_k|^2 = \int \overline{\psi(y)}R(y,x)\psi(x) dx dy \] (283)
\[ = \int \chi(x)\psi(x) dx \] (284)
\[ \chi(x) = \int \overline{\psi(y)}R(y,x) dy \] (285)
\[ = \sum_n c_n\chi_n(x) \] (286)
The role of the states and their duals is interchanged upon constructing the dual metric
\[ \tilde{R} = \sum_k |\psi_k \rangle \langle \psi_k| = \tilde{R}^\dagger \] (287)
In terms of this operator we have
\[ \langle \psi | = \langle \tilde{\psi} | \tilde{R} = \sum_n c_n \langle \psi_n | \] (288)
\[ |\psi\rangle = \tilde{R} \tilde{\psi} = \sum_k c_k |\psi_k\rangle \] (289)
so \( \tilde{R} \) is effectively the inverse to \( R \), and vice versa, at least on the appropriate subspaces. Correspondingly, we have the formal relations
\[ \tilde{R}R = \sum_k |\psi_k \rangle \langle \tilde{\psi}_k| \] (290)
\[ RR\tilde{R} = \sum_k |\tilde{\psi}_k \rangle \langle \psi_k| \] (291)
\[ \tilde{R}R = \tilde{R}R\tilde{R} , \quad R\tilde{R} = R\tilde{R}R \] (292)
The first of these acts as the unit operator on the span of \( \{ |\psi_k\rangle \} \) while the second acts as the unit on the span of \( \{ |\tilde{\psi}_k\rangle \} \). So \( \tilde{R}R \) and \( R\tilde{R} \) are at least projection operators onto those respective spaces, but they are not necessarily \( I \) acting on all functions. The effect of \( \tilde{R}R \) acting on the span of \( \{ |\tilde{\psi}_k\rangle \} \) is not clearly discernible, nor is \( R\tilde{R} \) acting on the span of \( \{ |\psi_k\rangle \} \).

For many PT symmetric systems, the duals are proportional to the wave functions. When these are chosen to be PT eigenstates, as is usually the case, then in a broad class of situations \( \chi_n(x) = \pm \psi_n(x) \). Again, we stress the absence of complex conjugation. However, there are some interesting situations where the dual functions are not so simply related to the wave functions \[8, 9\]. This is particularly the case when dealing with eigenfunctions at so-called “spectral singularities.” These exceptional cases seem to illustrate more general features of non-hermitian systems, so we will focus on them in this discussion.

Perhaps the most interesting feature of these cases is that the wave functions dual to energy eigenfunctions do not also obey homogeneous equations when acted upon by the Hamiltonian. Rather, in Dirac notation, the structure is like this:
\[ H |E\rangle = E |E\rangle , \quad \langle E|H = E\langle E| + \langle I_E| \] (293)
The inhomogeneity \( \langle I_E| \) varies with \( E \), usually, but for the situation of interest to us here, it is orthogonal to all the energy eigenstates, \( \langle I_E| E'\rangle = 0 \). For the PT symmetric theories of interest, the energy eigenvalues are real.

It is evident at this point that we may have some freedom in our construction of density operators, given that we may have \( |\tilde{\psi}_k\rangle \neq |\psi_k\rangle \), etc. This is so. There are four choices in general. For pure states in the various spans of the basis vectors, as specified above, but otherwise arbitrary, we have
\[ \rho = |\psi\rangle \langle \psi| = \rho^\dagger \] (294)
\[ \tilde{\rho} = |\tilde{\psi}\rangle \langle \tilde{\psi}| = \tilde{\rho}^\dagger \] (295)
as well as the less symmetrical
\[ (\tilde{\rho}) = |\tilde{\psi}\rangle \langle \psi| = \tilde{\rho}^\dagger \] (296)
\[ (\rho^\dagger) = |\psi\rangle \langle \tilde{\psi}| = (\rho^\dagger)^\dagger \] (297)
These are all interrelated by the \( R \) and \( \tilde{R} \) operators.
\[ (\rho^\dagger) = \rho R = \tilde{R} \tilde{\rho} \] (298)
\[ (\tilde{\rho}) = R\rho = \tilde{\rho} \tilde{R} \] (299)
\[ \rho = \tilde{R} \tilde{\rho} R = (\rho^\dagger) \tilde{R} = \tilde{R} (\rho^\dagger) \] (300)
\[ \tilde{\rho} = R\rho R = (\rho^\dagger) R = R(\rho^\dagger) \] (301)
When $\langle \psi | \psi \rangle = 1$, the various density operators also obey several variants of the standard pure state condition.

\[
\rho \tilde{\rho} = (\rho^*) \quad (302)
\]

\[
\tilde{\rho} \rho = (\tilde{\rho}) \quad (303)
\]

\[
(\rho^*) (\rho^*) = (\rho^*) \quad (304)
\]

\[
(\tilde{\rho}) (\tilde{\rho}) = (\tilde{\rho}) \quad (305)
\]

\[
\tilde{\rho} (\rho^*) = \tilde{\rho} \quad (306)
\]

\[
(\tilde{\rho}) \rho = \rho \quad (307)
\]

\[
(\rho^*) \rho = \rho \quad (308)
\]

\[
\rho (\tilde{\rho}) = \rho \quad (309)
\]

\[
\text{full equivalence is then given by}
\]

\[
e_{k,n} \sim |\phi_k \rangle \langle \phi_n| \quad (311)
\]

\[
f_{k,n} \sim |\psi_k \rangle \langle \psi_n| \quad (312)
\]

\[
f_{k,n} \sim \tilde{|\psi_k \rangle \langle \psi_n|} \quad (313)
\]

\[
g_{k,n} \sim |\psi_k \rangle \langle \psi_n| \quad (314)
\]

\[
h_{k,n} \sim |\phi_k \rangle \langle \phi_n| \quad (315)
\]

\[
h_{k,n} \sim \tilde{|\psi_k \rangle \langle \psi_n|} \quad (316)
\]

The star products of the previous Appendix correspond to the operator products of these dyadics, in an obvious way.

When the metric $R$ is positive definite and hence invertible, the other set of states could be realized through its hermitian square root $S$, with

\[
R = S^2 \quad (317)
\]

In this case we may take

\[
|\phi_k \rangle = S |\psi_k \rangle \quad (318)
\]

\[
\langle \psi_n | = \langle \phi_n | S \quad (319)
\]

\[
|\psi_k \rangle = S^{-1} |\phi_k \rangle \quad (320)
\]

\[
\langle \phi_n | = \langle \psi_n | S^{-1} \quad (321)
\]

If $R$ is positive but not definite, only the first two of these are guaranteed valid, but then the role of $S^{-1}$ to give the latter two relations may be effectively played on the appropriate subspace by the “other” square root, $\tilde{S}$, where

\[
\tilde{R} = \tilde{S}^2 \quad (322)
\]

\[
|\psi_k \rangle = \tilde{S} |\phi_k \rangle \quad (323)
\]

\[
\langle \phi_n | = \langle \psi_n | \tilde{S} \quad (324)
\]

However, we note that $\rho^2 \neq \rho$ and $\tilde{\rho}^2 \neq \tilde{\rho}$, in general, but rather

\[
\rho R \rho = \rho \, , \quad \tilde{\rho} \tilde{R} \tilde{\rho} = \tilde{\rho} \quad (310)
\]

as implicitly stated in the previous set of equations.

The results of the previous Appendix can be recast into the density operator language. A full equivalence with the results there can be achieved provided there is another set of states and their duals, $\{|\phi_k \rangle, \langle \phi_n |\}$, in one-to-one correspondence with $\{|\psi_k \rangle, \langle \psi_n |\}$, where the other set is equipped with a trivial metric (i.e. $R = 1$), as would be the case for a biorthogonal system which admits a similarity transformation to an hermitian Hamiltonian. The
Appendix F. Operator expressions from Weyl transforms

Real Liouville theory  (adapted from [10]) Given the factorized phase-space generating function

$$G(z; x, p) = K_{ip}(e^z) \exp \left( \frac{1}{2} e^{2x-z} \right)$$  \hspace{1cm} (325)

what is the operator corresponding to it? According to Weyl’s prescription the associated operator is

$$G(z) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \ G(z; x, p) \exp(i\tau(p - p) + i\sigma(x - x))$$  \hspace{1cm} (326)

$$= \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \ \exp(i\tau p + i\sigma x) \exp \left( -\frac{1}{2} e^{2x-z} - i\sigma x \right) K_{ip}(e^z) \exp(-i\tau p).$$

The integrals over $x$ and $p$ may be evaluated separately, if the $\sigma$ contour is first shifted slightly above the real axis, $\sigma \to \sigma + i\epsilon$, thereby suppressing contributions to the $x$-integral as $x \to -\infty$. Now $s \equiv \frac{1}{2} e^{2x-z}$ gives

$$\int_{-\infty}^{+\infty} dx \ \exp \left( -\frac{1}{2} e^{2x-z} - i(\sigma + i\epsilon) x \right) = \int_0^\infty ds \ \frac{2s e^z}{2s} (2se^z)^{-i(\sigma+i\epsilon)/2} \exp(-s)$$

$$= \frac{1}{2} e^{-i(z+\ln 2)\sigma/2} \Gamma(-i(\sigma + i\epsilon)/2) \hspace{1cm} (327)$$

But then again

$$\int_{-\infty}^{\infty} dp \ K_{ip}(e^z) \exp(-i\tau p) = \frac{1}{2} \int_{-\infty}^{\infty} dX e^{-e^z \cosh X} 2\pi \delta(X - \tau) = \pi e^{-e^z \cosh \tau} \hspace{1cm} (328)$$

So

$$G(z) = \frac{1}{8\pi} \int d\tau d\sigma \ e^{-i(z+\ln 2)\sigma/2} \Gamma(-i(\sigma + i\epsilon)/2) e^{-e^z \cosh \tau} \exp(i\tau p + i\sigma x) \hspace{1cm} (329)$$

The shifted $\sigma$ contour avoids the pole in $\Gamma$ at the origin.

Re-ordering with all $p$s to the right (thereby departing from Weyl ordering but without actually changing the operator) yields $\exp(i\tau p + i\sigma x) = \exp(i\sigma x) \exp(i\tau p/2) \exp(i\tau p)$. Performing the $\sigma$ integration before the $\tau$ integration permits taking the limit $\epsilon \to 0$ to obtain

$$G(z) = \frac{1}{8\pi} \int d\tau \ \left( \int d\sigma \Gamma(-i(\sigma + i\epsilon)/2) \exp(i\sigma x + i\tau p/2 - i\sigma(z + \ln 2)/2) \right) e^{-e^z \cosh \tau} \exp(i\tau p)$$

$$= \frac{1}{8\pi} \int d\tau \ \left( 4\pi \exp \left( -e^{2x+\tau-(z+\ln 2)} \right) \right) e^{-e^z \cosh \tau} \exp(i\tau p)$$

$$= \frac{1}{2} \int d\tau e^{-e^z \cosh \tau} \ \exp \left( -\frac{1}{2} e^{2x+\tau-z} \right) \exp(i\tau p) \hspace{1cm} (330)$$

This is the operator correspondent of $G(z; x, p)$. In this form it is straightforward to show that

$$[p^2 + e^{2x}, G(z)] = 0 \hspace{1cm} (331)$$

\[\text{\footnotesize \textsuperscript{15}The operator corresponding to } O \text{ will be distinguished from it by using the same letter but sans serif: } O.\]
for all values of $z$, which suggests an interpretation of $G(z)$ as a propagator and a concomitant interpretation of $z$ as a generalized “time.”

The form (330) also leads to a more intuitive Hilbert space representation. Acting to the right of a position eigen-bra, we have $\langle x| x = \langle x| x$, while the subsequent exponential of the momentum operator just translates, $\langle x| \exp(i\tau p) = \langle x + \tau|$. So the full right-operation of $G$ is

$$\langle x| G(z) = \frac{1}{2} \int d\tau \langle x + \tau| \exp \left( -\frac{1}{2} e^{2x+\tau-z} - \frac{1}{2} e^{z+\tau} - \frac{1}{2} e^{z-\tau} \right)$$

(332)

Inserting $1 = \int dx |x| \langle x|$ gives $G(z) = \int dx |x| \langle x| G(z)$, and leads to a coordinate space realization of the operator involving an $x, y$-symmetric kernel, in which form it is clear that $G(z) = G(z)^\dagger$ for real $z$.

$$G(z) = \frac{1}{2} \int dxdy |x| \exp \left( -\frac{1}{2} e^{x+y-z} - \frac{1}{2} e^{x+y-z} - \frac{1}{2} e^{z-y+x} \right)$$

(333)

All the operator characteristics are now carried by the dyadics $|x\rangle \cdots \langle y|$. The composition law of this operator parallels its phase-space isomorph.

$$G(u)G(v) = \frac{1}{2} \int dw \exp \left( -\frac{1}{2} (e^{u+v-w} + e^{u-v+w} + e^{-u+v+w}) \right) G(w) = G(v)G(u)$$

(334)

**Imaginary Liouville theory** Let us apply these same methods to the imaginary Liouville case. Here, we can just write down the final answer from what we know about real Liouville theory, as given above, and then work backwards, a procedure often followed in [8]. It is easily seen that the following operator similarity transformation is hermitian, for real $s$, and converts operator $H$ into operator $H^\dagger$.

$$F(s) = \frac{1}{2} \int dxdy |x| \exp \left( \frac{1}{2s} e^{(x+y)} - \frac{s}{2} e^{-(x+y)} - \frac{s}{2} e^{i(x+y)} \right)$$

(335a)

$$F(s)H = H^\dagger F(s)$$

(335b)

$$H = p^2 + e^{2ix} \quad H^\dagger = p^2 + e^{-2ix} \quad \text{where } x, p \text{ are operators}$$

(335c)

However, it is not obvious that $F(s)$ is positive definite, even on a restricted range of $p$ eigenstates.

Acting on a position eigenbra, $\langle x|$, it can be seen that $F$ is given by the parity operator multiplying a single parametric integral, whose integrand can be written in factorized form as a product of a spatial translation and an exponential of a momentum translation, just like the real Liouville case. That is to say

$$F(s) = \frac{1}{2} \mathbb{P} \int d\sigma e^{-s \cos \sigma} \exp \left( \frac{1}{2s} e^{i(2\pi+\sigma)} \right) \exp (i\sigma p)$$

(336)
where again \( x \) and \( p \) are position and momentum operators, and where \( P \) is the usual parity operator:
\[
\langle x | P = \langle -x |, \ P x P = -x, \ P p P = -p.
\]
The range of the \( \sigma \) integration is whatever is needed for completeness of position eigenstates:
\[
1 = \int d\sigma \ |\sigma \rangle \langle \sigma |.
\]
The form (330) should be compared to (330), with the identification \( s = -e^z \).

The presence of \( P \) in \( F(s) \) has interesting consequences. From (330) it follows for real \( s \) that we can write
\[
F(s) = \int d\sigma \ e^{-s \cos \sigma} \ D(s) \ P \ D^\dagger(s) \quad (337)
\]
where
\[
D(s) \equiv \exp \left( -\frac{1}{2} i \sigma p \right) \exp \left( \frac{1}{4s} e^{-2ix} \right), \quad D^\dagger(s) \equiv \exp \left( \frac{1}{4s} e^{2ix} \right) \exp \left( \frac{1}{2} i \sigma p \right) \quad (338)
\]
From the form (337), it would seem that \( F \) is not positive, although it is manifestly hermitian. From the form (335) we can also check that \( F = F^\dagger \) for real \( s \), but not so easily as from (335a) or from (337).

In the course of obtaining (336) from (335a) it is remarkably easy to see how the parity operator appears. No prior knowledge of its presence is needed. However, if we remove \( P \) from the RHS of (336), we see that
\[
[H, PF(s)] = 0 = [H^\dagger, F(s) P] \quad (339)
\]
which is similar to (331). All this again suggests an interpretation of \( PF(s) \), or \( F(s) P \), as a propagator for the system governed by \( H \), or \( H^\dagger \), as well as an interpretation of \( s \) as a generalized time. Moreover, from (336) the phase-space correspondent of \( PF(s) \) is obviously given by
\[
F(s; x, p) = \frac{1}{2} \int d\sigma \ e^{-s \cos \sigma} \ \exp \left( \frac{1}{2s} e^{i(2x+\sigma)} \right) \ * \ \exp (i\sigma p) \quad (340)
\]
\[
= \frac{1}{2} \int d\sigma \ e^{-s \cos \sigma} \ \exp (i\sigma p) \ \exp \left( \frac{1}{2s} e^{2ix} \right) \quad (341)
\]
This is nicely factorized into a function of \( p \) times a function of \( x \). For integer \( p \), with \( \sigma \) integrated over \([0, 2\pi]\), it is simply
\[
F(s; x, p \in \mathbb{N}) = \pi I_p (-s) \exp \left( \frac{1}{2s} e^{2ix} \right) \quad (342)
\]
This should be compared to (325), again with the identification \( s = -e^z \).

We leave as exercises for the interested reader to determine the operators that result from taking the Weyl transforms of the phase-space metrics and/or the dual metrics.

\[\text{LaTeX}\]