Monogamy and ground-state entanglement in highly connected systems

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We consider the ground-state entanglement in highly connected many-body systems, consisting of harmonic oscillators and spin-1/2 systems. Varying their degree of connectivity, we investigate the interplay between the enhancement of entanglement, due to connections, and its frustration, due to monogamy constraints. Remarkably, we see that in many situations the degree of entanglement in a highly connected system is essentially of the same order as in a low connected one. We also identify instances in which the entanglement decreases as the degree of connectivity increases.

Entanglement theory has experienced an impressive development in the last decade, mainly due to the key role quantum correlations play in quantum information science. The novel concepts and mathematical methods developed in this new research area are beginning to reveal their usefulness also in different contexts. A striking example of this tendency is given by the physics of quantum many-body systems. As an instance, the analysis of the role played by entanglement in quantum phase transitions allowed for a deeper understanding of this purely quantum phenomena [1, 2, 3]. In this scenario, entanglement theory is also giving a fundamental contribution in the development of new methods capable of simulating efficiently strongly interacting systems [4, 5, 6].

Clearly, the correlations between different parts of a many-body system are originated by their mutual interaction. In this sense it is natural to expect that the ground state of a strongly interacting and connected quantum system will exhibit a high degree of entanglement. However, this intuition has to be taken cautiously, since the shareability properties of quantum correlations are especially non trivial and without classical analogue. One of the main differences between classical and quantum correlations is the so called monogamy of the latter [4]. In the classical scenario, the fact that two systems share some correlations does not prevent them from being correlated with a third party. On the contrary, two maximally entangled quantum systems can share no correlation at all with a third one. More generally, quantum correlations are not infinitely sharable, and the more the entanglement the less the number of systems with which it can be shared.

Consider now two similar Hamiltonians consisting of the same interacting terms between pair of particles, the only difference being the degree of connectivity. One of them, for instance, has only nearest-neighbor interactions, while the second has also next-to-nearest-neighbor interactions. Let us focus on the ground-state entanglement between two halves of the system. Naively, the more connected hamiltonian is expected to have a larger entanglement, since there are more bonds connecting the two halves. However, in the more connected system, each particle has to share the quantum correlations with a larger number of particles, so the connecting bonds may give a smaller amount of entanglement. Therefore, it is unclear which geometry leads to a larger ground-state entanglement.

In this work we analyze the interplay between the enhancement of the ground-state entanglement due to connections and its suppression due to monogamy constraints. We consider spin-1/2 and infinite dimensional (harmonic oscillators) systems of varying geometries with two-body interactions and focus on the bipartite entanglement between two halves of the system. Remarkably, we see that in many situations the degree of entanglement in a highly connected system is essentially of the same order as in the case of a low connected one. Actually, we can even individuate systems for which the entanglement decreases as the degree of connectivity increases.

Before proceeding, let us mention that there exist some works studying how the monogamy of entanglement affects the ground state properties of Hamiltonians with nearest-neighbor interactions, see for example [3, 4]. The ground-state entanglement of a highly symmetric and connected system, the so-called Lipkin-Meshkov-Glick model, was also computed in [10].

As said, we consider two paradigmatic systems, namely interacting spin-1/2 and bosonic particles. Concerning the former, we study a system of $n$ spin-1/2 particles under the XY Hamiltonian,

$$H = \sum_{i,j} t_{ij} [\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y],$$

(1)

where $\sigma_k^i (k = x, y, z)$ denote the Pauli matrices referred to the $i$-th particle. The coupling $t_{ij}$ will be set different from zero when the $i-j$ couple directly interacts, i.e. in dependence on the topology and connectivity of the system. The actual value of the nonzero $t_{ij}$ will be chosen randomly, in order to have averaged properties and avoid the dependence of our results on the details of the interaction. Interactions of the type (1) may model highly connected physical systems, such as quantum spin glasses [11]. The entanglement between two parts of the system will be measured by the entropy of entanglement $E$, namely the von Neumann entropy $S(\rho) = - \text{Tr}[\rho \log_2 \rho]$ of one of the reduced subsystems [12].

Concerning the bosonic case, we consider systems consisting of $n$ harmonic oscillators with quadratic cou-
pling. Such systems may model discrete versions of Klein-Gordon fields, or vibrational modes in crystal lattices, ion traps and nanomechanical oscillators. We define the vector $R$ of quadrature operators by $R_j = \hat{X}_j$ and $R_{n+j} = \hat{P}_j$ ($1 \leq j \leq n$), where $\hat{X}_j$ and $\hat{P}_j$ are the position and linear momentum operator respectively. For simplicity, we consider only a coupling via the different position operators, in which case the Hamiltonian is of the form $H = R^T (V/2 \oplus I_n/2) R$, where $I_n$ denotes the $n \times n$ identity matrix. The potential matrix $V$ is defined via the harmonic coupling between oscillator $i$ and $j$, namely $\alpha(\hat{X}_i - \hat{X}_j)^2/2$. For each geometry considered in the following, we denote by $C$ the $n \times n$ adjacency matrix of the corresponding graph, with elements $c_{ij} = c_{ji} = 1$ if the $i$-th and $j$-th oscillator are coupled and $c_{ij} = c_{ji} = 0$ otherwise. Then, the potential matrix $V$ is given by $V_{ij} = -\alpha c_{ij}$ ($i \neq j$) and $V_{ii} = 1 + \alpha \sum_{j=1}^n c_{ij}$. The ground state of the system is a Gaussian state characterized by the covariance matrix $\gamma = (\gamma_x \oplus \gamma_p)/2$, with $\gamma_x = V^{-1/2}$ and $\gamma_p = V^{1/2}$.

We use as entanglement measure the logarithmic negativity $\mathcal{E}$, $N_i$, between two generic group $A$ and $B$. It can be shown that $N_i$ is given by

$$N_i = -\frac{1}{m} \log_2 \min[1, \Lambda_j(\gamma_x \oplus \gamma_p)], \quad \text{(2)}$$

where $\Lambda_j(M)$ is the $j$-th eigenvalue of matrix $M$. We denote by $P$ the $n \times n$ diagonal matrix with $j$-th diagonal entry given by 1 or $-1$, depending on whether the oscillator on position $1 \leq j \leq n$ belongs to group $A$ or $B$, respectively.

—Chains with neighbor coupling. The first configuration that we consider is a one-dimensional (1D) chain of $n$ particles, in which each of them can interact with $n_c$ of its neighbors. Thus, $n_c$ is the parameter that characterizes the degree of connectivity in this setting. We consider a distance-independent interaction, in order to avoid any dependence on the particular scaling of the interaction strength with the distance. In particular, given the ground state, we calculate the entanglement between the two halves of the system (groups $A$ and $B$) as a function of the number of interacting neighbors $n_c$. The typical behavior in the case of a XY system is reported in Fig. 1 for the case of $n = 22$ spins. The exact calculation of the ground state was performed using the SPINPACK package. The solid line represents the averaged ground-state entanglement, where the Hamiltonian parameters $t_{ij}$ between pair of particles are randomly chosen in the interval $[0, 1]$, while the dashed line gives the largest entanglement obtained. We clearly see that the entanglement grows only slightly and, in particular, the fully connected chain has a degree of entanglement comparable to the nearest-neighbor coupled chain. Note that, by contrast, the number of bonds connecting the two halves of the chain increases as $n_c^2$. The same behavior is observed for different Hamiltonian operators, consisting of other interaction terms, and smaller sizes.

FIG. 1: For a closed chain of $n = 22$ spin–$1/2$ particles with XY interaction $\Omega$, the entropy of entanglement is plotted versus the number of connected neighbors $n_c$, averaged over 100 realizations. The dashed line gives the largest entanglement obtained. Inset: for an open chain of $n = 100$ oscillators the logarithmic negativity $N_i$ is plotted versus $n_c$. From top to bottom the coupling constant $\alpha$ is given by $\alpha = 10, 1, 0$.

We consider now the same configuration for the case of a chain of harmonic oscillators. As said above, the interactions between the particles simply correspond to oscillators of coupling constant $\alpha$. The entanglement between the two halves of an open chain consisting of $n = 100$ oscillators is shown in the inset of Fig. 1, where the logarithmic negativity is plotted versus the number of coupled neighbors, $n_c$. One clearly see that the entanglement increases (almost linearly) as far as $n_c \lesssim n/2$, whereas for higher connected systems the entanglement is frustrated. The frustration mechanism is indeed stronger than in the spin case, the entanglement decreasing at some point as the number of connections increases. Notice the quite universal behavior of these curves: the position of the maximum does not depend on the coupling constant $\alpha$ and, as one can expect, the entanglement increases with $\alpha$, for fixed $n_c$.

Both the examples reported here confirm that the monogamy of entanglement plays a predominant role for highly connected systems. As said, as the connectivity increases, each particle of, e.g., set $A$ becomes as well entangled with many other particles of the same set. This in turn limits, for monogamy reasons, the entanglement with the particles of set $B$.

Up to now we considered the behavior of the entanglement for fixed system size. An analysis exploiting the dependence of the entanglement on the size of the system is reported in Fig. 2. In particular, we focused on the results corresponding to i) nearest-neighbor coupling and ii) the optimal configuration in which the number of connections $n_c$ is chosen in order to give the maximal amount of entanglement. For a closed harmonic chain,
we observe that the entanglement remains constant in the nearest-neighbor case (as we expect from the results of Ref. [13]) whereas it increases only logarithmically in the optimally connected case. Remarkably, the behavior is similar for a closed spin chain, in which the optimal number of connections \( n_{c_{\text{opt}}} \) is always given by \( n_{c_{\text{opt}}} = n \) (i.e., by the fully connected scenario). Although our computations are not very conclusive, they suggest a sub-linear increase in this case too.

**—Bipartite graph.** We exploited also different configurations, e.g. random graphs corresponding to disordered systems, always observing that monogamy of entanglement strongly suppresses the entanglement in highly connected systems. Perhaps the configuration in which the effects of the monogamy show up more impressively is given by the case in which the system can be represented by a random bipartite graph. The latter is constituted by two sets \( A \) and \( B \) (of \( n/2 \) particles each) for which particles belonging to the same set never interact directly, whereas the probability that a generic particle in \( A \) interacts with a particle in \( B \) is given by \( c_p \) (connectivity parameter). For example in the fully connected case \( (c_p = 1) \) each particle in \( A \) is coupled to each particle in \( B \). We report here the results for the bosonic case. For a fixed coupling constant we look for the optimal \( c_p^{\text{opt}} \) such that the entanglement is maximized. As shown in Fig. 2 we see that \( c_p^{\text{opt}} \neq 1 \) in general, depending non-trivially on \( \alpha \). For high values of \( \alpha \) the maximum entanglement is provided by Hamiltonian with few connections for each oscillator. Vice-versa, for low values of \( \alpha \) the completely connected case tends to maximize the entanglement. Note again that particles belonging to the same set do not directly interact. However, these particles become entangled through the common interaction with the particles of the other set. This, at the same time, limits the amount of entanglement between the two sets because of monogamy.

![FIG. 2: For an XY spin system the entropy of entanglement \( E \) is plotted as a function of the system size \( n \) (averaged over 100 realizations). Inset: corresponding graph for the negativity \( N_1 \) in a closed harmonic chain with \( \alpha = 1 \) for nearest-neighbor coupling (dashed line) and optimal coupling (solid line, see text).](image)

**—Monogamy inequality.** To quantify how the monogamy of entanglement acts in the considered scenarios we can refer to monogamy inequalities. In particular, for spin system, we consider the inequality [7, 16]

\[
\tau_{1:2\ldots n}^{1:j} \geq \frac{1}{\tau} \sum_{j=2}^n \tau_{i:j}^{1:j},
\]

where \( \tau \) is the square of the concurrence, or tangle, a measure of entanglement. Thus, \( \tau_{1:2\ldots n}^{1:j} \) quantifies the amount of entanglement between particle 1 and \( j \), whereas \( \tau_{1:2\ldots n} \) refers to the entanglement between particle 1 and the rest. Fig. 3 shows, for the bipartite graph with \( n_c \) connected neighbors, the quantities \( \sum_{j=2}^n \tau_{i:j}^{1:j} \) and \( \tau_{1:2\ldots n} \), whose difference indicates the presence of multipartite entanglement. For any value of the connectivity, any spin is maximally entangled with the rest, since \( \tau_{1:2\ldots n} = 1 \).

As a final remark, we would like to notice the difference between our findings and related results which investigate ground-state entanglement in the context of an area law [18]. For instance, it is shown in Refs. [3, 19] that, for non-critical systems with nearest-neighbor coupling, the entanglement between a distinguished part of a system and the rest increases as the area of the boundary between them, hence as the number of connections. In these works the Hamiltonian of the global system is kept fixed, whereas the size of the distinguished region varies.
FIG. 4: For a bipartite graph of XY spin systems with \( n = 14 \), the values of \( \tau_{12,n} \) (solid) and \( \sum_{j=2}^{n} \tau_{1j} \) (dashed) are shown versus the number of connections \( n_c \) (averaged over 100 realizations). Inset: corresponding results for the harmonic oscillators case (for \( \alpha = 1 \) and \( n = 100 \)), where the Gaussian tangle \( \tau_G \) is used as entanglement measure.

It is clear that the monogamy constraints do not act significantly, since the connectivity of the systems remains unchanged with the size. In most of our previous analysis we instead kept fixed the size of the distinguished region, whereas the connections between particles were modified (see Figs. 1, 3, and 4). As far as for the results of Fig. 2 is concerned, the connectivity is kept fixed, whereas the size of the distinguished region is varied (as well as the size of the total system), in a fashion resembling the works on area law. We see that the entanglement remains constant in the nearest-neighbor case, in agreement with the area law. In the case of highly connected systems, the boundary area (given by the connections between the two halves of the system) only gives an upper bound to the entanglement [20]. Thus, our results reveal that the entanglement can actually scale sensibly slower than the area law. In the case of highly connected systems, the ground-state entanglement and the connectivity in spin-

Renormalization Group (DMRG). As a matter of fact, the latter method has recently been efficiently applied to a specific highly connected model, in order to analyze quantum phase transitions in spin glass systems [22]. The efficiency of DMRG in this scenario is not trivial and is related to the fact that the specific system analyzed in Ref. [22] turns out to be only slightly entangled. We show here that a large variety of systems may have such a character, due to the fundamental constrain imposed by the monogamy of entanglement. Our results, then, may encourage the search for novel classical algorithms able at simulating highly connected quantum systems.

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[15] [http://www.e.uni-magdeburg.de/jshulen/spin/](http://www.e.uni-magdeburg.de/jshulen/spin/)