New Lepton Family Symmetry and Neutrino Tribimaximal Mixing

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Abstract

The newly proposed finite symmetry $\Sigma(81)$ is applied to the problem of neutrino tribimaximal mixing. The result is more satisfactory than those of previous models based on $A_4$ in that the use of auxiliary symmetries (or mechanisms) may be avoided. Deviations from the tribimaximal pattern are expected, but because of its basic structure, only $\tan^2 \theta_{12}$ may differ significantly from 0.5 (say 0.45) with $\sin^2 2\theta_{23}$ remaining very close to one, and $\theta_{13}$ very nearly zero.
Based on present neutrino-oscillation data, the neutrino mixing matrix $U_{\alpha i}$ linking the charged leptons ($\alpha = e, \mu, \tau$) to the neutrino mass eigenstates ($i = 1, 2, 3$) is determined to a large extent \cite{1}. In particular, a good approximate description is that of the so-called tribimaximal mixing of Harrison, Perkins, and Scott \cite{2}, i.e.

$$U_{\alpha i} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}. \quad (1)$$

Using the discrete lepton family symmetry group $A_4$ \cite{3,4}, this pattern has been discussed in a number of recent papers with varying additional assumptions \cite{5,6,7,8,9,10,11,12,13,14,15,16}. In particular, auxiliary symmetries (or mechanisms) beyond $A_4$ are required to enforce the following conflicting alignment of vacuum expectation values: $(1,1,1)$ for a $3$ representation which couples to charged leptons, and $(1,0,0)$ for a $3$ representation which couples to neutrinos. As shown below, this problem may be alleviated if $A_4$ is replaced by another finite discrete symmetry $\Sigma(81)$, which was recently proposed \cite{17}.

Consider the basis $(a_1, a_2, a_3)$ and the $Z_3$ transformation

$$a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_1. \quad (2)$$

If this is supplemented with the $Z_2$ transformation

$$a_{1,2} \rightarrow -a_{1,2}, \quad a_3 \rightarrow a_3, \quad (3)$$

then the group generated is $A_4$, which is the symmetry group of the even permutation of 4 objects, and that of the perfect tetrahedron \cite{18}. It is also a subgroup of $SU(3)$, denoted as $\Delta(12)$. If Eq. (2) is supplemented instead with another $Z_3$, i.e.

$$a_1 \rightarrow \omega a_1, \quad a_2 \rightarrow \omega a_2, \quad a_3 \rightarrow \omega^2 a_3, \quad (4)$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$, then the group generated is $\Delta(27)$ \cite{19,20,21}, which is also a subgroup of $SU(3)$. If Eq. (4) is replaced with

$$a_1 \rightarrow \omega a_1, \quad a_{2,3} \rightarrow a_{2,3}, \quad (5)$$

$$a_{2,3} \rightarrow -a_{2,3},$$

$$a_1 \rightarrow a_1, \quad a_2 \rightarrow -a_2, \quad a_3 \rightarrow -a_3, \quad (6)$$

then the group generated is $A_4$, which is the symmetry group of the even permutation of 4 objects. Therefore, $A_4$ is a good candidate for the symmetry of the lepton mixing matrix.
then the group generated, call it $\Sigma(81)$, contains $\Delta(27)$. It is a subgroup of $U(3)$ but not $SU(3)$. It has 9 one-dimensional irreducible representations $1_1(i = 1, \ldots, 9)$ and 8 three-dimensional ones $3_A, \overline{3}_A, 3_B, \overline{3}_B, 3_C, \overline{3}_C, 3_D, \overline{3}_D$. Its $17 \times 17$ character table and the 81 matrices of its defining representation $3_A$ are given in Ref. [17].

Consider the supersymmetric extension of the Standard Model with 3 lepton families. Under $\Sigma(81)$, let

$$L_i = (\nu_i, l_i) \sim 3_A, \quad l_i^c \sim 1_{1,2,3}, \quad \Phi = (\phi^0, \phi^-) \sim 1_1, \quad \Phi = (\phi^0, \phi^-) \sim 1_1,$$

$$\sigma_i \sim 3_A, \quad \bar{\sigma}_i \sim 3_A, \quad \chi_i \sim 3_B, \quad \bar{\chi}_i \sim \overline{3}_B, \quad \xi = (\xi^+, \xi^+, \xi^0) \sim 1_1. \quad (7)$$

Using the multiplication rules given in the Appendix, the allowed quadrilinear Yukawa terms are $(L_1 \bar{\sigma}_1 + L_2 \bar{\sigma}_2 + L_3 \bar{\sigma}_3)\Phi$, $(L_1 \bar{\sigma}_1 + \omega^2 L_2 \bar{\sigma}_2 + \omega L_3 \bar{\sigma}_3)\Phi$, $(L_1 \bar{\sigma}_1 + \omega^2 L_2 \bar{\sigma}_2 + \omega L_3 \bar{\sigma}_3)\Phi$, $(L_1 L_2 \chi_1 + L_3 L_3 \chi_1 + L_3 L_1 \chi_2)\xi$. As shown below, the singlet superfields $\sigma_i, \bar{\sigma}_i, \chi_i, \bar{\chi}_i$ will acquire vacuum expectation values without breaking the supersymmetry. The desirable solutions $(1,1,1)$ for $\sigma_i, \bar{\sigma}_i$, and $(1,0,0)$ for $\chi_i$ and $\bar{\chi}_i$ may then be obtained in a natural symmetry limit, for which the mismatch between the charged-lepton and neutrino mass matrices will exhibit tribimaximal mixing.

The most general superpotential of the singlet superfields invariant under $\Sigma(81)$ is given by

$$W = m_\sigma(\sigma_1 \bar{\sigma}_1 + \sigma_2 \bar{\sigma}_2 + \sigma_3 \bar{\sigma}_3) + m_\chi(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2 + \chi_3 \bar{\chi}_3)$$

$$+ \frac{1}{3} f(\sigma_1^3 + \sigma_2^3 + \sigma_3^3) + \frac{1}{3} \bar{f}(\bar{\sigma}_1^3 + \bar{\sigma}_2^3 + \bar{\sigma}_3^3) + \frac{1}{3} h(\chi_1^3 + \chi_2^3 + \chi_3^3) + \frac{1}{3} \bar{h}(\bar{\chi}_1^3 + \bar{\chi}_2^3 + \bar{\chi}_3^3) + \lambda(\chi_1 \sigma_2 \sigma_3 + \chi_2 \sigma_3 \sigma_1 + \chi_3 \sigma_1 \sigma_2) + \bar{\lambda}(\bar{\chi}_1 \bar{\sigma}_2 \bar{\sigma}_3 + \bar{\chi}_2 \bar{\sigma}_3 \bar{\sigma}_1 + \bar{\chi}_3 \bar{\sigma}_1 \bar{\sigma}_2). \quad (8)$$

The resulting scalar potential has a supersymmetric minimum ($V = 0$) if

$$0 = m_\sigma \bar{\sigma}_1 + f \sigma_1^2 + \lambda(\chi_2 \sigma_3 + \chi_3 \sigma_2) = m_\sigma \sigma_1 + \bar{f} \sigma_1^2 + \bar{\lambda}(\bar{\chi}_2 \bar{\sigma}_3 + \bar{\chi}_3 \bar{\sigma}_2), \quad (9)$$

$$0 = m_\sigma \bar{\sigma}_2 + f \sigma_2^2 + \lambda(\chi_3 \sigma_1 + \chi_1 \sigma_3) = m_\sigma \sigma_2 + \bar{f} \sigma_2^2 + \bar{\lambda}(\bar{\chi}_3 \bar{\sigma}_1 + \bar{\chi}_1 \bar{\sigma}_3), \quad (10)$$
and all previous such models, the predicted value of \(\tan^2 \theta_{12} = 0.5\) is not the central value of

\[
0 = m_\sigma \bar{\sigma}_3 + f \sigma_3^2 + \lambda (\chi_1 \sigma_2 + \chi_2 \sigma_1) = m_\sigma \bar{\sigma}_3 + f \sigma_3^2 + \lambda (\bar{\chi}_1 \bar{\sigma}_2 + \bar{\chi}_2 \bar{\sigma}_1), \quad (11)
\]

\[
0 = m_\chi \bar{\chi}_1 + h \chi_1^2 + \lambda \sigma_2 \sigma_3 = m_\chi \bar{\chi}_1 + h \bar{\chi}_2^2 + \bar{\lambda} \bar{\sigma}_3 \bar{\sigma}_3, \quad (12)
\]

\[
0 = m_\chi \bar{\chi}_2 + h \chi_2^2 + \lambda \sigma_3 \sigma_1 = m_\chi \bar{\chi}_2 + h \bar{\chi}_2^2 + \bar{\lambda} \bar{\sigma}_3 \bar{\sigma}_1, \quad (13)
\]

\[
0 = m_\chi \bar{\chi}_3 + h \chi_3^2 + \lambda \sigma_1 \sigma_2 = m_\chi \bar{\chi}_3 + h \bar{\chi}_3^2 + \bar{\lambda} \bar{\sigma}_3 \bar{\sigma}_2. \quad (14)
\]

In the limit \(\lambda = \bar{\lambda} = 0\), the symmetry of \(W\) is enlarged to \(\Sigma(81) \times \Sigma(81)\). Thus it is natural to expect \(\lambda, \bar{\lambda} < f, f, h, \bar{h}\), and as a first approximation, a possible solution of \(V = 0\) is

\[
\langle \sigma_{1,2,3} \rangle_0 = -m_\sigma (f^2 \bar{f})^{-1/3}, \quad \langle \bar{\sigma}_{1,2,3} \rangle_0 = -m_\sigma (f^2 f)^{-1/3},
\]

\[
\langle \chi_1 \rangle_0 = -m_\chi (h^2 \bar{h})^{-1/3}, \quad \langle \bar{\chi}_1 \rangle_0 = -m_\chi (h^2 h)^{-1/3}, \quad \langle \chi_{2,3} \rangle_0 = \langle \bar{\chi}_{2,3} \rangle_0 = 0,
\]

where \(\lambda = \bar{\lambda} = 0\) has been assumed. This results in the desirable Yukawa terms \((l_1 + l_2 + l_3)l_i^c \phi^0, (l_1 + \omega^2 l_2 + \omega l_3)l_i^c \phi^0, (l_1 + \omega l_2 + \omega^2 l_3)l_i^c \phi^0, (\nu_1 \nu_1 + \nu_2 \nu_2 + \nu_3 \nu_3)\xi^0,\) and \(\nu_2 \nu_3 \xi^0\), leading to tribimaximal mixing [5]. Specifically

\[
\mathcal{M}_l = \begin{pmatrix}
 h_e & h_\mu & h_\tau \\
 h_e & \omega^2 h_\mu & \omega h_\tau \\
 h_e & \omega h_\mu & \omega^2 h_\tau
\end{pmatrix}
\]

\[
v = \frac{1}{\sqrt{3}} \begin{pmatrix}
 1 & 1 & 1 \\
 1 & \omega^2 & \omega \\
 1 & \omega & \omega^2
\end{pmatrix}
\]

\[
\mathcal{M}_l = \begin{pmatrix}
 a & 0 & 0 \\
 a & d & 0 \\
 0 & d & a
\end{pmatrix}
\]

\[
\mathcal{M}_\nu = \begin{pmatrix}
 0 & 1 & 0 \\
 1/\sqrt{2} & 0 & i/\sqrt{2} \\
 1/\sqrt{2} & 0 & -i/\sqrt{2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
 0 & 1/\sqrt{2} & 1/\sqrt{2} \\
 -1/\sqrt{6} & \sqrt{1/3} & -\sqrt{1/2} \\
 -1/\sqrt{6} & \sqrt{1/3} & \sqrt{1/2}
\end{pmatrix}
\]

This is thus another version of a successful derivation of tribimaximal mixing, but as in all previous such models, the predicted value of \(\tan^2 \theta_{12} = 0.5\) is not the central value of
present experimental data: \( \tan^2 \theta_{12} = 0.45 \pm 0.05 \). To obtain a deviation from \( \tan^2 \theta_{12} = 0.5 \) in the context of \( \Sigma(81) \) alone, consider now \( \lambda, \bar{\lambda} \neq 0 \) but small. In that case, 

\[
\frac{\langle \chi_{23} \rangle}{\langle \chi_1 \rangle} \simeq \frac{\lambda (\bar{h} \bar{h})^{1/3} m_\sigma}{(f^2 f)^{2/3} m_\chi^2}, \quad \frac{\langle \bar{\chi}_{23} \rangle}{\langle \bar{\chi}_1 \rangle} \simeq \frac{\lambda (\bar{h} \bar{h})^{1/3} m_\sigma}{(f^2 f)^{2/3} m_\chi^2},
\]

\( \delta \langle \sigma_1 \rangle \simeq 0, \quad \frac{\delta \langle \sigma_{2,3} \rangle}{\langle \sigma \rangle_0} \simeq -\frac{m_\chi}{3m_\sigma} \left[ \frac{\bar{\lambda} f^{1/3}}{(h^2 h f)^{1/3}} + \frac{2\lambda f^{1/3}}{(h^2 h f)^{1/3}} \right], \]

\( \delta \langle \bar{\sigma}_1 \rangle \simeq 0, \quad \frac{\delta \langle \bar{\sigma}_{2,3} \rangle}{\langle \bar{\sigma} \rangle_0} \simeq -\frac{m_\chi}{3m_\sigma} \left[ \frac{\lambda f^{1/3}}{(h^2 h f)^{1/3}} + \frac{2\bar{\lambda} f^{1/3}}{(h^2 h f)^{1/3}} \right]. \)

Since \( \langle \bar{\sigma}_1 \rangle \neq \langle \bar{\sigma}_{2,3} \rangle \), the charged-lepton mass matrix is modified. Instead of Eq. (17), it is now of the form

\[
\mathcal{M}_l = \begin{pmatrix}
    h_e v_1 & h_\mu v_1 & h_\tau v_1 \\
    h_e v_2 & \omega^2 h_\mu v_2 & \omega h_\tau v_2 \\
    h_e v_3 & \omega h_\mu v_2 & \omega^2 h_\tau v_2
\end{pmatrix}
\]

(23)

Using the phenomenological hierarchy \( h_e \ll h_\mu \ll h_\tau \), it is easily shown \cite{15}, to first approximation, that the tribimaximal \( U_{ai} \) of Eq. (1) is multiplied on the left by

\[
R = \begin{pmatrix}
    1 & -r & -r \\
    r & 1 & -r \\
    r & r & 1
\end{pmatrix}, \quad r \simeq \frac{v_1 - v_2}{v_1 + 2v_2} \simeq -\frac{\delta \langle \bar{\sigma}_{2,3} \rangle}{3\langle \sigma \rangle_0}.
\]

(24)

The neutrino mass matrix of Eq. (18) is also changed, i.e.

\[
\mathcal{M}_\nu = \begin{pmatrix} a & e & e \\
    e & a + b & d \\
    e & d & a + b
\end{pmatrix},
\]

(25)

where \( |b| \ll |a| \) and \( |c| \ll |d| \). This leads to a correction of Eq. (1) on the right by the matrix

\[
R' = \begin{pmatrix} 1 & -r' & 0 \\
    r' & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}, \quad r' \simeq \frac{\sqrt{2} e}{d} \simeq \frac{\sqrt{2} \langle \chi_{2,3} \rangle}{\langle \chi_1 \rangle_0}.
\]

(26)

Hence the corrected mixing matrix is given by

\[
U_{ai} \simeq \begin{pmatrix} \sqrt{2/3}(1 + r + r' / \sqrt{2}) & \sqrt{1/3}(1 - 2r - \sqrt{2} r') & 0 \\
    -\sqrt{1/6}(1 - 3r - \sqrt{2} r') & \sqrt{1/3}(1 + r' / \sqrt{2}) & -\sqrt{1/2}(1 + r) \\
    -\sqrt{1/6}(1 - r - \sqrt{2} r') & \sqrt{1/3}(1 + 2r + r' / \sqrt{2}) & \sqrt{1/2}(1 - r)
\end{pmatrix}.
\]

(27)

5
Therefore,
\[
\tan^2 \theta_{12} \simeq \frac{1}{2} - 3(r + r'/\sqrt{2}), \quad \tan^2 \theta_{23} \simeq 1 + 4r, \quad \theta_{13} \simeq 0.
\] (28)

For example, let \( r = r' = 0.01 \), then \( \tan^2 \theta_{12} \simeq 0.45 \), whereas \( \tan^2 \theta_{23} \simeq 1.04 \) which is equivalent to \( \sin^2 2\theta_{23} \simeq 0.9996 \). A better match to the data is thus obtained.

In conclusion, it has been shown in this paper that neutrino tribimaximal mixing is a natural limit in a supersymmetric model based on \( \Sigma(81) \) alone. Whereas corrections are expected, they are such that only \( \tan^2 \theta_{12} \) may deviate significantly from 0.5 without affecting much the predictions \( \sin^2 2\theta_{23} = 1 \) and \( \theta_{13} = 0 \).

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Appendix

The 9 one-dimensional irreducible representations together with \( 3_D, \bar{3}_D \) behave as in \( \Delta(27) \), i.e. [21]
\[
3_D \times 3_D = \bar{3}_D + \bar{3}_D + 3_D, \quad 3_D \times \bar{3}_D = 1_{1,2,3} + 1_{4,5,6} + 1_{7,8,9}.
\] (29)

The \( 3_A, 3_B, 3_C \) representations are cyclically equivalent, as are their conjugates. Their multiplication rules are
\[
3_A \times \bar{3}_A = 3_B \times \bar{3}_B = 3_C \times \bar{3}_C = 1_{1,2,3} + 3_D + \bar{3}_D, \quad (30)
3_B \times \bar{3}_A = 3_C \times \bar{3}_B = 3_A \times \bar{3}_C = 1_{4,5,6} + 3_D + \bar{3}_D, \quad (31)
3_C \times \bar{3}_A = 3_A \times \bar{3}_B = 3_B \times \bar{3}_C = 1_{7,8,9} + 3_D + \bar{3}_D, \quad (32)
3_A \times 3_A = 3_B \times 3_C = \bar{3}_A + \bar{3}_B + \bar{3}_C, \quad (33)
3_B \times 3_B = 3_C \times 3_A = \bar{3}_B + \bar{3}_C + \bar{3}_C, \quad (34)
3_C \times 3_C = 3_A \times 3_B = \bar{3}_C + \bar{3}_A + \bar{3}_A. \quad (35)
\]
References


