Bianchi spacetimes in noncommutative phase-space

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Abstract

The effects of noncommutativity in the phase-space of the classical and quantum cosmology of
Bianchi models are investigated. Exact solutions in both commutative and noncommutative cases
are presented and compared. Further, the Noether symmetries of the Bianchi class A spacetimes
are studied in both cases and similarities and differences are discussed.

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1 Introduction

Since 1947 when noncommutativity between spacetime coordinates was first introduced by Snyder
[1], a great deal of interest has been generated in this area of research [2]-[4]. This interest has been
gathering pace in recent years because of strong motivations in the development of string and M-
theories, [5, 6]. However, noncommutative theories may also be justified in their own right because
of the interesting predictions they have made in particle physics, a few example of which are: the
IR/UV mixing and non-locality [7], Lorentz violation [8] and new physics at very short distance scales
[9]-[11]. Noncommutative versions of ordinary quantum [12] and classical mechanics [13] have also
been studied and shown to be equivalent to their commutative versions if an external magnetic field
is added to the Hamiltonian. The behavior of relativistic quantum particles (fermions and bosons)
in a noncommutative space is investigated in [14].

It is a generally accepted practice to introduce noncommutativity either through the coordinates
or fields, which may be called geometrical or phase-space noncommutativity respectively. Noncom-
mutative ordinary (quantum) field theories where the geometry is considered as noncommutative
are interesting to study since they could provide an effective theory bridging the gap between ordi-
nary quantum field theory and string theory, currently considered as the most important choice for
quantization of gravity [9]. A different approach to noncommutativity is through the introduction
of noncommutative fields [15], that is, fields or their conjugate momenta are taken as noncommuting.
These effective theories can address some of the problems in ordinary quantum field theory, e.g. reg-
ularization [15] and, predict new phenomenon, such as Lorentz violation [16], considered as one of
the general predictions of quantum gravity theories [17].

Since cosmology can test physics at energies that are much higher than those which the experi-
ments on earth can achieve, it seems natural that the effects of quantum gravity could be observed in
this context. Therefore, until a completely satisfactory theory regarding cosmology can be afforded
by string theory, the study of the general properties of quantum gravity through cosmological systems
such as the universe seems reasonably promising. There are several approaches in considering the
notion of noncommutativity in cosmology, which, as was mentioned above could be the best alterna-
tive in the absence of a comprehensive and satisfactory theory from string theory. These ideas have
been studied in different works, an example of which can be found in [18, 19]. There, taking coordinates as noncommuting, it has been shown that noncommutativity affects the spectrum of Cosmic Microwave Background [18, 19]. In [18], noncommutative geometry suggests a non-local inflaton field that changes the gaussianity and isotropy properties of fluctuations. In this approach the parameter of noncommutativity is a dynamical quantity. Introduction of noncommutativity in [19] is based on the stringy spacetime uncertainty principle (SSUR), see also [20]. This approach results in the appearance of a critical time for each mode at which SSUR is saturated, and which is taken to be the time the mode is generated. In this case, in contrast to [18], non-locality appears in time. In both the above approaches the spacetime is taken to be the isotropic FRW spacetime. We follow a phase-space approach to noncommutativity in cosmology, like that presented in [15] which is different from what has been studied in [18, 19] and start with Bianchi models as anisotropic spacetimes rather than an isotropic universe.

In cosmological systems, since the scale factors, matter fields and their conjugate momenta play the role of dynamical variables of the system, introduction of noncommutativity by adopting the approach discussed above is particularly relevant. The resulting noncommutative classical and quantum cosmology of such models have been studied in different works [21]. These and similar works have opened a new window through which some of problems related to cosmology can be looked at and, hopefully, resolved. For example, an investigation of the cosmological constant problem can be found in [22]. In [23] the same problem is carried over to the Kaluza-Klein cosmology. The problem of compactification and stabilization of the extra dimensions in multidimensional cosmology may also be addressed using noncommutative ideas [24].

In this paper we deal with noncommutativity in Bianchi class A models. The classical and quantum solutions of Bianchi models have been studied by many authors, see for example [25]-[28]. Since the Bianchi models have different scale factors in different directions, they are suitable candidates for studying noncommutative cosmology. Here, our aim is to introduce noncommutative scale factors in Bianchi spacetimes and compare and contrast their solutions to that of the commutative case at both the classical and quantum levels. We study the effects of noncommutativity on the underlying symmetries of these models [29]-[31] and, show that noncommutativity manifests itself in changing the number of such symmetries. It should be emphasized that when we speak of noncommutativity in this work, we mean noncommutativity in the fields (scale factors) and not the coordinates, that is to say that we study noncommutativity within the context of phase-space only.

2 The Model

Let us make a quick review of some of the important results in the Bianchi class A models and obtain their Lagrangian and Hamiltonian in the ADM decomposition, for more details see [25]. The Bianchi models are the most general homogeneous cosmological solutions of the Einstein field equations which admit a 3-dimensional isometry group, i.e. their spatially homogeneous sections are invariant under the action of a 3-dimensional Lie group. In the Misner [32] notation, the metric of the Bianchi models can be written as

\[ ds^2 = -N^2(t)dt^2 + e^{2u(t)}e^{2\beta_{ij}(t)}\omega^i \otimes \omega^j, \]

(1)

where \( N(t) \) is the lapse function, \( e^{2u(t)} \) is the scale factor of the universe and \( \beta_{ij} \) determine the anisotropic parameters \( v(t) \) and \( w(t) \) as follows

\[ \beta_{ij} = \text{diag} \left( v + \sqrt{3}w, v - \sqrt{3}w, -2v \right). \]

(2)

Also, in metric (1), the 1-forms \( \omega^i \) represent the invariant 1-forms of the corresponding isometry group and satisfy the following Lie algebra

\[ d\omega^i = -\frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k, \]

(3)
where $C_{jk}^i$ are the structure constants. Indeed, the Bianchi models are grouped by their structure constants into classes A and B. Because of the difficulty in formulating the class B Bianchi models in the context of the ADM decomposition and canonical quantization [33], it is usually the case that one confines attention to the class A models where the structure constants obey the relation $C_{ji}^j = 0$.

The Einstein-Hilbert action is given by (we work in units where $c = \hbar = 16\pi G = 1$)

$$S = \int d^4x \sqrt{-g} (\mathcal{R} - \Lambda),$$

where $g$ is the determinant of the metric, $\mathcal{R}$ is the scalar curvature of the spacetime metric (1) and $\Lambda$ is the cosmological constant. In terms of the ADM variables, action (4) can be written as [34]

$$S = \int dt d^3x N \sqrt{h} \left( K_{ij} K^{ij} - K^2 + R - \Lambda \right),$$

where $K_{ij}$ are the components of extrinsic curvature (second fundamental form) which represent how much the spatial space $h_{ij}$ is curved in the way it sits in the spacetime manifold. Also, $h$ and $R$ are the determinant and scalar curvature of the spatial geometry $h_{ij}$ respectively, and $K$ represents the trace of $K_{ij}$. The extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N} \left( N_{ij} + N_{ji} - \frac{\partial h_{ij}}{\partial t} \right),$$

where $N_{ij}$ represents the covariant derivative with respect to $h_{ij}$. Using (1) and (2) we obtain the non-vanishing components of the extrinsic curvature and its trace as follows

$$K_{11} = -\frac{1}{N} (\dot{u} + \dot{v} + \sqrt{3} \dot{w}) e^{2(u+v+\sqrt{3}w)},$$

$$K_{22} = -\frac{1}{N} (\dot{u} + \dot{v} - \sqrt{3} \dot{w}) e^{2(u+v-\sqrt{3}w)},$$

$$K_{33} = -\frac{1}{N} (\dot{u} - 2\dot{v}) e^{2(u-2v)},$$

$$K = -3 \frac{\dot{u}}{N},$$

where a dot represents differentiation with respect to $t$. The scalar curvature $R$ of a spatial hypersurface is a function of $v$ and $w$ and can be written in terms of the structure constants as [27]

$$R = C_{jk}^i C_{mn}^l h_{il} h^{km} h^{jn} + 2C_{jk}^i C_{li}^j h^{ij}.$$  (8)

The Lagrangian for the Bianchi class A models may now be written by substituting the above results into action (5), giving

$$\mathcal{L} = \frac{6e^{3u}}{N} \left( -\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) + Ne^{3u} (R - \Lambda).$$  (9)

The momenta conjugate to the dynamical variables are given by

$$p_u = \frac{\partial \mathcal{L}}{\partial \dot{u}} = -\frac{12}{N} \dot{u} e^{3u}, \quad p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \frac{12}{N} \dot{v} e^{3u}, \quad p_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \frac{12}{N} \dot{w} e^{3u},$$

leading to the following Hamiltonian

$$\mathcal{H} = \frac{1}{24} Ne^{-3u} \left( -p_u^2 + p_v^2 + p_w^2 \right) - Ne^{3u} (R - \Lambda).$$  (11)

The preliminary set-up for writing the dynamical equations at both the classical and quantum levels is now complete. In what follows, we will study these equations in commutative and noncommutative cases.
3 Classical cosmology

3.1 Commutative case

The classical and quantum solutions of the Bianchi models are studied in many works [25]-[28]. Since our aim here is to compare these solutions to the solutions of the noncommutative model, in the following two sections we consider only the simplest Bianchi class A model, namely type I. The structure constants of the Bianchi type I are all zero, that is \( C_{ijk} = 0 \). It then follows from equation (8) that \( R = 0 \). Thus, with the choice of the harmonic time gauge \( N = e^{3u} \) [35], the Hamiltonian can be write as

\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) + \Lambda e^{6u}.
\]  

(12)

The Poisson brackets for the classical phase-space variables are

\[
\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij},
\]  

(13)

where \( x_i(i = 1, 2, 3) = u, v, w \) and \( p_i(i = 1, 2, 3) = p_u, p_v, p_w \). Therefore, the equations of motion become

\[
\dot{u} = \{u, \mathcal{H}\} = -\frac{1}{12} p_u, \quad \dot{p}_u = \{p_u, \mathcal{H}\} = -6\Lambda e^{6u},
\]  

(14)

\[
\dot{v} = \{v, \mathcal{H}\} = \frac{1}{12} p_v, \quad \dot{p}_v = \{p_v, \mathcal{H}\} = 0,
\]  

(15)

\[
\dot{w} = \{w, \mathcal{H}\} = \frac{1}{12} p_w, \quad \dot{p}_w = \{p_w, \mathcal{H}\} = 0.
\]  

(16)

Equations (15) and (16) can be immediately integrated to yield

\[
p_v = p_{0v}, \quad v(t) = \frac{1}{12} p_{0v} t + v_0,
\]  

(17)

\[
p_w = p_{0w}, \quad w(t) = \frac{1}{12} p_{0w} t + w_0,
\]  

(18)

where \( v_0, w_0, p_{0v} \) and \( p_{0w} \) are integrating constants. To integrate equations (14) we note that the first integral of this system gives

\[
\dot{p}_u = 4A^2 - \frac{1}{4} p_u^2,
\]  

(19)

where \( A \) is a constant. Now, integration of this equation depends on the sign of the cosmological constant \( \Lambda \). If \( \Lambda > 0 \), then from the second equation of (14) we have \( \ddot{p}_u < 0 \), and therefore equations (19) and (14) result in

\[
p_u(t) = 4A \coth A(t + t_0),
\]  

(20)

\[
u(t) = \frac{1}{6} \ln \frac{2A^2}{3\Lambda} \left[ \coth^2 A(t + t_0) - 1 \right],
\]  

(21)

where \( t_0 \) is another constant of integration. In the case of a negative cosmological constant, \( \Lambda < 0 \), we have \( \ddot{p}_u > 0 \), and integration of equations (19) and (14) leads to

\[
p_u(t) = 4A \tanh A(t + t_0),
\]  

(22)

\[
u(t) = \frac{1}{6} \ln \frac{2A^2}{3\Lambda} \left[ 1 - \tanh^2 A(t + t_0) \right].
\]  

(23)

Finally for a universe with zero cosmological constant, \( \Lambda = 0 \), we have \( \ddot{p}_u = 0 \) and the equation of motion for \( u \) is similar to that of \( v \) and \( w \)

\[
p_u = p_{0u}, \quad u(t) = -\frac{1}{12} p_{0u} t + u_0.
\]  

(24)

For a non-zero cosmological constant \( \Lambda \) these solutions show a singularity at \( u \to -\infty \) (\( e^{2u} \to 0 \)) as \( t \to \infty \), i.e. the universe evolves to a vanishing spatial size. In the case of a zero cosmological constant this behavior depends on the sign of \( p_{0u} \). For positive \( p_{0u} \) we have again a similar singularity as before, but for a negative value of \( p_{0u} \), the scale factor goes to infinity as \( t \to \infty \). While independent of the cosmological constant’s value, the anisotropic parameters increase to infinity.
3.2 Noncommutative case

Let us now concentrate on the noncommutativity concepts in classical cosmology. Noncommutativity in classical physics [13] is described by a deformed product, also known as the Moyal product law between two arbitrary functions of position and momenta as

\[ (f \ast_\alpha g)(x) = \exp \left[ \frac{1}{2} \alpha^{ab} \partial_a \partial_b \right] f(x_1)g(x_2) |_{x_1=x_2=x}, \]

such that

\[ \alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix}, \]

where the \( N \times N \) matrices \( \theta \) and \( \beta \) are assumed to be antisymmetric with \( 2N \) being the dimension of the classical phase-space and represent the noncommutativity in coordinates and momenta respectively.

With this product law, the deformed Poisson brackets can be written as

\[ \{f, g\}_\alpha = f \ast_\alpha g - g \ast_\alpha f. \]

A simple calculation shows that

\[ \{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\}_\alpha = \beta_{ij}. \]

Now, consider the following transformations on the classical phase-space

\[ x'_i = x_i - \frac{1}{2} \theta_{ij} p^j, \quad p'_i = p_i + \frac{1}{2} \beta_{ij} x^j. \]

It can easily be checked that if \((x_i, p_j)\) obey the usual Poisson algebra (13), then

\[ \{x'_i, x'_j\} = \theta_{ij}, \quad \{x'_i, p'_j\} = \delta_{ij} + \sigma_{ij}, \quad \{p'_i, p'_j\} = \beta_{ij}, \]

where \( \sigma_{ij} = -\frac{1}{8} \left( \theta^k_i \beta_{kj} + \beta^k_i \theta_{kj} \right) \). These commutative relations are the same as (28). Consequently, for introducing noncommutativity, it is more convenient to work with Poisson brackets (30) than \( \alpha \)-star deformed Poisson brackets (28). It is important to note that the relations represented by equations (28) are defined in the spirit of the Moyal product given above. However, in the relations defined by (30), the variables \((x_i, p_j)\) obey the usual Poisson bracket relations so that the two sets of deformed and ordinary Poisson brackets represented by relations (28) and (30) should be considered as distinct.

In this work we consider a noncommutative phase-space in which \( \beta_{ij} = 0 \) and so \( \sigma_{ij} = 0 \), i.e. the Poisson brackets of the phase space variables are as follows

\[ \{u_{nc}, v_{nc}\} = \theta_1, \quad \{v_{nc}, w_{nc}\} = \theta_2, \quad \{w_{nc}, u_{nc}\} = \theta_3, \]

\[ \{x_{inc}, p_{jnc}\} = \delta_{ij}, \quad \{p_{inc}, p_{jnc}\} = 0. \]

With the noncommutative phase-space defined above, we consider the Hamiltonian of the noncommutative model as having the same functional form as equation (12), but in which the dynamical variables satisfy the above deformed Poisson brackets, that is

\[ H_{nc} = \frac{1}{24} \left( -p_{u_{nc}}^2 + p_{v_{nc}}^2 + p_{w_{nc}}^2 \right) + \Lambda e^{6u_{nc}}. \]

Therefore, the equations of motion read

\[ \dot{u}_{nc} = \{u_{nc}, H_{nc}\} = -\frac{1}{12} p_{u_{nc}}, \quad \dot{p}_{u_{nc}} = \{p_{u_{nc}}, H_{nc}\} = -6\Lambda e^{6u_{nc}}, \]
\[ \dot{v}_{nc} = \{ v_{nc}, \mathcal{H}_{nc} \} = \frac{1}{12} p_{v_{nc}} - 6 \theta_1 \Lambda e^{6u_{nc}}, \quad \dot{w}_{nc} = \{ w_{nc}, \mathcal{H}_{nc} \} = \frac{1}{12} p_{w_{nc}} + 6 \theta_3 \Lambda e^{6u_{nc}} \]  

Equations (33) are similar to equations (14) in the commutative case. Their solutions are therefore as follows

\[ u_{nc}(t) = \frac{1}{6} \ln \frac{2A^2}{3\Lambda} \left[ \cosh^2 A(t + t_0) - 1 \right], \]  

\[ p_{u_{nc}}(t) = 4A \coth A(t + t_0), \]  

for positive cosmological constant, and

\[ u_{nc}(t) = \frac{1}{6} \ln \frac{2A^2}{3\Lambda} \left[ 1 - \tanh^2 A(t + t_0) \right], \]  

\[ p_{u_{nc}}(t) = 4A \tanh A(t + t_0), \]  

in the case of a negative cosmological constant. Also for a zero cosmological constant the solutions can be written as

\[ u_{nc}(t) = -\frac{1}{12} p_{0_{nc}} t + u_0, \quad p_{u_{nc}} = p_{0_{nc}}. \]  

Substituting the above results in equations (34) and (35) yields the following equations for \( v_{nc}(t) \) and \( w_{nc}(t) \)

\[ v_{nc}(t) = \frac{1}{12} p_{0_{nc}} t + 4 \theta_1 A \coth A(t + t_0), \quad p_{v_{nc}} = p_{v_{0_{nc}}}, \]  

\[ w_{nc}(t) = \frac{1}{12} p_{0_{nc}} t - 4 \theta_3 A \coth A(t + t_0), \quad p_{w_{nc}} = p_{w_{0_{nc}}}, \]  

if \( \Lambda > 0 \), and

\[ v_{nc}(t) = \frac{1}{12} p_{0_{nc}} t + 4 \theta_1 A \tanh A(t + t_0), \quad p_{v_{nc}} = p_{v_{0_{nc}}}, \]  

\[ w_{nc}(t) = \frac{1}{12} p_{0_{nc}} t - 4 \theta_3 A \tanh A(t + t_0), \quad p_{w_{nc}} = p_{w_{0_{nc}}}, \]  

if \( \Lambda < 0 \). Finally if \( \Lambda = 0 \), the dynamical equations for \( v(t) \) and \( w(t) \) are again similar to the commutative case with solutions

\[ v_{nc}(t) = \frac{1}{12} p_{0_{nc}} t + v_0, \quad p_{v_{nc}} = p_{v_{0_{nc}}}, \]  

\[ w_{nc}(t) = \frac{1}{12} p_{0_{nc}} t + w_0, \quad p_{w_{nc}} = p_{w_{0_{nc}}}. \]  

As mentioned before, instead of dealing with the noncommutative variables we can construct, with the help of the transformations (29), a set of commutative dynamical variables \( \{ u, v, w, p_u, p_v, p_w \} \) obeying the usual Poisson brackets (13) which, for the problem at hand read

\[ p_{u_{nc}} = p_u, \quad p_{v_{nc}} = p_v, \quad p_{w_{nc}} = p_w, \]

\[ u_{nc} = u - \frac{1}{2} \theta_1 p_v + \frac{1}{2} \theta_3 p_w, \]

\[ v_{nc} = v + \frac{1}{2} \theta_1 p_u - \frac{1}{2} \theta_2 p_w, \]

\[ w_{nc} = w - \frac{1}{2} \theta_3 p_u + \frac{1}{2} \theta_2 p_v. \]
In terms of these commutative variables the Hamiltonian takes the form
\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) + \Lambda e^{6(u - \frac{1}{2} \theta_1 p_v + \frac{1}{2} \theta_3 p_w)}.
\] (43)

Therefore, we have the following equations of motion
\[
\dot{u} = \{u, \mathcal{H}\} = -\frac{1}{12} p_u, \quad \dot{p}_u = \{p_u, \mathcal{H}\} = -6\Lambda e^{6(u - \frac{1}{2} \theta_1 p_v + \frac{1}{2} \theta_3 p_w)},
\] (44)
\[
\dot{v} = \{v, \mathcal{H}\} = \frac{1}{12} p_v - 3\Lambda \theta_1 e^{6(u - \frac{1}{2} \theta_1 p_v + \frac{1}{2} \theta_3 p_w)}, \quad \dot{p}_v = \{p_v, \mathcal{H}\} = 0,
\] (45)
\[
\dot{w} = \{w, \mathcal{H}\} = \frac{1}{12} p_w + 3\Lambda \theta_3 e^{6(u - \frac{1}{2} \theta_1 p_v + \frac{1}{2} \theta_3 p_w)}, \quad \dot{p}_w = \{p_w, \mathcal{H}\} = 0.
\] (46)

The solutions of the above equations can be straightforwardly obtained in the same manner as that of the system (14)-(16). It is easy to check that the action of transformations (42) on the solutions of system (44)-(46) recovers solutions (36)-(41).

The differences between classical commutative and noncommutative cosmologies are now notable. It is clear from solutions in these two cases that both models have a similar singularity at \( t \to +\infty \). Thus, noncommutativity cannot remove this singular behavior from the commutative solutions. The other feature of these solutions is related to their isotropic behavior. With the choice of special initial conditions \( v(0) = p_v(0) = 0 \) and \( w(0) = p_w(0) = 0 \) for the case \( \Lambda \neq 0 \), the commutative classical cosmology predicts an isotropic universe, while applying the same initial conditions on the noncommutative solutions yields an anisotropic cosmology.

4 Quantum cosmology

4.1 Commutative case

Now, let us quantize the model described above. The quantum cosmology of Bianchi models is well studied, see [25]-[28]. Here, for comparison purposes and studying the noncommutativity effects on the solutions, we first discuss the commutative quantum cosmology of our model. For this purpose, we quantize the dynamical variables of the model with the use of the canonical quantization procedure that leads to the Wheeler-DeWitt (WD) equation \( \mathcal{H} \Psi = 0 \). Here, \( \mathcal{H} \) is the operator form of the Hamiltonian given by (12) and \( \Psi \) is the wave function of the universe, a function of spatial geometry and matter fields, if they exist. With the replacement \( p_u \to -i \frac{\partial}{\partial u} \) and similarly for \( p_v \) and \( p_w \) in (12) the WD equation reads
\[
\left[ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial w^2} + 24\Lambda e^{6u} \right] \Psi(u, v, w) = 0.
\] (47)

The solutions of the above differential equation are separable and may be written in the form
\[
\Psi(u, v, w) = U(u)V(v)W(w),
\]
leading to
\[
\frac{1}{V} \frac{d^2 V}{du^2} = \pm \eta^2, \quad \frac{1}{W} \frac{d^2 W}{dw^2} = \pm \kappa^2,
\] (48)
\[
\frac{d^2 U}{du^2} + \left( 24\Lambda e^{6u} + 9\nu^2 \right) U = 0,
\] (49)
where \( \eta \) and \( \kappa \) are separation constants and \( 9\nu^2 = \eta^2 + \kappa^2 \). Equations (48) have simple solutions in the form of exponential functions \( e^{\pm i\eta v} \) (or \( e^{\pm i\eta w} \)) and \( e^{\pm i\kappa w} \) (or \( e^{\pm i\kappa w} \)). Also the solutions of equation (49) can be written in terms of the Bessel functions as
\[
U(u) = J_\nu \left( 2\sqrt{\frac{2\Lambda}{3}} e^{3u} \right),
\]
for a positive cosmological constant and

\[ U(u) = K_{i\nu} \left( 2 \sqrt{2} \sqrt{3} e^{3u} \right), \]

for a negative cosmological constant. Thus, the eigenfunctions of the WD equation can be written as

\[ \Psi_{\nu}(u, v, w) = e^{-(\eta v + \kappa w)} J_\nu \left( 2 \sqrt{2} \sqrt{3} e^{3u} \right), \quad \Lambda > 0, \tag{50} \]

\[ \Psi_{\nu}(u, v, w) = e^{i(\eta v + \kappa w)} K_{i\nu} \left( 2 \sqrt{2} \sqrt{3} e^{3u} \right), \quad \Lambda < 0, \tag{51} \]

where for having well defined functions we use the separation constants (48) with plus sign when \( \Lambda > 0 \) and minus sign in the case \( \Lambda < 0 \). We may now write the general solutions to the WD equations as a superposition of the eigenfunctions

\[ \Psi(u, v, w) = \int_{-\infty}^{+\infty} C(\nu) \Psi_{\nu}(u, v, w) d\nu, \tag{52} \]

where \( C(\nu) \) can be chosen as a shifted Gaussian weight function \( e^{-a(\nu-b)^2} \) [21].

### 4.2 Noncommutative case

To study noncommutativity at the quantum level we follow the same procedure as before, namely the canonical transition from classical to quantum mechanics by replacing the Poisson brackets with the corresponding Dirac commutators as \{ \} \rightarrow -i \[ . \] Thus, the commutation relations between our dynamical variables should be modified as follows

\[ [u_{nc}, v_{nc}] = i \theta_1, \quad [v_{nc}, w_{nc}] = i \theta_2, \quad [w_{nc}, u_{nc}] = i \theta_3, \quad [u_{nc}, p_u] = [v_{nc}, p_v] = [w_{nc}, p_w] = i. \tag{53} \]

The corresponding WD equation can be obtained by the modification of the operator product in (47) with the Moyal deformed product [21]

\[ \left[ -p_u^2 + p_v^2 + p_w^2 + 24 \Lambda e^{6u} \right] \Psi(u, v, w) = 0. \tag{54} \]

Using the definition of the Moyal product (25), it may be shown that [21]

\[ f(u, v, w) \Psi(u, v, w) = f(u_{nc}, v_{nc}, w_{nc}) \Psi(u, v, w), \tag{55} \]

where the relations between the noncommutative variables \( u_{nc}, v_{nc}, w_{nc} \) and commutative variables \( u, v, w \) are given by (42). Therefore, the noncommutative version of the WD equation can be written as

\[ \left[ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial w^2} + 24 \Lambda e^{6(u-\frac{1}{2}\theta_1 p_v+\frac{1}{2}\theta_3 p_w)} \right] \Psi(u, v, w) = 0. \tag{56} \]

We again separate the solutions into the form \( \Psi(u, v, w) = e^{i(\eta v + \kappa w)} U(u) \). Noting that

\[ e^{6(u-\frac{1}{2}\theta_1 p_v+\frac{1}{2}\theta_3 p_w)} \Psi(u, v, w) = e^{6u} \Psi(u, v + 3i\theta_1, w - 3i\theta_3) = e^{6u} U(u) e^{i(\nu+3i\theta_1)} e^{i(\nu-3i\theta_3)} U(u) = e^{6u} e^{-3(\eta \theta_1 - \kappa \theta_3)} \Psi(u, v, w), \tag{57} \]
Figure 1: The figure on the left shows the square of the commutative wave function while the figure on the right, the square of the noncommutative wave function. We choose \( w = 0 \) for simplicity.

Equation (56) takes the form

\[
\left[ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial w^2} + 24\Lambda e^{6u} e^{-3(\eta \theta_1 - \kappa \theta_3)} \right] \Psi(u, v, w) = 0.
\]

(58)

Following the solutions to equation (47), we can find the general solutions for the above equation as

\[
\Psi(u, v, w) = \int_{-\infty}^{\infty} A(\nu) \Psi_\nu(u, v, w) d\nu,
\]

(59)

where again, \( A(\nu) \) is chosen as a shifted Gaussian function and \( \Psi_\nu(u, v, w) \) are the eigenfunctions of equation (58), that is

\[
\Psi_\nu(u, v, w) = e^{-(\eta v + \kappa w)} J_\nu  2 \Lambda^3 e^{3(u - \frac{1}{2} \eta \theta_1 + \frac{1}{2} \kappa \theta_3)}, \quad \Lambda > 0,
\]

(60)

\[
\Psi_\nu(u, v, w) = e^{i(\eta v + \kappa w)} K_\nu \left( \frac{2}{3} \Lambda^3 e^{3(u - \frac{1}{2} \eta \theta_1 + \frac{1}{2} \kappa \theta_3)}, \quad \Lambda < 0.
\]

(61)

where again \( 9\nu^2 = \eta^2 + \kappa^2 \). Figure 1 shows the square of wave functions of the commutative and noncommutative universes. The comparison of the figures show that the noncommutativity causes a shift in the minimum of the values of \( u \), corresponding to the spatial volume. The emergence of new peaks in the noncommutative wave packet may be interpreted as a representation of different quantum states that may communicate with each other through tunnelling. This means that there are different possible universes (states) from which our present universe could have evolved and tunneled in the past, from one universe (state) to another (see the first reference in [21]).

5 Symmetries of commutative and noncommutative Bianchi class A spacetimes

In this section we investigate the Noether Symmetries of the Bianchi class A models and study the effects of noncommutativity on these symmetries. Following [29] and [30], we define the Noether symmetry of the spacetime as a vector field \( X \) on the tangent space of the phase-space-like \((u, v, w, \dot{u}, \dot{v}, \dot{w})\) through

\[
X = \alpha \frac{\partial}{\partial u} + \beta \frac{\partial}{\partial v} + \gamma \frac{\partial}{\partial w} + \frac{d \alpha}{dt} \frac{\partial}{\partial \dot{u}} + \frac{d \beta}{dt} \frac{\partial}{\partial \dot{v}} + \frac{d \gamma}{dt} \frac{\partial}{\partial \dot{w}},
\]

(62)
such that the Lie derivative of the Lagrangian with respect to this vector field vanishes

$$L_X L = 0.$$  \hspace{1cm} (63)

In (62), \((\alpha, \beta, \gamma)\) are linear functions\(^1\) of \((u, v, w)\) and \(\frac{d}{dt}\) represents the Lie derivative along the dynamical vector field, that is

$$\frac{d}{dt} = \dot{u} \frac{\partial}{\partial u} + \dot{v} \frac{\partial}{\partial v} + \dot{w} \frac{\partial}{\partial w}.$$ \hspace{1cm} (64)

It is easy to find the constants of motion corresponding to such symmetry. Indeed, equation (63) can be rewritten as

$$L_X L = \left( \alpha \frac{\partial L}{\partial u} + \frac{d\alpha}{dt} \frac{\partial L}{\partial \dot{u}} \right) + \left( \beta \frac{\partial L}{\partial v} + \frac{d\beta}{dt} \frac{\partial L}{\partial \dot{v}} \right) + \left( \gamma \frac{\partial L}{\partial w} + \frac{d\gamma}{dt} \frac{\partial L}{\partial \dot{w}} \right) = 0.$$ \hspace{1cm} (65)

Noting that \(\frac{\partial L}{\partial q} = \frac{dp}{dt} q\), we have

$$\left( \alpha \frac{dp}{dt} u + \frac{d\alpha}{dt} p_u \right) + \left( \beta \frac{dp}{dt} v + \frac{d\beta}{dt} p_v \right) + \left( \gamma \frac{dp}{dt} w + \frac{d\gamma}{dt} p_w \right) = 0,$$ \hspace{1cm} (66)

which yields

$$\frac{d}{dt} (\alpha p_u + \beta p_v + \gamma p_w) = 0.$$ \hspace{1cm} (67)

Thus the constants of motion are found as

$$Q = \alpha p_u + \beta p_v + \gamma p_w.$$ \hspace{1cm} (68)

In order to obtain the functions \(\alpha, \beta\) and \(\gamma\) we use equation (65). However, since in the noncommutative case we use the Hamiltonian formalism, equation (66) is better suited in finding these coefficients which, equivalently, can be written as

$$\alpha \{p_u, \mathcal{H}\} + \beta \{p_v, \mathcal{H}\} + \gamma \{p_w, \mathcal{H}\} + \left[ \frac{\partial \alpha}{\partial u} \{u, \mathcal{H}\} + \frac{\partial \alpha}{\partial v} \{v, \mathcal{H}\} + \frac{\partial \alpha}{\partial w} \{w, \mathcal{H}\} \right] p_u +$$

$$+ \left[ \frac{\partial \beta}{\partial u} \{u, \mathcal{H}\} + \frac{\partial \beta}{\partial v} \{v, \mathcal{H}\} + \frac{\partial \beta}{\partial w} \{w, \mathcal{H}\} \right] p_v + \left[ \frac{\partial \gamma}{\partial u} \{u, \mathcal{H}\} + \frac{\partial \gamma}{\partial v} \{v, \mathcal{H}\} + \frac{\partial \gamma}{\partial w} \{w, \mathcal{H}\} \right] p_w = 0.$$ \hspace{1cm} (69)

In general, the expression above gives a quadratic polynomial in terms of momenta with coefficients being partial derivatives of \(\alpha, \beta\) and \(\gamma\) with respect to the configuration variables \(u, v\) and \(w\). Thus, the expression is identically equal to zero if and only if these coefficients are zero, leading to a system of partial differential equations for \(\alpha, \beta\) and \(\gamma\). In the following subsections we obtain such symmetries for Bianchi class A models in the commutative and noncommutative cases.

### 5.1 The Bianchi type I model

This model is the simplest in that all the structure constants are zero. The Hamiltonian in the gauge \(N = e^{3u}\) is given by equation (12). Substituting the Poisson brackets (14)-(16) in equation (69), we get

$$-6\Lambda e^{6u} - \frac{1}{12} \frac{\partial \alpha}{\partial u} p_u^2 + \frac{1}{12} \frac{\partial \beta}{\partial u} p_v^2 + \frac{1}{12} \frac{\partial \gamma}{\partial w} p_w^2 + \frac{1}{12} \left( \frac{\partial \alpha}{\partial v} + \frac{\partial \beta}{\partial u} \right) p_u p_v + \frac{1}{12} \left( \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} \right) p_u p_w$$

$$+ \frac{1}{12} \left( \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} \right) p_v p_w = 0.$$ \hspace{1cm} (70)

\(^1\)One may, of course, seek more complicated symmetries. However, we have chosen the linear case for simplicity.
which leads to the following system

\[\begin{align*}
\alpha &= 0, \quad \frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial v} = \frac{\partial \gamma}{\partial w} = 0, \\
\frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} &= \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} = \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} = 0.
\end{align*}\]  

(71)

The above system has the general solutions [29, 30]

\[\begin{align*}
\alpha &= 0, \quad \beta = aw + b, \quad \gamma = -av + b',
\end{align*}\]  

(72)

where \(a, b\) and \(b'\) are constants. Thus, according to (68) the independent constants of motion are

\[\begin{align*}
Q_1 &= p_v, \quad Q_2 = p_w, \quad Q_3 = wp_v - vp_w,
\end{align*}\]  

(73)

which correspond to the following symmetries

\[\begin{align*}
X_1 &= \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial w}, \quad X_3 = w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w} + \dot{w} \frac{\partial}{\partial \dot{v}} - \dot{v} \frac{\partial}{\partial \dot{w}}.
\end{align*}\]  

(74)

These symmetries satisfy the Lie algebra

\[\begin{align*}
[X_1, X_2] &= 0, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.
\end{align*}\]  

(75)

Now, let us see which of the above symmetries survive in the noncommutative case. Here, the Hamiltonian is given by (43) and the required Poisson brackets are given by equations (44)-(46). Substituting these Poisson brackets into relation (69) gives

\[\begin{align*}
-6\Lambda e^{6u-3\theta_1 p_v + 3\theta_3 p_w} - \frac{1}{12} \frac{\partial \alpha}{\partial u} p_u^2 + \frac{1}{12} \frac{\partial \beta}{\partial v} p_v^2 + \frac{1}{12} \frac{\partial \gamma}{\partial w} p_w^2 + \frac{1}{12} \left( \frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} \right) p_u p_v + \frac{1}{12} \left( \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} \right) p_u p_w + \frac{1}{12} \left( \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} \right) p_v p_w + 3\Lambda e^{6u-3\theta_1 p_v + 3\theta_3 p_w} \left[ \left( \theta_1 \frac{\partial \alpha}{\partial v} - \theta_3 \frac{\partial \alpha}{\partial w} \right) p_u + \left( \theta_1 \frac{\partial \beta}{\partial v} - \theta_3 \frac{\partial \beta}{\partial w} \right) p_v + \left( \theta_1 \frac{\partial \gamma}{\partial v} - \theta_3 \frac{\partial \gamma}{\partial w} \right) p_w \right] = 0.
\end{align*}\]  

(76)

Thus, in addition to equations (71) we have a number of extra restrictions on \(\alpha, \beta\) and \(\gamma\) as

\[\begin{align*}
\theta_1 \frac{\partial \alpha}{\partial v} - \theta_3 \frac{\partial \alpha}{\partial w} = \theta_1 \frac{\partial \beta}{\partial v} - \theta_3 \frac{\partial \beta}{\partial w} = \theta_1 \frac{\partial \gamma}{\partial v} - \theta_3 \frac{\partial \gamma}{\partial w} = 0,
\end{align*}\]  

(77)

which should hold for all values of \(\theta_1\) and \(\theta_3\). Imposing relations (77) on the solutions (71) results in \(\beta\) and \(\gamma\) becoming constants. Therefore, in the noncommutative case the only constants of motions are

\[\begin{align*}
(Q_1)_{nc} &= p_v, \quad (Q_2)_{nc} = p_w,
\end{align*}\]  

(78)

which correspond to the symmetries

\[\begin{align*}
(X_1)_{nc} &= \frac{\partial}{\partial v}, \quad (X_2)_{nc} = \frac{\partial}{\partial w}, \quad [X_1, X_2] = 0.
\end{align*}\]  

(79)

Consequently, the third symmetry is absent in the noncommutative case. A quick look at the Hamiltonians (12) and (43) shows that the variables \(v\) and \(w\) are cyclic and consequently the corresponding momenta are constants of motion. In addition, Hamiltonian (12) is rotationally invariant about the \(u\)-axis, that is \(v^2 + \dot{w}^2\). Thus, the angular momentum about this axis is conserved. This symmetry is absent in the noncommutative Hamiltonian (43).

At this point, it is appropriate to investigate the symmetries when the cosmological constant is zero. In this case there is no difference between commutative and noncommutative Bianchi type I
satisfying the following Lie algebra

\[ \alpha = av + bw + c, \quad \beta = au + b'w + c', \quad \gamma = bu - b'v + c'', \]

without the constraint \( \alpha \) models. It is clear from (70) that the symmetries in this case are obtained from the system (71) which correspond to the following symmetries

\[ P_\text{s} \text{oisson brackets are} \]

\[ \alpha \]

In order to obtain the symmetries, we have to solve equation (69) for \( a \) with the gauge \( N \)

Thus, the Hamiltonian for the model can be written from equation (11) with the result (again we use \( \alpha \))

which upon substitution into relation (8), yields the scalar curvature of the corresponding 3-geometry

Therefore, the non-vanishing structure constants may be obtained from equation (3)

\[ C_{23}^1 = -C_{32}^1 = 1, \]

which, upon substitution into relation (8), yields the scalar curvature of the corresponding 3-geometry as

Thus, the Hamiltonian for the model can be written from equation (11) with the result (again we use the gauge \( N = e^{3u} \))

In order to obtain the symmetries, we have to solve equation (69) for \( \alpha, \beta \) and \( \gamma \). The required Poisson brackets are

\[ \{ p_u, H \} = -[6\Lambda e^{6u} - 8e^{4(u+v+\sqrt{3}w)}], \]

\[ \{ p_v, H \} = 8e^{4(u+v+\sqrt{3}w)}, \quad \{ p_w, H \} = 8\sqrt{3}e^{4(u+v+\sqrt{3}w)}, \]

\[ \{ u, H \} = -\frac{1}{12}p_u, \quad \{ v, H \} = \frac{1}{12}p_v, \quad \{ w, H \} = \frac{1}{12}p_w. \]
These Poisson brackets make equation (69) read

\[ -6\alpha \Lambda e^{6\alpha} + 8(\alpha + \beta + \sqrt{3}\gamma)e^{4(u+v+\sqrt{3}w)} - \frac{1}{12} \frac{\partial \alpha}{\partial u} p_u^2 + \frac{1}{12} \frac{\partial \beta}{\partial v} p_v^2 + \frac{1}{12} \frac{\partial \gamma}{\partial w} p_w^2 + \frac{1}{12} \left( \frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} \right) p_u p_v + \frac{1}{12} \left( \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} \right) p_u p_w + \frac{1}{12} \left( \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} \right) p_v p_w = 0, \]  

which, in turn, results in the following system

\[ \alpha = 0, \quad \alpha + \beta + \sqrt{3}\gamma = 0, \quad \frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial v} = \frac{\partial \gamma}{\partial w} = 0, \]  

\[ \frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} = 0, \quad \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} = 0, \quad \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} = 0. \]  

The solutions of this system are similar to the solutions of the system (71) with the additional constraint \( \alpha + \beta + \sqrt{3}\gamma = 0 \). Therefore, we obtain \( \alpha = 0, \beta \) and \( \gamma = \text{const.} \) as solutions, i.e. we have \( \alpha = 0, \beta = -\sqrt{3} \) and \( \gamma = 1 \). Consequently, there is only one symmetry

\[ X = -\sqrt{3} \frac{\partial}{\partial v} + \frac{\partial}{\partial w}, \]  

with the corresponding constant of motion

\[ Q = -\sqrt{3} p_v + p_w. \]  

In the noncommutative case we have the system of equations (90) plus the additional equations in terms of the noncommutativity parameters and partial derivative of \( \alpha, \beta \) and \( \gamma \), similar to the last term in (76). It is easy to see that these additional equations do not have any effect on the solutions of system (90), and thus, also hold in the noncommutative phase-space.

Now, let us consider this model with a zero cosmological constant. In this case the restriction \( \alpha = 0 \) is removed from the system of equations (90) and it may be shown that the general solutions can be written as

\[ \alpha = \sqrt{3}av - aw - c - \sqrt{3}c', \quad \beta = \sqrt{3}au + aw + c, \quad \gamma = -au - av + c', \]  

where \( a, c \) and \( c' \) are constant. Therefore, in this case we have three independent constants of motion

\[ Q_1 = -\sqrt{3} p_u + p_w, \quad Q_2 = -p_u + p_v, \quad Q_3 = \left( \sqrt{3}v - w \right) p_u + \left( \sqrt{3}u + w \right) p_v - (u + v)p_w, \]  

which correspond to the following symmetries

\[ X_1 = -\sqrt{3} \frac{\partial}{\partial u} + \frac{\partial}{\partial w}, \quad X_2 = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad X_3 = \left( \sqrt{3}v - w \right) \frac{\partial}{\partial u} + \left( \sqrt{3}u + w \right) \frac{\partial}{\partial v} - (u + v) \frac{\partial}{\partial w}, \]  

satisfying the Lie algebra given by

\[ [X_1, X_2] = 0, \quad [X_1, X_3] = \sqrt{3} X_1 - 2 X_2, \quad [X_2, X_3] = -\sqrt{3} X_2. \]  

The noncommutative phase-space with zero cosmological constant has the Hamiltonian

\[ \mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) - 2e^{4u - 2\theta_1 p_u + 2\theta_3 p_w} e^{4v + 2\theta_1 p_u - 2\theta_2 p_w} e^{\sqrt{3}(4w - 2\theta_3 p_u + 2\theta_2 p_w)}. \]
To construct equation (69) we need the following Poisson brackets

\[
\{p_u, \mathcal{H}\} = \{p_v, \mathcal{H}\} = \frac{1}{\sqrt{3}} \{p_w, \mathcal{H}\} = 8e^A e^B e^C,
\]

\[
\{u, \mathcal{H}\} = -\frac{1}{12} p_u - 4 \left( \theta_1 - \sqrt{3} \theta_3 \right) e^A e^B e^C,
\]

\[
\{v, \mathcal{H}\} = \frac{1}{12} p_v + 4 \left( \theta_1 - \sqrt{3} \theta_3 \right) e^A e^B e^C,
\]

\[
\{w, \mathcal{H}\} = \frac{1}{12} p_w - 4 \left( \theta_3 - \theta_2 \right) e^A e^B e^C,
\]

where \( A = 4u - 2\theta_1 p_v + 2\theta_3 p_w \), \( B = 4v + 2\theta_1 p_u - 2\theta_2 p_w \) and \( C = \sqrt{3} (4w - 2\theta_3 p_u + 2\theta_2 p_v) \). Substitution of these relations into equation (69) shows that solutions (93) should satisfy the following additional constraints

\[
\left( \theta_1 - \sqrt{3} \theta_3 \right) \frac{\partial \alpha}{\partial v} - \left( \theta_3 - \theta_2 \right) \frac{\partial \alpha}{\partial w} = 0,
\]

\[
\left( \theta_1 - \sqrt{3} \theta_3 \right) \frac{\partial \beta}{\partial u} + \left( \theta_3 - \theta_2 \right) \frac{\partial \beta}{\partial w} = 0,
\]

\[
\frac{\partial \gamma}{\partial u} - \frac{\partial \gamma}{\partial v} = 0.
\]

Imposing these conditions on solutions (93), we are led to

\[
a \left( \sqrt{3} \theta_1 - \theta_2 - 2\theta_3 \right) = 0.
\]

Now if \( a = 0 \), i.e. \( \alpha, \beta \) and \( \gamma \) are constant, then solutions (93) automatically satisfy conditions (99). Therefore, the noncommutative case has the symmetries \( X_1 \) and \( X_2 \) only, given by (95). On the other hand if \( a \neq 0 \), there is a special combination of \( \theta \)'s, that is

\[
\sqrt{3} \theta_1 - \theta_2 - 2\theta_3 = 0,
\]

for which all the symmetries (95) in the commutative case can also be recovered in the noncommutative case. This phenomena can be understood by noting that in this case Hamiltonian (97) takes the form

\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) - 2e^{2(\theta_1 - \sqrt{3} \theta_3)}(2Q_2 - \sqrt{3} Q_1) e^{4(u+v+\sqrt{3}w)},
\]

Since \( Q_1 \) and \( Q_2 \) are constants of motion the above Hamiltonian differs from the Hamiltonian (87) (for \( \Lambda = 0 \) only) by a constant factor in the potential term. As it is clear from equation (69) which is the starting point in our procedure for finding the symmetries, this constant factor does not have any effect on the symmetries. Thus in the special case (101), Hamiltonians (102) and (87) have the same structure as far as the symmetry considerations are concerned.

### 5.3 The Bianchi types VI\(_0\) and VII\(_0\)

These types of Bianchi models are characterized by the 1-forms

\[
\omega^1 = \cosh z dx \mp \sinh z dy, \quad \omega^2 = - \sinh z dx + \cosh z dy, \quad \omega^3 = dz,
\]

with the corresponding structure constants

\[
C_{23}^1 = -C_{32}^1 = \pm 1, \quad C_{31}^2 = -C_{13}^2 = -1,
\]

where the upper and lower signs denotes the Bianchi type VI\(_0\) and VII\(_0\) respectively. The scalar curvature of the corresponding 3-geometry can be evaluated from equation (8) with the result

\[
R = 4e^{u+4v} \left[ \cosh(4\sqrt{3}w) \pm 1 \right],
\]
leading to the Hamiltonian
\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) + \Lambda e^{6u} - 4e^{4(u+v)} \left[ \cosh(4\sqrt{3}w) \pm 1 \right].
\] (106)

To find the symmetries, we need the following Poisson brackets, obtained from relation (69)
\[
\begin{align*}
\{p_u, \mathcal{H}\} &= -6\Lambda e^{6u} + 16e^{4(u+v)} \left[ \cosh(4\sqrt{3}w) \pm 1 \right], \\
\{p_v, \mathcal{H}\} &= 16e^{4(u+v)} \left[ \cosh(4\sqrt{3}w) \pm 1 \right], \\
\{p_w, \mathcal{H}\} &= 16\sqrt{3}e^{4(u+v)} \sinh(4\sqrt{3}w), \\
\{u, \mathcal{H}\} &= -\frac{1}{12}p_u, \quad \{v, \mathcal{H}\} = \frac{1}{12}p_v, \quad \{w, \mathcal{H}\} = \frac{1}{12}p_w.
\end{align*}
\] (107)

Substitution of the above Poisson brackets into equation (69) leads to the following equation
\[
\begin{align*}
-6\alpha \Lambda e^{6u} + 16(\alpha + \beta) e^{4(u+v)} \left[ \cosh(4\sqrt{3}w) \pm 1 \right] + \\
16\sqrt{3}\gamma e^{4(u+v)} \sinh(4\sqrt{3}w) & - \frac{1}{12} \frac{\partial \alpha}{\partial u} p_u^2 + \frac{1}{12} \frac{\partial \beta}{\partial v} p_v^2 + \frac{1}{12} \frac{\partial \gamma}{\partial w} p_w^2 + \\
\frac{1}{12} \left( \frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} \right) p_u p_v & + \frac{1}{12} \left( \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} \right) p_u p_w + \frac{1}{12} \left( \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} \right) p_v p_w = 0,
\end{align*}
\] (108)

which for a non-zero cosmological constant immediately yields \( \alpha = \beta = \gamma = 0 \), i.e. Bianchi types VI\(_0\) and VII\(_0\) universes with non-zero cosmological constants have no symmetries and, therefore, no constants of motion. It is clear that this also holds when we consider such spacetimes in a noncommutative scenario.

Let us now assume that the cosmological constant in these models is equal to zero. In this case, from equation (108) we have
\[
\begin{align*}
\gamma &= 0, \quad \alpha + \beta = 0, \quad \frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial v} = \frac{\partial \gamma}{\partial w} = 0, \\
\frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u} &= 0, \quad \frac{\partial \alpha}{\partial w} - \frac{\partial \gamma}{\partial u} = 0, \quad \frac{\partial \beta}{\partial w} + \frac{\partial \gamma}{\partial v} = 0,
\end{align*}
\] (109)

which admit the solution \( \alpha = -\beta = \text{const} \). Therefore, the only symmetry and its corresponding constant of motion are obtained as
\[
X = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \quad Q = p_u - p_v.
\] (110)

The Hamiltonian in the noncommutative model for \( \Lambda = 0 \) is
\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 + p_v^2 + p_w^2 \right) - 4e^{(4u-2\theta_1 p_u+2\theta_3 p_w)} e^{(4v+2\theta_1 p_u-2\theta_2 p_w)} \left[ \cosh(4\sqrt{3}w) - 2\sqrt{3}\theta_3 p_u + 2\sqrt{3}\theta_2 p_v \right] \pm 1, \quad (111)
\]
yielding the following Poisson brackets
\[
\begin{align*}
\{p_u, \mathcal{H}\} &= \{p_v, \mathcal{H}\} = 16e^{A+B} (\cosh C \pm 1), \quad \{p_w, \mathcal{H}\} = 16\sqrt{3} \sinh C, \\
\{u, \mathcal{H}\} &= -\frac{1}{12}p_u - 8\theta_1 e^{A+B} (\cosh C \pm 1) + 8\theta_3 \sqrt{3} e^{A+B} \sinh C, \\
\{v, \mathcal{H}\} &= \frac{1}{12}p_v + 8\theta_1 e^{A+B} (\cosh C \pm 1) - 8\theta_2 \sqrt{3} e^{A+B} \sinh C, \\
\{w, \mathcal{H}\} &= \frac{1}{12}p_w - 8(\theta_3 - \theta_2) e^{A+B} (\cosh C \pm 1).
\end{align*}
\] (112)
Using these relations in equation (69), we conclude that the solutions of the system (109) should also satisfy the following additional conditions:

\[
\frac{\partial \alpha}{\partial v} - \frac{\partial \alpha}{\partial u} = 0, \quad \theta_3 \frac{\partial \alpha}{\partial u} - \theta_2 \frac{\partial \alpha}{\partial v} = 0, \quad (\theta_3 - \theta_2) \frac{\partial \alpha}{\partial w} = 0, \\
\frac{\partial \beta}{\partial v} - \frac{\partial \beta}{\partial u} = 0, \quad \theta_3 \frac{\partial \beta}{\partial u} - \theta_2 \frac{\partial \beta}{\partial v} = 0, \quad (\theta_3 - \theta_2) \frac{\partial \beta}{\partial w} = 0.
\]

(113)

If \( \theta_2 \neq \theta_3 \), these equations have the previous solutions \( \alpha = -\beta = \text{const.} \), and thus the spacetime has the same symmetry as given by (110). However, in the case where \( \theta_2 = \theta_3 \), the above system admits the solutions

\[
\alpha = -\beta = aw + b,
\]

(114)

with constants \( a \) and \( b \). The independent constants of motions are then given by

\[
Q_1 = p_u - p_v, \quad Q_2 = w(p_u - p_v),
\]

(115)

corresponding to the following symmetries

\[
X_1 = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \quad X_2 = w \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) + w \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right), \quad [X_1, X_2] = 0.
\]

(116)

Here, we have an additional constant of motion relative to the commutative case. This is notable in Hamiltonian (111) in that when \( \theta_2 = \theta_3 \), it takes the form

\[
\mathcal{H} = \frac{1}{24} \left( -p_v^2 + p_v^2 + p_w^2 \right) - 4e^{2\theta_1 Q_1} e^{4(u+v)} \left[ \cosh(4\sqrt{3}w - 2\theta_2 \sqrt{3}Q_1) \pm 1 \right].
\]

(117)

The above Hamiltonian differs from Hamiltonian (106) (for \( \Lambda = 0 \)) not only by a constant factor in the potential term (which has no effect on the number of symmetries like the Bianchi type II case) but also in the argument of the “cosh” term. This argument is modified by a constant shift which results in a new symmetry (compare (115) with (110)).

### 5.4 The Bianchi types VIII and IX

In the Bianchi type VIII model the invariant 1-forms are

\[
\omega^1 = \cosh y \cos zdz - \sin y \sin zdz, \\
\omega^2 = \cosh y \sin zdz + \cos y \sin zdz, \\
\omega^3 = \sinh y dx + dz,
\]

(118)

and thus the corresponding non-vanishing structure constants can be evaluated from (3) with the result

\[
C_{23}^2 = -C_{32}^1 = 1, \quad C_{23}^1 = -C_{13}^2 = -1, \quad C_{12}^3 = -C_{21}^3 = 1.
\]

(119)

The scalar curvature of the spatial hypersurface can be obtained by substituting the above structure constants in (8), yielding

\[
R = -4e^{-2u+4v} \cos(4\sqrt{3}w) - 2e^{-2a-8v} + 4e^{-2u-2v} \cos(2\sqrt{3}w) - 2e^{-2u+4v}.
\]

(120)

Therefore, we can write the Hamiltonian using (12)

\[
\mathcal{H} = \frac{1}{24} \left( -p_u^2 - p_v^2 + p_w^2 \right) + \Lambda e^6u + 4e^{4(u+v)} \cos(4\sqrt{3}w) - 4e^{2(2u-v)} \cos(2\sqrt{3}w) + 2e^{4(u-2v)} + 2e^{4(u+v)}.
\]

(121)
As it is clear from the above form, the spacetime has no symmetry. Indeed, evaluating the required Poisson brackets and constructing equation (69), we see that this system admits only the trivial solution $\alpha = \beta = \gamma = 0$, i.e. there are no symmetries neither in the commutative nor noncommutative models.

Our final goal is to address the same issues in the Bianchi type IX model, characterized by the structure constants

$$C^{i}_{jk} = \varepsilon_{ijk}. \quad (122)$$

Calculations similar to the previous models lead to the following Hamiltonian

$$\mathcal{H} = \frac{1}{24} \left( -p_{u}^{2} + p_{v}^{2} + p_{w}^{2} \right) + \Lambda e^{6u} + e^{4(u-2v)} - 4e^{2(2u-v)} \cosh(2\sqrt{3}w) + 2e^{4(u+v)} \left[ \cosh(4\sqrt{3}w) - 1 \right]. \quad (123)$$

Again, a glance at the equations resulting from the above Hamiltonian shows that this spacetime has no symmetry, neither in the commutative nor in its noncommutative version, even if the cosmological constant is equal to zero.

6 Conclusions

In this paper we have studied the Bianchi type I classical and quantum cosmology in both commutative and noncommutative scenarios. We have obtained exact solutions of the vacuum gravitational field equations in three cases when the cosmological constant is positive, negative or zero. The corresponding classical cosmology shows a singularity at $t \to \infty$, together with increasing anisotropic parameters $v$ and $w$. We have seen that noncommutativity can not exclude such behavior from the solutions. The corresponding quantum cosmology of the model in both commutative and noncommutative cases is obtained from the exact solutions of the WD equation, showing how such solutions differ.

We have also studied the Noether symmetries of the Bianchi class A models and investigated their noncommutative counterparts. In Bianchi type I, we have shown that for a non-zero cosmological constant, there are three Noether symmetries, two of which retain their character in the noncommutative case. For a zero cosmological constant, there is no difference between commutative and noncommutative cases and the number of symmetries is six. The constants of motion in this case are the momenta in $u$, $v$ and $w$ directions and their corresponding angular momenta. However, we note that because of the wrong sign (minus) of $p_{u}$ in the Hamiltonians, this dynamical variable behaves like a ghost field and thus the angular momenta-like constants $Q_{5}$ and $Q_{6}$ do not have the usual sign. The Bianchi type II model with a non-zero cosmological constant in the commutative and noncommutative cases has only one symmetry, which is a Killing vector field. If the cosmological constant is zero, this spacetime has three symmetries when studied in its noncommutative form. We have shown that when the phase-space has a noncommutative structure, one of its symmetries is removed except when the noncommutative parameters satisfy a certain relation. With this relation between the noncommutative parameters, although the Hamiltonians are different in the commutative and noncommutative cases, they show the same structure and thus exhibit the same symmetries.

The Bianchi types VI$^{0}$ and VII$^{0}$ with non-zero cosmological constants have no symmetries neither in the commutative nor noncommutative versions. But in the case of a zero cosmological constant, these spacetimes exhibit one symmetry if their phase-space is commutative. This symmetry remains unchanged in the noncommutative phase-space as well. It is interesting to note that if the noncommutative parameters $\theta_{2}$ and $\theta_{3}$ are equal, these models have an additional symmetry. In this case the corresponding Hamiltonians of the commutative and noncommutative phase-space have the same structure but differ in a shift term in the variable $w$. In the last section of the paper we have dealt with the Bianchi type VIII and IX models and shown that they have no symmetries at all.

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