Bouncing and Accelerating Solutions in Nonlocal Stringy Models


Abstract

A general class of cosmological models driven by a non-local scalar field inspired by string field theories is studied. In particular cases the scalar field is a string dilaton or a string tachyon. A distinguished feature of these models is a crossing of the phantom divide. We reveal the nature of this phenomena showing that it is caused by an equivalence of the initial non-local model to a model with an infinite number of local fields some of which are ghosts. Deformations of the model that admit exact solutions are constructed. These deformations contain locking potentials that stabilize solutions. Bouncing and accelerating solutions are presented.
1 Introduction

Field theories which violate the null energy condition (NEC) [1, 2] are of interest for the solution of the cosmological singularity problem and for models of cosmological dark energy with the equation of state parameter \( w < -1 \).

One of the first attempts to apply string theory to cosmology [3] was related to the problem of the cosmological singularity [2]. A possible way to avoid cosmological singularities consists of dealing with nonsingular bouncing cosmological solutions. In these scenarios the Universe contracts before the bounce [4]. Such models have strong coupling and higher-order string corrections are inevitable. It is important to construct nonsingular bouncing cosmological solutions in order to make a concrete prediction of bouncing cosmology.

Present cosmological observations [5] do not exclude an evolving dark energy (DE) state parameter \( w \), whose current value is less than \(-1\), which leads to violation of the NEC (see [6, 7] for a review of DE problems and [8] for a search for a super-acceleration phase of the Universe).

A simple possibility to violate NEC is just to deal with a phantom field. The phantom field is unstable. There are general arguments that coupled scalar-gravity models violating the NEC are unstable ([9]-[13] and refs. therein). At the same time a phantom model could be an approximation to a non-local model that has no problems with instability [14]. A simple example of such a model is a model with a scalar nonlocal action \((e^{-\Box_g} \phi)^2\). In the second order derivative approximation \(e^{-\Box_g} \approx 1 - \Box_g\) this model is equivalent to a phantom one but does not have problems with instability. This type of models does appear in String Field Theory (SFT)(see [15] for a review) and in the p-adic string models [16]. The model with the particular kinetic term mentioned above is in fact realized in the p-adic string near a perturbative vacuum and is expected to be realized in the Vacuum String Field Theory (VSFT) [17].

The purpose of this paper is the study of this type of models. As a model we consider a SFT inspired nonlocal dilaton action. Distinguished features of the model are the invariance of the action under the shift of the dilaton field to a constant as well as a presence of infinite number of higher derivatives terms. A more general family of nonlocal models loosing the invariance of the nonlocal dilaton is also considered. For special values of the parameters the models describe linear approximations to the cubic bosonic or nonBPS fermionic SFT nonlocal tachyon models, or p-adic string models [14], [18]-[23]. The NonBPS fermionic string field tachyon nonlocal model has been considered as a candidate for the dark energy [14]. Several string-inspired and braneworld dark energy models have been recently proposed (see for example [24]-[26] and refs. therein). About a study of the tachyon dynamics with the Born-Infeld action see [27, 28, 29].

We discuss a possibility to stabilize the model that violates the NEC in the flat space-time at the cost of adding extra interaction terms in the Friedmann background. One of the lessons from a study non-local dynamics in the flat case is a sensitivity of the stability problem to the form of the interaction term [30, 31, 32, 33, 34, 35]. We use the Weierstrass product to present the nonlocal field in terms of an infinite number of local fields [21]. Some of these local fields are ghosts, which violate the NEC and are unstable. The model is linear and admits exact solutions in the flat space-time. In non-flat case we get the same exact
solutions after a deformation of the model. We used a similar approach to construct effective SFT inspired phantom models \[36, 18, 37\].

Another recently proposed model which violates the NEC and has higher derivatives is the ghost-condensation model \[38\]. It is discussed in \[21\]. Vector-scalar and tensor-scalar models that violate NEC and are stable in some region have been proposed in \[39, 40\], respectively.

The paper is organized as follows. In Section 2 we describe our strategy to the study of stringy inspired models. In Section 3 we present general solutions of the models in the flat case. Then we use some approximation to study these dynamics in the Friedmann metric and discuss cosmological properties of the constructed solutions.

2 Set up

In this paper we consider a model of gravity coupling with a nonlocal scalar field

\[
S = \int d^4x \sqrt{-g} \left( \frac{m^2}{2} R + \phi F(-\Box_g)\phi \right),
\]

where \( \Box_g \) is the d’Alembertian. The form of the function \( F \) is inspired by a nonlocal action appeared in string field theories. In a particular case

\[
F(z) = -\xi^2 z + 1 - ce^{-2z},
\]

we can rewrite \( S \) as follows

\[
\int d^4x \sqrt{-g} \left( \frac{m^2}{2} R + \frac{\xi^2}{2} \phi \Box_g \phi + \frac{1}{2} (\phi^2 - c\Phi^2) \right).
\]

where \( \Phi = e^{\Box_g} \phi \) and \( \xi \) is a real parameter and \( c \) is a positive constant.

The form of the term \( (e^{\Box_g} \phi)^2 \) is analogous to the form of the interaction in the action for the string field tachyon in non-flat background \[14\], which is a generalization of the SFT tachyon interaction term in a flat background \[11, 12, 15, 31, 32\]. At some particular values of \( \xi^2 \) and \( c \) this action appears in a linear approximation to SFT actions \[43-47\] and in a non-flat background has been considered in \[14, 18, 19, 20, 23\].

We consider in detail action \( (3) \) at \( c = 1 \), which is invariant under translation \( \phi \rightarrow \phi + \text{const} \).

We take the metric in the form

\[
ds^2 = -dt^2 + a^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right)
\]

and get the following equation of motion for the space homogeneous scalar field \( \phi \):

\[
F(\mathcal{D})\phi = 0,
\]

where

\[
\mathcal{D} \equiv -\partial_t^2 - 3H(t)\partial_t, \quad \text{and} \quad H = \frac{\dot{a}}{a}.
\]
The Friedmann equations have the following form

\[3H^2 = \frac{1}{m_p^2} \mathcal{E},\]
\[3H^2 + 2\dot{H} = -\frac{1}{m_p^2} \mathcal{P},\]  \hspace{1cm} (7)

where the energy and the pressure are obtained from the action (1) using standard formula

\[T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \quad T_{\mu\nu} = \text{diag}\{\mathcal{E}, \mathcal{P}, \mathcal{P}, \mathcal{P}\}.\]  \hspace{1cm} (8)

For the case of \(F\) given by (2) the energy and the pressure have additional non-local terms \(\mathcal{E}_{nl1}\) and \(\mathcal{E}_{nl2}\) \[32, 48, 49\]

\[\mathcal{E} = \mathcal{E}_k + \mathcal{E}_p + \mathcal{E}_{nl2} + \mathcal{E}_{nl1},\]
\[\mathcal{P} = \mathcal{E}_k - \mathcal{E}_p + \mathcal{E}_{nl2} - \mathcal{E}_{nl1}.\]  \hspace{1cm} (9)

Non-local term \(\mathcal{E}_{nl1}\) plays a role of an extra potential term and \(\mathcal{E}_{nl2}\) a role of an extra kinetic term. The explicit form of the terms in the R.H.S. of (9) is

\[\mathcal{E}_k = \frac{\xi^2}{2} (\partial \phi)^2, \quad \mathcal{E}_p = -\frac{1}{2} \left(\phi^2 - c(e^D \phi)^2\right),\]
\[\mathcal{E}_{nl1} = c \int_0^1 (\psi^{(1+\rho)D} \phi) \left(-\mathcal{D} e^{(1-\rho)D} \phi\right) d\rho,\]  \hspace{1cm} (10)
\[\mathcal{E}_{nl2} = -c \int_0^1 (\partial \psi^{(1+\rho)D} \phi) (\partial e^{(1-\rho)D} \phi) d\rho.\]

Our strategy is the following:
First, we study the dynamics of the model (3) in the flat case.

- We show that solutions have the form of plane waves. There are special values of parameters for which plane waves should be multiplied on linear functions. We calculate the energy and pressure on the corresponding solutions.

- We present dynamics of the nonlocal model in terms of an infinite number of local fields \[21\] and show that the energy and pressure densities of the nonlocal model are reproduced by the energy and pressure densities of the corresponding local models. For this purpose we use the Weierstrass product representation for the function \(F\) in (11),

\[F(z) = e^{f(z)} \prod_n \left(1 - \frac{z}{\alpha_n^2}\right),\]  \hspace{1cm} (11)

where \(\alpha_n^2\) are complex numbers, and represent the flat analog of (11) as

\[S_{flat} = \frac{1}{2} \int d^4x \phi \mathcal{F}(-\Box) \phi \sim \frac{1}{2} \sum [\epsilon_n \psi_n e^{f(-\Box)}(\Box + \alpha_n^2) \psi_n + \text{c.c.}],\]  \hspace{1cm} (12)

where \(\Box\) is the d’Alembertian in the flat space-time.
• We consider approximated models obtained by a truncation of number of local fields.

Then we consider the Friedmann Universe. There are two ways to study dynamics in the Friedmann metric:

• One can use the found expressions for the energy and pressure in the flat case, \( E(\phi_0) \) and \( P(\phi_0) \), to calculate the corresponding Hubble parameter \( H_0 \), then using this Hubble parameter calculate a perturbation of the flat solution of equation of motion and so on:
\[
\phi = \phi_0 + \phi_1(H_0) + ..., \quad H = H_0(\phi_0) + H_1(\phi_0, \phi_1) + ...
\]  
\[ (13) \]
• One can search for deformations of the model that admit the same exact solutions as in the flat case and try to argue that the deformed models describe the initial model with a good accuracy.

Both ways permit to find the first approximations to the models (5), (7). The first way have been used in [20]. In this paper we will follow the second way.

To this goal we use a representation of non-local dynamics given by action (11) in terms of local fields
\[
\int \sqrt{-g} \left( \frac{m^2}{2} R + \frac{1}{2} \sum [\epsilon_n \psi_n e^{f(-\Box)}(\Box g + \alpha_n^2)\psi_n + c.c.] \right).
\]  
\[ (14) \]
We perform a deformation of this model by several steps. First, we consider an approximation to the model (14) in the form
\[
\int \sqrt{-g} \left( \frac{m^2}{2} R + \sum \left[ \frac{\epsilon_n}{2} \psi_n e^{f(\alpha_n^2)}(\Box g + \alpha_n^2)\psi_n + c.c. \right] \right).
\]  
\[ (15) \]
Second, we restrict a number of local fields and third, we add potentials of the order \( 1/m_p^2 \):
\[
\int \sqrt{-g} \left( \frac{m^2}{2} R + \sum \left[ \frac{\epsilon_n}{2} \psi_n e^{f(\alpha_n^2)}(\Box g + \alpha_n^2)\psi_n - \mathcal{V}_n(\psi_n) + c.c. \right] \right).
\]  
\[ (16) \]
such that solutions of the field equations in the non-flat case are the same as the flat case.

Finally, we find the corresponding scale factor \( a(t) \) and study cosmological properties of approximated solutions to our model.

3 Flat Dynamics

3.1 General Solutions

3.1.1 Roots of the Characteristic Equation

In the flat case the action (11) has the following form:
\[
S_{\text{flat}} = \frac{1}{2} \int d^4x \phi F(-\Box)\phi.
\]  
\[ (17) \]
Equation of motion on the space-homogeneous configurations (5) is reduced to the following linear equation:

$$F(\partial^2)\phi = 0.$$  \hspace{1cm} (18)

A plane wave

$$\phi = e^{\alpha t}$$  \hspace{1cm} (19)

is a solution of (18) if $\alpha$ is a root of the characteristic equation

$$F(\alpha^2) = 0.$$  \hspace{1cm} (20)

If the characteristic equation has only real simple roots $m_n, n = 1, ...$ and pairs of complex conjugated simple roots, $\alpha_n, \alpha_n^*, n = 1, ...$, then real solutions to equation (18) have the form

$$\phi = \sum_n R_n e^{m_n t} + \sum_n \left(C_n e^{\alpha_n t} + C_n^* e^{\alpha_n^* t}\right),$$  \hspace{1cm} (21)

where $R_n$ and $C_n$ are arbitrary real and complex numbers respectively. In the case of a degeneration of the root $\alpha_{n_0}$, i.e. $F(\alpha_{n_0}) = 0$, $F'(\alpha_{n_0}) = 0$, ..., $F^{(p)}(\alpha_{n_0}) = 0$, $F^{(p+1)}(\alpha_{n_0}) \neq 0$,

$$C_{n_0} e^{\alpha_{n_0} t} \rightarrow P_p(t) e^{\alpha_{n_0} t},$$  \hspace{1cm} (22)

where $P_p(t)$ is the p-th degree polynomial with arbitrary coefficients.

For a case of $F$ given by (2) equation (18) has the following form

$$-\xi^2 \partial^2 \phi + \phi - c e^{-2\phi^2} \phi = 0.$$  \hspace{1cm} (23)

This equation has an infinite number of derivatives and can be treated as a pseudodifferential as well as an integral equation [34].

The corresponding characteristic equation:

$$F(\alpha^2) \equiv -\xi^2 \alpha^2 + 1 - c e^{-2\alpha^2} = 0$$  \hspace{1cm} (24)

has the following solutions

$$\alpha_n = \pm \frac{1}{2\xi} \sqrt{4 + 2\xi^2 W_n \left( -\frac{2c e^{-2/\xi^2}}{\xi^2} \right)}, \hspace{1cm} n = 0, \pm 1, \pm 2, ...$$  \hspace{1cm} (25)

where $W_n$ is the n-s branch of the Lambert function satisfying a relation $W(z)e^{W(z)} = z$. The Lambert function is a multivalued function, so eq. (24) has an infinite number of roots.

Parameters $\xi$ and $c$ are real, therefore if $\alpha_n$ is a root of (24), then the adjoined number $\alpha_n^*$ is a root as well. Note that if $\alpha_n$ is a root of (24), then $-\alpha_n$ is a root too. In other words, equation (24) has quadruples of complex roots

$$\alpha_{n,\pm} = \pm \text{Re}(\alpha_n) \pm i\text{Im}(\alpha_n).$$  \hspace{1cm} (26)

It is easy to show that all complex roots of $F(z)$, given by (2), are simple. Indeed, if $z = z_0$ is a multiple root, then at this point $F(z_0) = 0$ and $F'(z_0) = 0$. These equations give that

$$z_0 = \alpha_0^2 = \frac{1}{2} - \frac{1}{\xi^2};$$  \hspace{1cm} (27)
i.e. $z_0$ is a real root. The multiple roots $z_0$ exist if and only if

$$c = \frac{\xi^2 e^{2\xi^2}}{2e}.$$  (28)

Real roots for any $\xi$ and $c$, except $\xi^2 = 0$ and $c = \infty$, are no more then double degenerated, because $F''(z_0) \neq 0$.

Locations of roots of the characteristic equation in the complex $\alpha$-plane are presented in Fig.1 (compare with [20]). All roots $\alpha_n$, except roots (27), are simple.

\[ \begin{align*}
\Re(\alpha) \\
\Im(\alpha)
\end{align*} \]

Figure 1: Roots for $c = 1$ and $\xi = 0$ (big crosses), $\xi = 2$ (middle crosses) and $\xi = 15$ (small crosses).

Let us summarize:

- If $c \neq \frac{\xi^2 e^{2\xi^2}}{2e}$ and $c \neq 1$ then formula (21) gives the general real solution to (23).

- If $c = \frac{\xi^2 e^{2\xi^2}}{2e} > 1$, then to get the general real solution one has to add to (21)

$$\phi_0 = (R_{+1} + R_{+2}t)e^{m_0t} + (R_{-1} + R_{-2}t)e^{-m_0t}, \quad m_0 = \sqrt{1 - \frac{1}{\xi^2}}, \quad \text{if} \quad \xi^2 < 2, \quad (29)$$

$$\phi_0 = (C_{-1} + C_{+2}t)e^{ia_0t} + (C_{+1} + C_{+2}t)e^{-ia_0t}, \quad \alpha_0 = i\sqrt{1 - \frac{1}{2\xi^2}}, \quad \text{if} \quad \xi^2 > 2. \quad (30)$$

- If $c = 1$ then to get the general real solution one has to add to (21)

$$\phi_0 = C_1 t + C_0, \quad \text{if} \quad \xi^2 \neq 2, \quad (31)$$

$$\phi_0 = C_3 t^3 + C_2 t^2 + C_1 t + C_0, \quad \text{if} \quad \xi^2 = 2. \quad (32)$$
3.1.2 Real Roots of the Characteristic Equation

For some values of the parameters $\xi$ and $c$ eq. (24) has real roots. To mark out real values of $\alpha$ we will denote real $\alpha$ as $m$:

$$m = \alpha.$$  \hspace{1cm} (33)

To determine values of the parameters at which eq. (24) has real roots we rewrite this equation in the following form:

$$\xi^2 = g(m^2, c), \quad \text{where} \quad g(m^2, c) = \frac{e^{2m^2} - c}{m^2 e^{2m^2}}.$$ \hspace{1cm} (34)

The dependence of $g(m, c)$ on $m$ for different $c$ is presented in Fig. (2). This function has a maximum at $m_{max}^2$

$$m_{max}^2 = -\frac{1}{2} - \frac{1}{2} W_{-1}\left(-\frac{e^{-1}}{c}\right),$$ \hspace{1cm} (35)

provided $c$ is such that $W_{-1}\left(-\frac{e^{-1}}{c}\right) < -1$, in the other words $0 < c < 1$.

![Figure 2: The dependence of function $g(m^2, c)$, which is equal to $\xi^2$, on $m$ at $c = 1/2$ (left), $c = 1$ (center) and $c = 2$ (right).](image)

There are three different cases (see Fig. 2).

- If $c < 1$, then eq. (24) has two simple real roots: $m = \pm m_1$ for any values $\xi$.
- If $c = 1$, then eq. (24) has a zero root. Nonzero real roots exist if and only if $\xi^2 > 2$.
- If $c > 1$, then eq. (24) has
  - no real roots for $\xi^2 > \xi^2_{max}$, where
    $$\xi^2_{max} = \frac{1 - ce^{-2m^2_{max}}}{m^2_{max}} = -\frac{2}{W_{-1}(-e^{-1}/c)}$$ \hspace{1cm} (36)
  - two real double roots $m = \pm m_{max}$ for $\xi^2 = \xi^2_{max}$
  - four real single roots for $\xi^2 < \xi^2_{max}$. In this case we have the following restriction on real roots: $m^2 > \frac{1}{2} \ln c$. 

8
3.1.3 Pure Imaginary Roots of the Characteristic Equation

For some values of the parameters $\xi$ and $c$ eq. (24) has a pair of pure imaginary roots. Let us introduce a new real variable

$$\mu = i\alpha.$$  

From (23) we obtain

$$(\xi^2\mu^2 + 1)e^{-2\mu^2} = c.$$  

This equation is equivalent to

$$\xi^2 = \tilde{g}(\mu^2, c), \quad \text{where} \quad \tilde{g}(\mu^2, c) = \frac{c - e^{-2\mu^2}}{\mu^2 e^{-2\mu^2}}.$$  

The dependence of $\tilde{g}(\mu^2, c)$ on $\mu$ for different $c$ is presented in Fig. (3).

Figure 3: The dependence of function $\tilde{g}(\mu^2, c)$, which is equal to $\xi^2$, on $\mu$ at $c = 1/2$ (right), $c = 1$ (center) and $c = 2$ (left).

For different $c$ we have:

- For $c < 1$ there are two real simple roots $\mu = \pm \mu_1$,
- For $c = 1$ nonzero real roots exist only if $\xi^2 > 2$,
- For $c > 1$ real roots exist if and only if

$$\xi \geq \xi_{\text{min}} = -\frac{2}{W_0(-\frac{e^{-1}}{c})}.$$  

If $\xi = \xi_{\text{min}}$, then there exist two double real roots: $\mu = \pm \mu_{\text{min}}$, where

$$\mu_{\text{min}} = \frac{1}{2}\sqrt{2 + 2W_0\left(-\frac{e^{-1}}{c}\right)}.$$  

At $\xi > \xi_{\text{min}}$ eq. (38) has four real simple roots.
3.2 Energy Density and Pressure

3.2.1 General Formula

Equation (23) has the conserved energy (compare with \[48, 49, 18\]), which is defined by the formula that is a flat analog of (9). The energy density is as follows:

\[ E = E_k + E_p + E_{nl1} + E_{nl2}, \]  

(42)

where

\[ E_k = \frac{\xi^2}{2} (\partial \phi)^2, \quad E_p = -\frac{1}{2} \phi^2 + \frac{c}{2} \Phi^2, \]  

(43)

\[ E_{nl1} = c \int_0^1 \left( e^{-\rho \Phi} \left( \partial^2 (e^{\rho \Phi}) \right) \right) d\rho, \quad E_{nl2} = -c \int_0^1 \left( \partial (e^{-\rho \Phi}) \right) \left( \partial (e^{\rho \Phi}) \right) d\rho. \]  

(44)

For the pressure

\[ P = E_k - E_p - E_{nl1} + E_{nl2}, \]  

(45)

we have the following explicit form

\[ P = \frac{\xi^2}{2} (\partial \phi)^2 + \frac{1}{2} \phi^2 - \frac{c}{2} \Phi^2 - c \int_0^1 \left\{ \left( e^{-\rho \Phi} \right) \left( \partial^2 (e^{\rho \Phi}) \right) - \left( \partial (e^{-\rho \Phi}) \right) \left( \partial (e^{\rho \Phi}) \right) \right\} d\rho. \]  

(46)

Let us calculate the energy density and pressure for the following solution

\[ \phi = \sum_{n=1}^N C_n e^{\alpha_n t}, \]  

(47)

where \( N \) is a natural number, \( C_n \) are some constant and \( \alpha_n \) are solutions to eq. (24).

For \( N = 1 \) and

\[ \phi = Ce^{\alpha t} \]  

(48)

we obtain

\[ E(Ce^{\alpha t}) = 0, \quad P(Ce^{\alpha t}) = C^2 p_\alpha e^{2\alpha t}. \]  

(49, 50)

Hereafter we denote the energy density and pressure of function \( \phi(t) \) as the functionals \( E(\phi) \) and \( P(\phi) \), respectively, and use the following notation

\[ p_\alpha = \alpha^2 \left( \xi^2 - 2 + 2 \xi^2 \alpha^2 \right). \]  

(51)

For \( N = 2 \) and

\[ \phi = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}, \]  

(52)

where \( \alpha_1 \) and \( \alpha_2 \) are different roots of (24) we have (see Appendix A for details)

\[ E(C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}) = -2C_1 C_2 p_\alpha, \quad \text{at} \quad \alpha_2 = -\alpha_1, \]  

(53)
and
\[ E(C_1e^{\alpha_1 t} + C_2e^{\alpha_2 t}) = 0, \quad \text{at} \quad \alpha_2 \neq -\alpha_1. \quad (54) \]

The pressure \( P(\phi) \) for solution (52) is
\[ P(C_1e^{-\alpha_1 t} + C_2e^{\alpha_2 t}) = (C_1^2 e^{-2\alpha_1 t} + C_2^2 e^{2\alpha_2 t}) p_0. \quad (55) \]

In the general case we have
\[ E \left( \sum_{n=1}^{N} C_n e^{\alpha_n t} \right) = -2 \sum_{n=1}^{N} \sum_{k=n+1}^{N} C_nC_k p_{\alpha_n} \delta_{\alpha_n,-\alpha_k}, \quad (56) \]

where
\[ \delta_{\alpha_n,-\alpha_k} = \begin{cases} 1, & \alpha_n = -\alpha_k, \\ 0, & \alpha_n \neq -\alpha_k. \end{cases} \quad (57) \]

Note that all summands in (56) are integrals of motion, therefore, we explicitly show that \( E \left( \sum_{n=1}^{N} C_n e^{\alpha_n t} \right) \) is an integral of motion. From formula (56) we see that the energy density is a sum of the crossing terms. At the same time the pressure is a sum of "individual" pressures and has no crossing term. In the case of an arbitrary finite number of summands the pressure is as follows:
\[ P \left( \sum_{n=1}^{N} C_n e^{\alpha_n t} \right) = \sum_{n=1}^{N} C_n^2 p_{\alpha_n} e^{2\alpha_n t}. \quad (58) \]

If the parameters \( \xi^2 \) and \( c \) are such that the characteristic equation (24) have double roots, then eq. (23) has the following solution
\[ \phi_0(t) = B_1 e^{\alpha_0 t} + C_1 e^{\alpha_0 t} + B_2 e^{-\alpha_0 t} + C_2 e^{-\alpha_0 t}, \quad (59) \]

where \( B_1, B_2, C_1 \) and \( C_2 \) are constants, \( \alpha_0 \neq 0 \) is defined by (27). Using formulas (42) and (45) and substituting
\[ \alpha_0 = \frac{\sqrt{4 - 2\xi^2}}{2\xi}, \quad (60) \]

we obtain
\[ E(\phi_0) = -\frac{(\xi^2 - 2)}{3\xi^2} \left( 3C_1B_2\sqrt{4 - 2\xi^2}\xi - 3C_2B_1\sqrt{4 - 2\xi^2}\xi + 8B_1B_2 \left( 2\xi^2 - 1 \right) \right). \quad (61) \]

The pressure is as follows
\[ P(\phi_0) = -\frac{(\xi^2 - 2)}{3\xi^2} \left( B_2 e^{-\frac{\sqrt{4 - 2\xi^2}}{\xi}} \left( B_2 \left( 8\xi^2 - 3t\sqrt{4 - 2\xi^2}\xi - 4 \right) - 3C_2\xi\sqrt{4 - 2\xi^2} \right) + B_1 e^{-\frac{\sqrt{4 - 2\xi^2}}{\xi}} \left( 3C_1\sqrt{4 - 2\xi^2}\xi + B_1 \left( 8\xi^2 + 3t\sqrt{4 - 2\xi^2}\xi - 4 \right) \right) \right). \quad (62) \]
3.2.2 Energy Density and Pressure for real $\alpha$

As we have seen in Sect. 3.1.3 for some values of parameters $\xi$ and $c$ eq. (24) has real roots. We denote as $p_m$ the values of $p_\alpha$ for real $\alpha = m$,

$$p_m = m^2 \left( \xi^2 - 2 + 2\xi^2 m^2 \right),$$

(63)

where $\xi^2$ is given by (34). For different values of $c$ the function $p_m$ is presented in Fig. 4.

![Figure 4: The dependence of $p_m$ on $m$ at $c = 1/2$ (right), $c = 1$ (center) and $c = 2$ (left).](image)

If and only if $c > 1$, then there exists the interval of $0 < m^2 < m^2_{\text{max}}$ on which $p_m < 0$. Some part of this interval is not physical because on this part $g(m^2, c) < 0$. The straightforward calculations show that

$$p_m = -\frac{\partial g(m^2, c)}{\partial m^2}. \quad (64)$$

Therefore, at the point

$$m^2_{\text{max}} = -\frac{1}{2} - \frac{1}{2} W_{-1} \left( -\frac{e^{-1}}{c} \right), \quad (65)$$

we obtain $p_m(m_{\text{max}}) = 0$. We conclude that for $c > 1$ and $\xi^2 < \xi^2_{\text{max}}$ we have two positive roots of (24): $m_1$ and $m_2 > m_1$, with $p_{m_1} < 0$ and $p_{m_2} > 0$.

The energy density and the pressure for solutions with real $\alpha$ one can calculate using formulas (56) and (58) and results are presented in Table 1.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$E$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{\pm mt}$</td>
<td>0</td>
<td>$p_m e^{\pm 2mt}$</td>
</tr>
<tr>
<td>$\sinh(mt)$</td>
<td>$p_m/2$</td>
<td>$p_m \cosh(2mt)/2$</td>
</tr>
<tr>
<td>$\cosh(mt)$</td>
<td>$-p_m/2$</td>
<td>$p_m \cosh(2mt)/2$</td>
</tr>
</tbody>
</table>

Table 1: Solutions, densities of energy and pressures for real $\alpha$

We see from Table 1 that odd solutions are physically meaningful, $E > 0$, if $p_m$ is positive. Fig 4 shows that odd solutions are physical for $c < 1$ and any $\xi^2$ and for $c > 1$ only for $m^2 > m^2_{\text{max}}$. The pressure corresponding to this solution is always positive.

Even solutions are physically meaningful if $p_m$ is negative. Therefore, even solutions are physical only for $c > 1$ and $m^2 < m^2_{\text{max}}$. The pressure corresponding to this solution is always negative. The equation of the state parameter for this solution is

$$w = -\cosh(2mt) < -1. \quad (66)$$
3.2.3 Energy Density and Pressure for pure imaginary $\alpha$

As we have seen in Sect.3.1.4 for some values of parameters $\xi$ and $c$ eq. (24) has only pair of pure imaginary roots. These solutions correspond to

$$\phi = C \sin(\mu(t - t_0)).$$  \hspace{1cm} (67)

On the solutions (67) the energy and pressure are given by

$$E(C \sin(\mu(t - t_0))) = \frac{C^2}{2} p_\alpha \equiv \frac{C^2}{2} \pi_\mu,$$  \hspace{1cm} (68)

$$P(C \sin(\mu(t - t_0))) = -\frac{C^2}{2} \pi_\mu \cos(2\mu(t - t_0)),$$  \hspace{1cm} (69)

where

$$\pi_\mu \equiv p_{i\mu} = \mu^2 \left(2 + 2\xi^2 - \xi^2\right),$$  \hspace{1cm} (70)

where $\xi^2$ is given by (39). $\pi_\mu$ as a function of $\mu$ for different values of $c$ is presented in Fig (5).

![Figure 5: The dependence of $\pi_\mu$ on $\mu$ at $c = 1/2$ (right), $c = 1$ (center) and $c = 2$ (left).](image)

Note that $\pi_\mu$ for $c \leq 1$ is positive, however for $c > 1$ is positive only for $\mu^2 \geq \mu^2_{\text{min}} = \frac{1}{2} - \frac{1}{\xi^2}$. The equation of the state parameter for this solution is

$$w = -\cos(2\mu(t - t_0)), \quad |w| \leq 1.$$  \hspace{1cm} (71)

3.2.4 Energy and Pressure in the case $c = 1$

The energy density and pressure for solutions (31):

$$\phi_1(t) = C_1 t + C_0,$$  \hspace{1cm} (72)

where $C_0$ and $C_1$ are arbitrary constants, are as follows

$$E(\phi_1) = \left(\frac{\xi^2}{2} - 1\right) C_1^2, \quad P(\phi_1) = \left(\frac{\xi^2}{2} - 1\right) C_1^2$$  \hspace{1cm} (73)

and the state parameter $w \equiv P/E = 1$. 

13
The straightforward calculations show that pressure and energy density for more general solutions
\[ \phi = \sum_{n=1}^{N} \tilde{C}_n e^{\alpha_n t} + \phi_1 \] (74)
are
\[ E \left( \sum_{n=1}^{N} \tilde{C}_n e^{\alpha_n t} + \phi_1 \right) = E \left( \sum_{n=1}^{N} \tilde{C}_n e^{\alpha_n t} \right) + E (\phi_1) \] (75)
and
\[ P \left( \sum_{n=1}^{N} \tilde{C}_n e^{\alpha_n t} + \phi_1 \right) = P \left( \sum_{n=1}^{N} \tilde{C}_n e^{\alpha_n t} \right) + P (\phi_1). \] (76)

Let, for example,
\[ \phi (t) = \cosh (mt) + C_1 t + C_0. \]
The corresponding energy density and pressure are:
\[ E = -\frac{1}{2} p_m + \left( \frac{\xi^2}{2} - 1 \right) C_1^2, \quad P = \frac{1}{2} p_m \cosh (2mt) + \left( \frac{\xi^2}{2} - 1 \right) C_1^2. \]
In the case \( \xi^2 = 2 \) the root \( \alpha = 0 \) is a double root of eq. (24), so eq. (23) has solutions (32):
\[ \phi_2 = C_3 t^3 + C_2 t^2 + C_1 t + C_0. \]

We obtain:
\[ E (\phi_2) = 4 \left( 3C_1 C_3 - C_2^2 - 6C_3^2 \right), \quad P (\phi_2) = 72C_3^2 t^2 + 48C_2 C_3 t + 4 \left( C_2^2 + 3C_1 C_3 - 6C_3^2 \right). \]

### 3.2.5 Energy and pressure for complex \( \alpha = r + iv \)

For a decreasing solution
\[ \phi (t) = e^{-rt} \cos (vt) \] (77)
we have
\[ E \left( e^{-rt} \cos (vt) \right) = 0, \] (78)
\[ P \left( e^{-rt} \cos (vt) \right) = \frac{e^{-2rt}}{4} \left( p_\alpha e^{2ivt} + p_\alpha^* e^{-2ivt} \right). \] (79)

Using general formulas (56) and (58) it is easy to find the energy and pressure for even and odd real solutions with complex \( \alpha \). For example the energy and pressure for the even solution \( \phi = \cosh (rt) \cos (Im(\alpha)t) \) are as follows:
\[ E (\cosh (rt) \cos (vt)) = -\frac{1}{8} (p_\alpha + p_\alpha^*), \] (80)
\[ P (\cosh (rt) \cos (vt)) = \frac{p_\alpha}{16} (e^{-2\alpha t} + e^{2\alpha t}) + \frac{p_\alpha^*}{16} (e^{-2\alpha^*t} + e^{2\alpha^*t}). \] (81)

The equation of state parameter on the even solution is
\[ w = -\frac{p_\alpha + p_\alpha^*}{p_\alpha \cosh \alpha t + p_\alpha^* \cosh \alpha^* t}. \] (82)
3.3 Local Field Representation

3.3.1 Action in a Form of the Weierstrass Product

As in [21] we can present $F(-\Box)$ in the action (1) as the Weierstrass product. To this purpose let us recall that a function $F(z)$ of a complex variable $z$ such that its logarithmic derivative \( F'(z)/F(z) \) is a meromorphic function regular in the point $z = 0$, has simple poles and satisfies \( |F'(z)/F(z)| < C, z \in \Gamma_n, n = 1, 2, ... \), can be presented as

\[
F(z) = F(0) e^{F'(0) \cdot \frac{z}{F(0)}} \prod \left( 1 - \frac{z}{z_k} \right) e^{z/z_k} .
\] (83)

$\Gamma_n, n = 1, 2, ...$ is a set of special closed contours $\Gamma_n$ such that the point $z = 0$ is in all $\Gamma_n$, $\Gamma_n$ is in $\Gamma_{n+1}$, and $S_n/d_n \leq C$, where $S_n$ is a length of the contour $\Gamma_n$, and $d_n$ is its distance from zero [50].

In the case of a more week requirement \( |F'(z)/F(z)| < M|z|^p z \in \Gamma_n, n = 1, 2, ... \) we have

\[
F(z) = e^{f(z)} \prod \left( 1 - \frac{z}{z_k} \right) e^{P_k(z)} , \quad P_k(z) = \sum_{l=1}^{p} \frac{1}{l} \left( \frac{z}{z_k} \right)^l ,
\] (84)

where $f(z)$ is an entire function.

In the case of $F$ given by (2) the Weierstrass product can be written in the form

\[
F(\alpha^2) = e^{f(\alpha^2)} \prod_n (\alpha^2 - \alpha_n^2) ,
\] (85)

The function $f(z)$ in our case is

\[
f(z) = A + \beta z ,
\] (86)

where constants $A$ and $\beta$ are determined by $\xi$ and $c$.

It is convenient to pick out real roots in (85) and combine the complex conjugated roots:

\[
F(\alpha^2) = e^{f(\alpha^2)} \prod (\alpha^2 - m_k^2) \prod (\alpha^2 - \alpha_n^2)(\alpha^2 - \alpha_n^2) ,
\] (87)

where $m_k$ denote real roots. In Sect.3 we have found the cases when real roots do exist.

In the case of distinctive roots the Lagrangian up to a total derivative can be presented as a sum of an infinite number of fields [51]-[54]

\[
L = \frac{1}{2} \phi F(\partial^2) \phi \sim \frac{1}{2} \sum \epsilon_n \psi_n e^{f(\partial^2)} (-\partial^2 + \alpha_n^2) \psi_n + c.c. ,
\] (88)

where $\sim$ means equivalence up to a total derivative, $\epsilon_n$ are constants. Note that for complex roots $\psi_n$ are complex.

3.3.2 Energy Density and Pressure in Terms of $\psi$-fields

It is instructive to present the formula for energy and pressure obtained in section 3.2. in terms of $\psi$ fields. All considerations below take place for nondegenerate roots.
According to a general procedure of construction of the energy and pressure we write a generalization of (88) to a non-flat case

$$L_g = \sum L_g(\psi_n), \quad L_g(\psi_n) = \frac{\epsilon_n}{2} \sqrt{-g} \psi_n e^{f(-\Box_g)} (\Box_g + \alpha_n^2) \psi_n,$$

and find

$$E = \sum E_n, \quad E_n = \frac{\epsilon_n}{2} \left( \psi_n^2 - \alpha_n^2 \psi_n^2 \right) e^{f(\alpha_n^2)};$$

$$P = \sum P_n, \quad P_n = \frac{\epsilon_n}{2} \left( \psi_n^2 + \alpha_n^2 \psi_n^2 \right) e^{f(\alpha_n^2)}.$$

The E.O.M. for $\psi_n$ is

$$\left( \partial^2 - \alpha_n^2 \right) \psi_n = 0$$

and its solutions are

$$\psi_n = A_n e^{\alpha_n t} + B_n e^{-\alpha_n t}.$$

For solutions (93) we obtain

$$E = 2 \sum \alpha_n^2 \epsilon_n A_n B_n e^{2\alpha_n^2};$$

$$P = \sum \epsilon_n \alpha_n^2 \left( A_n^2 e^{2\alpha_n t} + B_n^2 e^{-2\alpha_n t} \right) e^{2\alpha_n^2}.$$

On the other hand according to (56) and (58) we have

$$E = -2 \sum A_n B_n \alpha_n^2 \left( \xi^2 - 2 + 2\xi^2 \alpha_n^2 \right);$$

$$P = \sum \left( A_n^2 e^{-2\alpha_n t} + B_n^2 e^{2\alpha_n t} \right) \alpha_n^2 \left( \xi^2 - 2 + 2\xi^2 \alpha_n^2 \right).$$

Comparing (94),(95) and (96),(97) and using equation (24) we see that $\epsilon_n$ can be presented as

$$\epsilon_n = -(2ce^{-2\alpha_n^2} + \xi^2) e^{-\beta \alpha_n^2},$$

that is in accordance with general formula for $\epsilon_n$ [53] [21].

### 3.4 Finite Order Derivative Approximation

#### 3.4.1 Two types of approximations

There are two different types of finite order derivative approximations:

- a direct finite order derivative approximation

$$L_{2k}(\alpha^2) \equiv \sum_{n=0}^{k} a_n \alpha_n^2;$$

16
• a finite number of terms in the Weierstrass product

\[ L_{2k}^{(W)}(\alpha^2) \equiv f(\alpha^2) \prod_{n}^{k}(\alpha^2 - \alpha_n^2). \]  

(100)

We label roots so that \(|\alpha_n| \leq |\alpha_{n+1}|\).

Let us consider the most simple case \(\xi = 0, c = 1\), that corresponds to

\[ L(\alpha^2) \equiv L_{0,1}(\alpha^2) = 1 - e^{-2\alpha^2}. \]  

(101)

In this case all roots can be written explicitly

\[ 1 - e^{-2\alpha^2} = 2\alpha^2 e^{-\alpha^2} \prod_{j=1}^{\infty} \left(1 + \frac{\alpha^4}{\pi^2 j^2}\right). \]  

(102)

### 3.4.2 Direct Finite Order Approximation in the case \(\xi = 0\) and \(c = 1\)

In the second order derivative approximation we should keep in (23) with \(c = 1, \xi = 0\) only the second derivatives

\[ 2\partial^2 \phi = 0. \]  

(103)

This equation has the following solutions

\[ \phi = C_1 t + C_2. \]  

(104)

These solutions correspond to roots \(\alpha_1 = 0\).

In the fourth order derivative approximation eq. (23) with \(c = 1, \xi = 0\) is as follows

\[ \partial^2 \phi - \partial^4 \phi = 0. \]  

(105)

Equation (24) in this approximation

\[ 2\alpha^2 - 2\alpha^4 = 0, \]  

(106)

has two solutions:

\[ \alpha^2 = 0, \quad \alpha^2 = 1. \]  

(107)

We see that the fourth order direct approximation the approximate equation (106) has an extra root \((\alpha = 1)\) that is absent in the initial equation (24).

The similar situation takes place for higher order approximations. The characteristic equation of the direct n-order approximation contains several artificial roots. The appearance of these roots is related to artificial roots of polynomial approximations of the function \(f\) in the Weierstrass product (85). In Fig. 6 we plot all roots of an n-order polynomial approximation of function \(L_{0,1}(\alpha^2)\) for n=20 and 40. We see that these polynomials have true roots as well as artificial roots that go to infinity when the order of the approximate polynomial increases.
3.4.3 Finite Order Weierstrass Product Approximation

We consider an approximation to $L_{0,1}(\alpha^2)$ that keeps only a finite number of terms in the Weierstrass product

$$L_{0,1,2+4k}^{(W)}(\alpha^2) \equiv 2\alpha^2 e^{-\alpha^2} \prod_{j=1}^{k} \left( 1 + \frac{\alpha^4}{\pi^2j^2} \right).$$

We call this approximation 1+2k-roots approximation. The corresponding Lagrangian is

$$\mathcal{L}_{0,1,2+4k}^{(W)}(\alpha^2) \equiv \frac{1}{2} \phi \partial^2 e^{-\partial^2 \phi} \prod_{j=1}^{k} \left( 1 + \frac{\partial^4}{\pi^2j^2} \right) \phi.$$

Let us consider one root approximation

$$\mathcal{L}_{0,1,2}^{(W)}(\alpha^2) \equiv \frac{1}{2} \phi \partial^2 e^{-\partial^2 \phi}.$$

E.O.M. is

$$\partial^2 e^{-\partial^2 \phi} = 0$$

and it has an unique solution (the same as E.O.M. of 2-nd order derivative approximation)

$$\phi = At + C.$$
E.O.M. is
\[ \partial^2 \left( 1 + \frac{\partial^4}{\pi^2} \right) e^{-\partial^2} \phi = 0 \] (114)
and it has the following solutions
\[ \phi_6(t) = A_1 t + \delta_1 + A_2 e^{\sqrt{\pi^2} t} \sin(\sqrt{\frac{\pi}{2}} t + \delta_2) + A_3 e^{-\sqrt{\pi^2} t} \sin(\sqrt{\frac{\pi}{2}} t + \delta_3), \] (115)
where \( A_k \) and \( \delta_k \) are arbitrary constants.

The next approximation contains 4 extra roots
\[ \mathcal{L}^{(W)}_{0,1,10} \equiv \frac{1}{2} \phi \partial^2 e^{-\partial^2} \left( 1 + \frac{\partial^4}{\pi^2} \right) \left( 1 + \frac{\partial^4}{4\pi^2} \right) \phi. \] (116)
E.O.M. is
\[ \partial^2 \left( 1 + \frac{\partial^4}{\pi^2} \right) \left( 1 + \frac{\partial^4}{4\pi^2} \right) e^{-\partial^2} \phi = 0 \] (117)
and it has the following solutions
\[ \phi_{10}(t) = \phi_6(t) + A_4 e^{\sqrt{\pi^2} t} \sin(\sqrt{\pi^2} t + \delta_4) + A_5 e^{-\sqrt{\pi^2} t} \sin(\sqrt{\pi^2} t + \delta_5). \] (118)

It is obvious that solutions of the Weierstrass approximate equations reproduce a finite number of modes of the full equation.

If we restrict ourself to decreasing solutions
\[ \phi_{10, \text{decr}}(t) = \delta_1 + A_3 e^{-\sqrt{\pi^2} t} \sin(\sqrt{\frac{\pi}{2}} t + \delta_3) + A_5 e^{-\sqrt{\pi^2} t} \sin(\sqrt{\pi^2} t + \delta_5) \] (119)
we see that the last term can be ignored as compared with term first two terms.

4 Non-flat Dynamics

4.1 Modified Action

Our starting point is the action
\[ \int \sqrt{-g} \left( \frac{m_p^2}{2} R + \sum \left[ \frac{\epsilon_n}{2} \psi_n e^{\beta \alpha_n^2 (\Box g + \alpha_n^2)} - V_n(\psi_n) + c.c. \right] \right). \] (120)
If we left only the time dependence and metric as a spatially flat Friedmann metric we have the following equation for \( \psi_n \)
\[ \epsilon_n (\mathcal{D} + \alpha_n^2) \psi_n - e^{-\beta \alpha_n^2} V_n'(\psi_n) = 0, \] (121)
where \( V_n' \) a derivative on the argument.
The energy and the pressure in the Friedmann metric have the form

\[ E_{\text{mod}} = E + \sum V_n, \]
\[ P_{\text{mod}} = P - \sum V_n. \]

where \( E \) and \( P \) are given by formula (90) and (91). This means that all extra terms \( V_n \) play a role of potential terms.

The Friedmann equations of motion are

\[ 3H^2 = \frac{1}{m_p^2} \left( E + \sum V_n \right), \]
\[ 3H^2 + 2\dot{H} = -\frac{1}{m_p^2} \left( P - \sum V_n \right), \]

where \( \dot{H} \equiv \partial H \), and therefore

\[ \dot{H} = -\frac{1}{2m_p^2}(P + E). \]

Now we assume that \( V_n(\psi_n) \) is found from the requirement that \( \psi_n \) in the non-flat case is the same as in the flat case. This means that we can find the contribution from each mode to the evolution of \( H \) and we get

\[ \dot{H} = \frac{1}{2m_p^2} \left( \sum 2A_n B_n p_{\alpha_n} - \sum \left( A_n^2 e^{-2\alpha_n t} + B_n^2 e^{2\alpha_n t} \right) p_{\alpha_n} \right). \]

Therefore,

\[ H(t) = \frac{1}{2m_p^2} \left( \sum 2A_n B_n p_{\alpha_n} t - \sum \left( -\frac{A_n^2}{2\alpha_n} e^{-2\alpha_n t} + \frac{B_n^2}{2\alpha_n} e^{2\alpha_n t} \right) p_{\alpha_n} \right) + H_0, \]

where \( H_0 \) is an integration constant and we assume the sum goes over the complex conjugated roots. Let us note that a non-zero \( H_0 \) it is convenient [20] to incorporate in a change of plane solutions

\[ \psi_n(t) = A_n e^{\alpha_n H_0 t} + B_n e^{-\alpha_n H_0 t}, \]

where \( \alpha_{n,H_0} \) is related with \( \alpha_n \) as

\[ \alpha_{n,H_0}^2 + 3H_0 \alpha_{n,H_0} = \alpha_n^2. \]

4.2 One Mode Solutions

4.2.1 One Real Root: Decreasing Solution

Let us consider a simplest particular case:

\[ \psi_1 = \exp(-mt), \]
From (127) we have
\[ H(t) = \frac{p_m}{4m_p^2}e^{-2mt} + H_0. \]  
(131)

Therefore to ensure that (130) and (131) solve the Friedmann equations we have to add to the action the following potential
\[ V(\psi_1) = \frac{3m^2}{16m_p^2}(\xi^2 - 2 + 2\xi^2m^2)^2\psi_1^4 + \frac{3m}{2}(\xi^2 - 2 + 2\xi^2m^2)H_0\psi_1^2 + 3m^2H_0^2. \]  
(132)

Let us note that one gets the same potential for the unbounded solution \( \psi_1 = \exp(mt) \).

4.2.2 One Real Root: Odd Solution

Let us consider an odd solution
\[ \psi_1 = \sinh(mt). \]  
(133)

From (127) we have
\[ H(t) = -\frac{p_m}{8m_p^2} \sinh(2mt) - \frac{p_m}{4m_p^2}t + H_0 \]  
(134)

and the potential for \( H_0 = 0 \) is
\[ V(\psi) = \frac{p_m}{2} \left( -1 + \frac{3p_m}{8m_p^2} \left( t + \frac{1}{2m} \sinh(2mt) \right)^2 \right). \]  
(135)

The potential as a function of \( \psi \) is given by
\[ V(\psi) = -\frac{p_m}{2} + \frac{3p_m^3}{16m_p^2m^2} \left[ \psi^2(1 + \psi^2) + 2\psi \sqrt{1 + \psi^2} \arcsinh(\psi) + \arcsinh^2(\psi) \right]. \]  
(136)

![Figure 7: A. Potential \( V(\psi) \) for odd solutions corresponding to the real root; \( |\psi| < 1 \) B. Potential \( V(\psi) \) for odd solutions corresponding to a pure imaginary root. Horizontal lines represent \( p_m \) or \( \pi_\mu \), respectively.](image_url)
4.2.3 One Pure Imaginary Root

Let us consider an odd solution

\[ \psi = \sin(\mu t). \]

(137)

From (68), (69) and (125) we get explicitly the Hubble parameter

\[ H(t) = \frac{\pi \mu}{4m_p^2} t + \frac{\pi \mu}{8\mu m_p^2} \sin 2\mu t + H_0. \]

(138)

The potential for \( H_0 = 0 \) as a function of \( \psi \) is given by

\[ V(\psi) = \frac{1}{2} \pi \mu + \frac{3\pi^2}{16m_p^2 \mu^2} \left[ \psi^2(1 - \psi^2) + 2\psi \sqrt{1 - \psi^2} \arcsin \psi + \arcsin^2 \psi \right]. \]

(139)

Note, that this formula is valued only for \( \psi^2 < 1 \). On this region the potential (139) is convex and it has an unique minimum (see Fig. 7).

4.2.4 Pair of Complex Roots

For the case of a complex root \( \alpha \) we have to consider the pair \( \alpha, \alpha^* \) and the solution \( \psi \) as a sum

\[ \psi = \psi_\alpha + \psi_{\alpha^*}, \quad \psi_\alpha = e^{-\alpha t}, \quad \psi_{\alpha^*} = e^{-\alpha^* t}, \]

(140)

we assume that \( \Re(\alpha) = r > 0 \). From (127) we have

\[ H(t) = \frac{1}{4m_p^2} \left[ \frac{p_\alpha}{\alpha} e^{-2\alpha t} + \frac{p_{\alpha^*}}{\alpha^*} e^{-2\alpha^* t} + H_0 \right] = \frac{1}{4m_p^2} \left[ \frac{p_\alpha}{\alpha} \psi_\alpha^2 + \frac{p_{\alpha^*}}{\alpha^*} (\psi_{\alpha^*})^2 + H_0 \right]. \]

(141)

This solution corresponds the following potential

\[ V = \frac{3}{16m_p^2} \left[ \alpha^2 \left( \xi^2 - 2 + 2\xi^2 \alpha^2 \right) \psi_\alpha^4 + \alpha^* \left( \xi^2 - 2 + 2\xi^2 \alpha^* \right) \psi_{\alpha^*}^4 + H_0^2 + 2\alpha \alpha^* \left( \xi^2 - 2 + 2\xi^2 \alpha^2 \right) \left( \xi^2 - 2 + 2\xi^2 \alpha^* \right) \psi_\alpha^2 \psi_{\alpha^*}^2 + 2H_0 \left( \frac{p_\alpha}{\alpha} \psi_\alpha^2 + \frac{p_{\alpha^*}}{\alpha^*} (\psi_{\alpha^*})^2 \right) \right]. \]

(142)

4.3 Cosmological Properties of One Mode Solutions

4.3.1 Real Root

The decreasing solution (130), (131) corresponds to the scale factor

\[ a(t) = a_0 \exp \left( -\frac{p_m}{8m_p^2 m^2} \exp(-2mt) + H_0 t \right). \]

(143)

This solution has no singularities at finite \( t \). It describes an increasing Universe only in the case \( p_m < 0 \). As we have seen in Sect 3.2.2 the negative pressure is possible if \( c > 1 \) and \( m^2 < m_{\text{max}}^2 \) that takes place if \( \xi^2 \leq \xi_{\text{max}}^2 \), where \( \xi_{\text{max}} \) is given by (36).

The odd increasing solution (133), (134) corresponds to the scale factor

\[ a(t) = a_0 \exp \left( -\frac{p_m}{16m_p^2 m^2} \left( \cosh(2mt) + 2m^2 t^2 \right) + H_0 t \right). \]

(144)

The scale factor (144) has no singularity. It increases at large time if \( p_m < 0 \).
4.3.2 Imaginary Root

The odd periodic solution (137), (138) corresponds to the scale factor

\[ a(t) = a_0 \exp \left( -\frac{\pi \mu}{16 \mu^2 m_p^2} \cos(2\mu t) + \frac{\pi \mu}{8 m_p^2} t^2 + H_0 t \right) \]

(145)

and we have an expansion with an acceleration if \( \pi \mu > 0 \).

4.3.3 Pair of Complex Roots

The Hubble parameter (141) corresponds to the following scale factor

\[ a(t) = a_0 \exp \left\{ -\frac{1}{8 m_p^2} \left[ \frac{p_\alpha}{\alpha^2} e^{-2\alpha t} + \frac{p_{\alpha^*}}{\alpha^{*2}} e^{-2\alpha^* t} \right] \right\}, \]

(146)

substituting the explicit formula for \( p_\alpha \) and \( p_{\alpha^*} \) we get

\[ a(t) = a_0 \exp \left\{ H_0 t - \frac{(\xi^2 - 2)}{4 m_p^2} e^{-2\nu t} \cos 2\nu t - \frac{\xi^2}{2 m_p^2} e^{-2\nu t} [(\nu^2 - \nu^2) \cos 2\nu t - 2r \nu \sin 2\nu t] \right\}. \]

(147)

We see that a late time expansion regime corresponds only to \( H_0 > 0 \) (compare with [20]).

4.4 Two Real Root Solutions

Above-mentioned solutions for real roots \( m_n \) correspond to monotonic behaviour of the Hubble parameter. To describe nonmonotonic behaviour let us consider the case of \( c > 1 \) and \( \xi^2 < \xi^2_{\text{max}} \). There exist two real roots of (24) \( m_1 \) and \( m_2 \) such that that \( |m_1| < |m_2| \). The corresponding solution to (23) is

\[ \phi = Ae^{m_1 t} + Be^{m_2 t}, \]

(148)

where \( A \) and \( B \) are constants. Without loss of generality we can put \( A = 1 \).

Using (126) we obtain

\[ \dot{H} = -\frac{1}{2 m_p^2} \left( p_{m_1} e^{2m_1 t} + B^2 p_{m_2} e^{2m_2 t} \right). \]

(149)

Let us analyze a possibility \( \dot{H} = 0 \), which correspond to crossing of the cosmological constant barrier for the state parameter \( w \). In Subsection 3.2.2 we have obtained that \( p_{m_1} < 0 \) and \( p_{m_2} > 0 \) (See Figs. 2 and 4). So, for any roots \( m_1 \) and \( m_2 \), there exist such real \( B \) that \( \dot{H} = 0 \) at the point \( t = t_1 \):

\[ B = \pm \sqrt{\frac{-p_{m_1} e^{2m_1 t_1}}{p_{m_2} e^{2m_2 t_1}}}. \]

(150)

We can conclude that solutions (148) correspond to cosmological models with the crossing of \( w = -1 \) barrier.
The Hubble parameter and the scale factor are as follows:

\[ H = -\frac{1}{4m_0^2m_1m_2} \left( p_{m_1} m_2 e^{2m_1 t} + B^2 p_{m_2} m_1 e^{2m_2 t} \right), \quad (151) \]

\[ a = a_0 \exp \left( -\frac{1}{4m_0^2m_1^2m_2^2} \left( p_{m_1} m_2 e^{2m_1 t} + B^2 p_{m_2} m_1 e^{2m_2 t} \right) \right). \quad (152) \]

If \( m_1 > 0 \) and \( m_2 < 0 \), then at late time

\[ \dot{H} \approx -\frac{1}{2m_0^2} p_{m_1} e^{2m_1 t} > 0 \quad (153) \]

and

\[ H \approx -\frac{1}{4m_0^2m_1} p_{m_1} e^{2m_1 t} > 0. \quad (154) \]

We can conclude that all solutions (148) correspond to cosmological models with the crossing of \( w = -1 \) barrier. Let us remind in this context that models with a crossing of the \( w = -1 \) barrier are a subject of recent studies [39, 40, 58, 59, 60, 61]. Simplest models include two scalar fields (one phantom and one usual field, see [37, 62, 63] and refs. therein). In our case a non-locality provides a crossing of the \( w = -1 \) barrier in spite of the presence of only one scalar field. This fact has a simple explanation. The crossing of the \( w = -1 \) in our case is driven by an equivalence of our non-local model to a set of local models some of which are ghosts.

5 Conclusions

We have studied linear non-local models which violate the NEC. The form of these models is inspired by the SFT models. These models have an infinite number of higher derivative terms and are characterized by two positive parameters, \( \xi^2 \) and \( c \).

The model with \( c = 1 \) is a toy nonlocal model for the dilaton coupling to the gravitation field. A distinguished feature of this model is the invariance under the shift of the dilaton field to a constant.

For particular cases of the parameters \( \xi^2 \) and \( c \) these models correspond to linear approximations to the bosonic [33, 31] and non-BPS fermionic [45] cubic SFT as well as to the nonpolynomial SFT [46, 47].

The case \( \xi = 0 \) describes a linear approximation to the p-adic string [10]. Let us note that recently a p-adics string inflation model has been considered [22].

In the flat case all solutions of the equation of motion are plane waves and are controlled by roots of the characteristic equation. Our characteristic equation has complex distinctive simple roots. In some particular cases there are single or double roots, which are real or pure imaginary. The energy on plane waves is equal to zero except for the cases of couples of roots \( (\alpha, -\alpha) \). The pressure is a sum of one mode pressures. The pressure for the one plane wave corresponding to a real root can be positive or negative depending on parameters of
the theory. For \( c \leq 1 \) the one mode pressure is positive and for \( c > 1 \) it could be negative or positive.

To study cosmological applications of these models we have investigated the behaviour of these models in the Friedmann background. We have performed this study within an approximation scheme. A simplest approximation is a local field approximation (or a mechanical analogous model in a terminology of [18]). On an example of the free flat case we have shown that in special cases we can use a local two derivatives approximation, but the next derivative approximations exhibit artifacts. Followed [21] we have used the Weierstrass product representation to study finite mode approximations to these models. As was noted in [21] a straightforward application of the Ostrogradski method to these approximations indicates that energies are unbounded (an eigenvalue problem for the unbounded hyperbolic Klein-Gordon equation on manifolds is solved in [55]) and it is expected [21] that an incorporation of non-flat metric or nonlinear terms could drastically change the situation.

A distinguished cosmological property of these models is a crossing of the phantom divide [20, 23]. But there are also possibilities for other types of behaviour. Namely, the toy dilaton model possesses decreasing solutions describing asymptotical flat Universes (adding a cosmological constant modifies these solutions to near-de Sitter solutions). It also has odd bouncing solutions describing a contracting Universes meanwhile even bouncing solutions are forbidden. For special values of parameters corresponding to tachyon SFT models there are even bouncing solutions with an accelerated expansion.

We have shown that for some particular cases there are deformations of the model such that exact solutions of the linear problem are inherited by nonlinear non-flat ones. This is similar to what was done before for local models [36, 56, 57]. A stability of exact solutions in local models has been studied in [36]. We will study stability of our solutions in the future work.

Acknowledgements

This research is supported in part by RFBR grant 05-01-00758. The work of I.A. and L.J. is supported in part by INTAS grant 03-51-6346 and Russian President’s grant NSh–672.2006.1. L.J. acknowledges the support of the Centre for Theoretical Cosmology, in Cambridge. S.V. is supported in part by Russian President’s grant NSh–8122.2006.2. I.A. and S.V. would like to thank A.S. Koshelev and I.V. Volovich for useful discussions. L.J. would like to thank David Mulryne for very helpful communications.

A Calculations of Nonlocal Energy Density and Pressure on Plane Waves

In this appendix we calculate the energy density and pressure for the following solution

\[
\phi = A e^{\alpha_1 t} + B e^{\alpha_2 t},
\]  

(155)
where \( \alpha_1 \) and \( \alpha_2 \) are different roots of (21). We have

\[
E(Ae^{\alpha_{1t}} + Be^{\alpha_{2t}}) = E(Ae^{\alpha_{1t}}) + E(Be^{\alpha_{2t}}) + E_{\text{cross}}(Ae^{\alpha_{1t}}, Be^{\alpha_{2t}}) = ABE_{\text{cross}}(e^{\alpha_{1t}}, e^{\alpha_{2t}}),
\]

where the functional \( E_{\text{cross}} \) is defined as follows:

\[
E_{\text{cross}}(\phi_1, \phi_2) = \xi^2 \partial \phi_1 \partial \phi_2 + \phi_1 \phi_2 + c \Phi_1 \Phi_2 + c \int_0^1 \left[ \left( e^{-\rho^2 \Phi_1} \right) \partial^2 \left( e^{-\rho^2 \Phi_2} \right) + \left( e^{-\rho^2 \Phi_2} \right) \partial^2 \left( e^{-\rho^2 \Phi_1} \right) \right] d\rho - c \int_0^1 \left[ \partial \left( e^{-\rho^2 \Phi_1} \right) \partial \left( e^{-\rho^2 \Phi_2} \right) + \partial \left( e^{-\rho^2 \Phi_2} \right) \partial \left( e^{-\rho^2 \Phi_1} \right) \right] d\rho,
\]

\[
\Phi_1 \equiv e^{-\rho^2} \phi_1, \quad \Phi_2 \equiv e^{-\rho^2} \phi_2.
\]

For \( \phi_1 = e^{\alpha_{1t}} \) and \( \phi_2 = e^{\alpha_{2t}} \) we have

\[
E_{\text{cross}}(e^{\alpha_{1t}}, e^{\alpha_{2t}}) = e^{(\alpha_1 + \alpha_2)t} \left\{ \alpha_1 \alpha_2 - 1 + ce^{a_1^2 - a_2^2} + \alpha_1 \alpha_2 \left( e^{a_2^2 - a_1^2} + e^{b_2^2 - b_1^2} \right) \right\} d\rho.
\]

If \( \alpha_1 = -\alpha_2 \), then it is easy to show that

\[
E_{\text{cross}}(e^{\alpha_{1t}}, e^{-\alpha_{1t}}) = -2p_a. \tag{157}
\]

where \( p_a \) is given by (51). We get

\[
E(Ae^{\alpha_{1t}} + Be^{-\alpha_{1t}}) = -2ABp_a. \tag{158}
\]

In the opposite case \( \alpha_1 \neq -\alpha_2 \)

\[
E_{\text{cross}}(e^{\alpha_{1t}}, e^{\alpha_{2t}}) = E_{\alpha_1, \alpha_2} e^{(\alpha_1 + \alpha_2)t}, \tag{159}
\]

\[
E_{\alpha_1, \alpha_2} = \alpha_1 \alpha_2 - 1 + \frac{\alpha_1 \alpha_2 \left( a_2 \alpha_1^2 - a_1 \alpha_2^2 - 2 \alpha_1 \alpha_2 \left( a_1^2 - a_2^2 \right) \right)}{\alpha_1^2 - \alpha_2^2}. \tag{160}
\]

Constants \( \alpha_1 \) and \( \alpha_2 \) are roots of (21), therefore,

\[
E_{\alpha_1, \alpha_2} = \frac{1}{\alpha_1^2 - \alpha_2^2} \left\{ \alpha_1 \alpha_2 \left( (\xi^2 a_2^2 - ce^{-2a_2}) - (\xi^2 a_1^2 - ce^{-2a_1}) \right) + \alpha_2^2 \xi^2 a_1^2 - \alpha_1^2 \xi^2 a_2^2 \right\} = 0. \tag{161}
\]

Note that, the equality \( E_{\text{cross}}(e^{\alpha_{1t}}, e^{\alpha_{2t}}) = 0 \) at \( \alpha_1 \neq -\alpha_2 \) also follows from the energy conservation law. Let us calculate the pressure \( P(\phi) \) for the solution \( \phi = Ae^{\alpha_{1t}} + Be^{\alpha_{2t}} \).

\[
P \left( Ae^{\alpha_{1t}} + Be^{\alpha_{2t}} \right) = P \left( Ae^{\alpha_{1t}} \right) + P \left( Be^{\alpha_{2t}} \right) + P_{\text{cross}} \left( Ae^{\alpha_{1t}}, Be^{\alpha_{2t}} \right), \tag{162}
\]

where

\[
P_{\text{cross}}(\phi_1, \phi_2) = E_{k_{\text{cross}}}(\phi_1, \phi_2) + E_{n_{2\text{cross}}}(\phi_1, \phi_2) - E_{p_{\text{cross}}}(\phi_1, \phi_2) - E_{n_{1\text{cross}}}(\phi_1, \phi_2), \tag{163}
\]

26
\[ E_{k\text{cross}} = \xi^2 \partial \phi_1 \partial \phi_2, \quad E_{p\text{cross}} = \phi_1 \phi_2 + c \Phi_1 \Phi_2, \quad (164) \]

\[ E_{nl1\text{cross}} = c \int_0^1 \left[ \left( e^{-\rho \partial^2} \Phi_1 \right) \partial^2 \left( e^{-\rho \partial^2} \Phi_2 \right) + \left( e^{-\rho \partial^2} \Phi_2 \right) \partial^2 \left( e^{-\rho \partial^2} \Phi_1 \right) \right] d\rho, \quad (165) \]

and

\[ E_{nl2\text{cross}} = -c \int_0^1 \left[ \partial \left( e^{-\rho \partial^2} \Phi_1 \right) \partial \left( e^{-\rho \partial^2} \Phi_2 \right) + \partial \left( e^{-\rho \partial^2} \Phi_2 \right) \partial \left( e^{-\rho \partial^2} \Phi_1 \right) \right] d\rho. \quad (166) \]

The functional \( E_{\text{cross}} = E_{k\text{cross}} + E_{nl2\text{cross}} + E_{p\text{cross}} + E_{nl1\text{cross}} \), proving that \( E_{\text{cross}} = 0 \), we obtain that

\[ E_{k\text{cross}} + E_{nl2\text{cross}} = 0 \quad \text{and} \quad E_{p\text{cross}} + E_{nl1\text{cross}} = 0. \quad (167) \]

Therefore

\[ P_{\text{cross}} \left( e^{\alpha_n t}, e^{\alpha_k t} \right) = 0, \quad \text{if} \quad \alpha_n \neq -\alpha_k. \quad (168) \]

The straightforward calculations give that

\[ P_{\text{cross}} (e^{\alpha_n t}, e^{-\alpha_n t}) = 0. \quad (169) \]

So, \( P_{\text{cross}} (e^{\alpha_n t}, e^{\alpha_k t}) = 0 \) for all \( \alpha_n \) and \( \alpha_k \neq \alpha_n \). The pressure is as follows:

\[ P \left( \sum_{n=1}^{2} C_n e^{\alpha_n t} \right) = \sum_{n=1}^{2} C_n^2 P \left( e^{\alpha_n t} \right), \quad (170) \]

where

\[ P \left( e^{\alpha t} \right) = p_{\alpha} e^{2\alpha t}. \quad (171) \]

**References**


W. Taylor, *Lectures on D-branes, tachyon condensation and string field theory*, hep-th/0301094


[17] L. Rastelli, A. Sen, B. Zwiebach, *String field theory around the tachyon vacuum*, hep-th/001225,


Th.G. Erler, Level Truncation and Rolling the Tachyon in the Lightcone Basis for Open String Field Theory, hep-th/0409179.


[57] E.O. Kahya, V.K. Onemli, Quantum Stability of a $w$ less than $-1$ Phase of Cosmic Acceleration, gr-qc/0612026.


