CRITICAL FLAVOR NUMBER IN THE THREE DIMENSIONAL THIRRING MODEL

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We present results of a Monte Carlo simulation of the three dimensional Thirring model with the number of fermion flavors $N_f$ varied between 2 and 18. By identifying the lattice coupling at which the chiral condensate peaks, simulations are be performed at couplings $g^2(N_f)$ corresponding to the strong coupling limit of the continuum theory. The chiral symmetry breaking phase transition is studied as $N_f$ is increased, and the critical number of flavors estimated as $N_{fc} = 6.6(1)$. The critical exponents measured at the transition do not agree with self-consistent solutions of the Schwinger-Dyson equations; in particular there is no evidence for the transition being of infinite order. Implications for the critical flavor number in QED$_3$ are briefly discussed.

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The study of quantum field theories in which the ground state shows a sensitivity to the number of fermion flavors $N_f$ is intrinsically interesting. According to certain approximate solutions of Schwinger-Dyson equations (SDEs), in $d = 3$ spacetime dimensions both quantum electrodynamics (QED$_3$) and the Thirring model display this phenomenon. Both models have been proposed as effective theories describing different regions of the cuprate phase diagram, Thirring describing the superconducting phase, while QED$_3$ supposedly describes the non-superconducting “pseudogap” behaviour seen in the underdoped regime \cite{1,2}. The Thirring model is a theory of fermions interacting via a current contact interaction:

\[ \mathcal{L} = \bar{\psi}_i(\partial + m)\psi_i + \frac{g^2}{2N_f}(\bar{\psi}_i \gamma_\mu \psi_i)^2, \]  \tag{1}

where $\psi_i, \bar{\psi}_i$ are four-component spinors, $m$ is a parity-conserving bare mass, and the index $i$ runs over $N_f$ distinct fermion flavors. In the chiral limit $m \to 0$ the Lagrangian \cite{10} shares the same global U(1) chiral symmetry $\psi \mapsto e^{i\alpha\gamma_5}\psi, \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}$ as QED$_3$. Since the coupling $g^2$ has mass dimension $-1$, naive power-counting suggests that the model is non-renormalisable. However \cite{1,4,5}, an expansion in powers of $1/N_f$, rather than $g^2$, is exactly renormalisable and the model has a well-defined continuum limit corresponding to a UV-stable fixed point of the renormalization group (RG). The $1/N_f$ expansion may not, however, describe the true behaviour of the model at small $N_f$. Chiral symmetry breaking, signalled by a condensate $\langle \bar{\psi}\psi \rangle \neq 0$, is forbidden at all orders in $1/N_f$, and yet is predicted by a self-consistent SDE approach \cite{5,6,7}. SDEs solved in the limit $g^2 \to \infty$ \cite{2} show that chiral symmetry is spontaneously broken for $N_f < N_{fc} \approx 4.32$, close to certain predictions of $N_{fc}$ for non-trivial IR behaviour in QED$_3$ \cite{5}. Based on these results, at $N_f = N_{fc}$ the model is expected to undergo an infinite order or conformal phase transition, originally discussed by Miranski \textit{et al} in the context of quenched QED$_4$ \cite{10,11}. Using different sequences of truncation, however, other SDE approaches have found values of $N_{fc}$ ranging between 3.24 \cite{4} and $\infty$ \cite{6}.

Since for $N_f < N_{fc}$ the chiral symmetry breaking transition corresponds to a UV-stable RG fixed point at a critical $g^2$, which can be analysed using finite volume scaling techniques, study using lattice Monte Carlo simulation has proved possible with relatively modest computer resources \cite{12,13,14,15}. It has been shown that for $N_f \leq 5$ the model is in the chirally broken phase at strong enough coupling $g^2 < \infty$, providing a lower bound on $N_{fc}$, and the critical exponents found to vary with $N_f$. This should be contrasted with numerical studies of QED$_3$: in this case the phase transition in the continuum limit at $N_f = N_{fc}$ is an IR-stable fixed point (see, eg. \cite{16,17}), and SDE predictions for the separation of scales parametrised by the dimensionless quantity $\langle \bar{\psi}\psi \rangle/\alpha^2$ are so small that enormous lattices are required to establish whether the model with a particular $N_f$ lies in the broken or symmetric phase \cite{18}. A determination of $N_{fc}$ for QED$_3$ (whose value may have profound implications for the cuprate phase diagram \cite{2}) by purely numerical means currently appears very difficult.

As already noted, Thirring and QED$_3$ share the same global symmetries; moreover in the strong coupling limit the $1/N_f$ expansion predicts the existence of a massless spin-1 $\bar{\psi}\psi$ bound state in the Thirring spectrum \cite{3,7}. The RG fixed point of both models is therefore a theory of massless fermions interacting via massless vector exchange between conserved currents. It is therefore plausible that the two models have the same $N_{fc}$, and that simulation of the Thirring model is the most effective means to determine its value. Here, we present results based on numerical simulations with $N_f = 2, \ldots, 18$ in an effort to determine $N_{fc}$ with unprecedented precision, exploiting a strategy of identifying the value of the lattice coupling $g^2$ corresponding to the strong coupling limit of the continuum theory.

The lattice action we have used is based on \cite{12} and is
as follows:

\[ S = \frac{1}{2} \sum_{x\mu} \bar{\chi}_i(x) \eta_\mu(x)(1 + iA_\mu(x))\chi_i(x + \hat{\mu}) - \text{h.c.} \]

\[ + m \sum_i \bar{\chi}_i(x) \chi_i(x) + \frac{N}{4g^2} \sum_{x\mu} A_\mu(x)^2 \]

where \( \chi, \bar{\chi} \) are staggered fermion fields and the flavor index \( i \) runs over \( N \) species. We have introduced an auxiliary real-valued link field \( A_\mu \), superficially resembling a photon field, so that the lattice action can be expressed as a bilinear form. The formulation is not unique; Gaussian integration over the \( A_\mu \) in (2) results in a lattice action resembling the continuum form (1) in that all interactions remain of the form \( \bar{\chi} \chi \bar{\chi} \chi \) for arbitrary \( N \), but some of which when re-expressed in terms of continuum-like Dirac spinors are of non-covariant form. As argued in [13], in the \( 1/N_f \) expansion these unwanted contributions are probably irrelevant, but more care may be needed when discussing the UV fixed-point theory.

For \( d = 3 \) \( N \) staggered fermion species describe \( N_f = 2N \) continuum flavors. We employed the Hybrid Monte Carlo (HMC) algorithm to simulate even \( N_f \) [13] and the Hybrid Molecular Dynamics (HMD) algorithm for odd or non-integer \( N_f \) [13]. For the HMD simulations we used a small enough fictitious time step \( \Delta \tau \leq 0.0025 \) to ensure that the \( O(N^2\Delta \tau^2) \) systematic errors in the molecular dynamics steps were smaller than the statistical errors of the various observables.

For the \( A_\mu \) propagator in the lattice Thirring model (2) however, the second diagram is absent, and the transversity condition (3) violated by a term of \( O(a^{-1}) \). Transversity of \( \Pi_{\mu\nu} \) is crucial to the renormalisability of the \( 1/N_f \) expansion; fortunately, the impact of the extra divergence can be absorbed by a wavefunction renormalisation of \( A_\mu \) and a coupling constant renormalisation

\[ g_R^2 = \frac{g^2}{1 - g^2J(m)} \]

where \( J(m) \) is the value of the integral contributed by the second diagram in Fig. 1. The physics described by continuum \( 1/N_f \) perturbation theory occurs for the range of couplings \( g_R^2 \in [0, \infty) \), i.e. for \( g^2 \in [0, g_{\text{lim}}^2] \) where to leading order in \( 1/N_f \) \( g_{\text{lim}}^2 = \frac{1}{2} \) for \( m = 0 \). The strong coupling limit is therefore recovered at \( g^2 = g_{\text{lim}}^2 \). For \( g^2 > g_{\text{lim}}^2 \) the auxiliary propagator becomes negative, and the lattice model no longer describes a unitary theory.

As already discussed, chiral symmetry breaking is absent in large-\( N_f \) calculations. It may well be that the value of the second diagram in Fig. 1 is considerably altered in the chirally broken vacuum expected for \( N_f < N_{f_c} \). In this study we use lattice simulations to estimate the value of \( g_{\text{lim}}^2 \) as a function of \( N_f \). Fig. 2 shows \( \langle \bar{\psi} \psi \rangle \) data as a function of \( g^{-2} \) for various \( N_f, m \) and lattice volumes. The condensate is non-monotonic, showing a clear maximum for \( N_f = 6 \) flavors at \( g^{-2} \approx 0.3 \). The peak position at \( g^2 = g_{\text{max}}^2 \), unlike the value of the condensate itself, is independent of both lattice volume and fermion mass, indicating that its origin is at the UV scale. Since in an orthodox description of chiral symmetry breaking \( \langle \bar{\psi} \psi \rangle \) is expected to increase with the strength of the interaction, we interpret the peak as the point where unitarity violation sets in, i.e. \( g_{\text{max}}^2 \approx g_{\text{lim}}^2 \), and hence identify the peak with the location of the strong coupling limit.

The figure also shows that \( g_{\text{lim}}^2 \) depends on \( N_f \) (Cf. Fig. 3 of Ref. [13]). For \( N_f < N_{f_c} \), it is possible to fit condensate data taken at \( g^2 < g_{\text{lim}}^2 \) to an equation of state (EoS) of the form [13]

\[ m = A(g^{-2} - g_c^{-2})\langle \bar{\psi} \psi \rangle^p + B\langle \bar{\psi} \psi \rangle^\delta, \]

FIG. 1: Leading order vacuum polarisation in lattice QED.

Next we review the discussion of [13] regarding the non-conservation of the vector current in the lattice Thirring model. The leading order \( 1/N \) quantum corrections to the photon propagator in lattice QED3 are sketched in Fig. 1. The second diagram, arising from the gauge-invariant form \( \bar{\chi}_i \gamma^\mu A^\mu \chi_i \), is peculiar to the lattice regularization, but is required to ensure transversity of the vacuum polarisation tensor, ie.

\[ \sum_{\mu} [\Pi_{\mu\nu}(x) - \Pi_{\mu\nu}(x - \hat{\mu})] = 0. \]

For the \( A_\mu \) propagator in the lattice Thirring model (2) however, the second diagram is absent, and the transversity condition (3) violated by a term of \( O(a^{-1}) \). Transversity of \( \Pi_{\mu\nu} \) is crucial to the renormalisability of the \( 1/N_f \) expansion; fortunately, the impact of the extra divergence can be absorbed by a wavefunction renormalisation of \( A_\mu \) and a coupling constant renormalisation

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The figure also shows that \( g_{\text{lim}}^2 \) depends on \( N_f \) (Cf. Fig. 3 of Ref. [13]). For \( N_f < N_{f_c} \), it is possible to fit condensate data taken at \( g^2 < g_{\text{lim}}^2 \) to an equation of state (EoS) of the form [13]

\[ m = A(g^{-2} - g_c^{-2})\langle \bar{\psi} \psi \rangle^p + B\langle \bar{\psi} \psi \rangle^\delta, \]

where \( J(m) \) is the value of the integral contributed by the second diagram in Fig. 1. The physics described by continuum \( 1/N_f \) perturbation theory occurs for the range of couplings \( g_R^2 \in [0, \infty) \), i.e. for \( g^2 \in [0, g_{\text{lim}}^2] \) where to leading order in \( 1/N_f \) \( g_{\text{lim}}^2 = \frac{1}{2} \) for \( m = 0 \). The strong coupling limit is therefore recovered at \( g^2 = g_{\text{lim}}^2 \). For \( g^2 > g_{\text{lim}}^2 \) the auxiliary propagator becomes negative, and the lattice model no longer describes a unitary theory.
Figs. 3 and 4 show the predictions of the 1/N_f expansion in the regime N_f > N_{fc}. In particular, there remains a disparity between g^2_{max}(N_f) and the large-N_f prediction \( g^2_{\text{lim}} = \frac{\alpha}{2} \), which may be due to subleading corrections in 1/N_f, or to large finite-volume corrections described by the conformal field theory expected in the limit N_f \rightarrow N_{fc}.

Fig. 4 is interesting because for the first time it presents lattice data in a form suitable for direct comparison with SDE predictions. One such study of chiral symmetry breaking in the d-dimensional Thirring model with d \in (2, 4) using this approach was by Itoh \textit{et al}. They calculated the dressed fermion propagator \( S(p) = [A(p^2)\tilde{p} + B(p^2)]^{-1} \), by first exploiting a hidden local symmetry to fix a gauge in which \( A \equiv 1 \), and then approximating the auxiliary propagator and fermion-auxiliary vertex by their forms to leading order in 1/N_f. This enabled a solution for the self-energy function \( \Sigma(x) = B(x)/\Lambda \), with \( \Lambda \) a UV cutoff and \( x = (p/\Lambda)^d - 2 \), in the strong coupling limit \( g^2 \rightarrow \infty \), yielding a dynamically-generated fermion mass \( M \):

\[
\left( \frac{M}{\Lambda} \right)^{d-2} \propto \exp \left( -\frac{2\pi}{\sqrt{\frac{N_f}{\Lambda}} - 1} \right),
\]

where \( N_{fc}(d = 3) = 128/3\pi^2 \approx 4.32 \). The chiral order parameter follows via the relation \( \langle \bar{\psi}\psi \rangle \propto \Sigma'(x = 1) \), leading to the strong coupling prediction

\[
\langle \bar{\psi}\psi \rangle \propto \Lambda^{d-2} M^{\frac{2}{d}}.
\]

Spontaneous chiral symmetry breaking occurs for \( N_f < N_{fc} \) in qualitative, but not quantitative agreement with Fig. 4. Eqn. (6) predicts an infinite-order phase transition. Its nature is further elucidated using the anomalous scaling dimension of the \( \bar{\psi}\psi \) bilinear \( \gamma_{\bar{\psi}\psi} = d\ln \langle \bar{\psi}\psi \rangle / d\ln \Lambda \) and the relations for the critical exponents \( \eta \) and \( \delta \):

\[
\eta = d - 2\gamma_{\bar{\psi}\psi}; \quad \delta = \frac{d + 2 - \eta}{d - 2 + \eta}.
\]

From (8) we obtain \( \gamma_{\bar{\psi}\psi} = (d - 2)/2 \) and hence \( \eta = 2, \delta = 1 \). This scenario has been termed a “conformal phase transition” in the strong coupling limit of the continuum theory, as argued above, it is natural to ask whether the transition at \( N_f \approx 6.5 \) has any correlation with the chiral order parameter. Fig. 4 shows \( \langle \bar{\psi}\psi \rangle \propto N_f \). This

Eqn. (6) is the main result of this Letter.

In order to refine this picture, further theoretical analysis is needed to establish contact between the results shown in Fig. 3 and the predictions of the 1/N_f expansion in the regime N_f > N_{fc}. In particular, there remains a disparity between g^2_{max}(N_f) and the large-N_f prediction g^2_{lim} = \frac{\alpha}{2}, which may be due to subleading corrections in 1/N_f, or to large finite-volume corrections described by the conformal field theory expected in the limit N_f \rightarrow N_{fc}.

The same chain of arguments leads to critical exponents \( \eta = 4 - d, \delta = d - 1 \), coincident with those of the d-dimensional Gross-Neveu model in the large-N_f limit.

\[
\langle \bar{\psi}\psi \rangle \propto \Lambda^{d-2} M.
\]
The nature of the transition predicted by SDEs thus appears sensitive to the order of the limits $g^2 \to \infty$, $N_f \to N_{fc}$.

We have been motivated by these considerations to attempt an EoS fit to the data of Fig. 4 of the form

$$m = A[(N_f - N_{fc}) + C L^{-\frac{2}{\nu}}] \langle \bar{\psi} \psi \rangle^p + B \langle \bar{\psi} \psi \rangle^\delta,$$

(11)

where $L$ is the linear extent of the system and the exponent $\nu$ is given by the hyperscaling relation $\nu = (\delta + 1)/d(\delta - p)$. The term proportional to $C$ accounts for finite volume corrections. Our best fit to data with $N_f \in [4, 8]$ yields $N_{fc} = 6.89(2), \delta = 6.90(3), p = 4.23(2)$, with $\chi^2/dof=129/15$.

Besides the relatively poor quality, and significant disagreement with Eq. (10), the fit should not be taken too seriously for two further reasons, namely the combination of “exact” HMC data with HMD containing a systematic error due to $\Delta T \neq 0$, and the as yet unquantified errors due to the identification of $\delta_{\text{max}}$ with $\delta_{\text{sim}}$. Nonetheless the disparity of the value of the exponent $\delta$ with the SDE prediction is striking. Moreover, if we plot the fitted values for $\delta$ and $\nu$ together with those from EoS fits to data from $N_f < N_{fc}$ compiled from Refs. [13, 14, 15], as shown in Fig. 5, we see that the estimate for $\delta(N_{fc})$ lies within a general trend of $\delta$ increasing with $N_f$, with no evidence for non-commutativity of the limits $g^2 \to \infty$ and $N_f \to N_{fc}$ (there is some indication for a jump from $\nu \approx 0.5$ to $\nu \approx 1$ as $N_f$ increases from 6 to $N_{fc}$, which needs to be confirmed in a more refined study). In any case, the fitted values of $\delta$ lie well above those predicted in the SDE approach.

In summary, for the first time we have been able using lattice Monte Carlo simulation to study a chiral phase transition as the number of fermion flavors $N_f$ is varied continuously. We find a continuous phase transition in the strong coupling limit at a critical flavor number $N_{fc} = 6.6(1)$, in qualitative, but not quantitative agreement with the predictions of an analysis using Schwinger-Dyson equations. In particular, the critical exponents are not those either of a conformal phase transition or the $3d$ Gross-Neveu model. It is plausible that our result may inform estimates of the corresponding critical number of flavors for chiral symmetry breaking in QED$_3$, where direct lattice simulations are hampered by a large separation of scales. Note that a recent perturbative analysis of RG flow equations in the large-$N_f$ limit of QED$_3$ predicts $N_{fc} = 6$ [20]. In future it will be interesting to check whether the same universal features of the strong coupling limit emerge using the alternative lattice formulations of the Thirring model reviewed in [13].

Acknowledgments

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