Cosmological magnetic fields from nonlinear effects

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In the standard cosmological model, magnetic fields and vorticity are generated during the radiation era via second-order density perturbations. In order to clarify the complicated physics of this second-order magnetogenesis, we use a covariant approach and present the electromagnetic-dynamical equations in the fully nonlinear regime. We use the tight-coupling approximation to analyze Thomson and Coulomb scattering. At the zero-order limit of exact tight-coupling, we show that the vorticity is zero and no magnetogenesis takes place at any nonlinear order. We show that magnetogenesis also fails at all orders if either protons or electrons have the same velocity as the radiation, and momentum transfer is neglected. At first-order in the tight-coupling approximation, magnetic fields and vorticity still cannot be generated even via nonlinear effects. However, at second-order both of them are generated, and we derive a closed set of nonlinear evolution equations that governs this generation.

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I. INTRODUCTION

The origin of cosmological magnetic fields is an important problem in cosmology [1]. Many mechanisms for primordial magnetogenesis (i.e., creation before structure formation) have been proposed. In order to generate fields on large scales, inflationary mechanisms are the best candidates, but they require uncertain modifications to standard physics in order to break the conformal invariance of Maxwell fields [2].

The generation of cosmological magnetic fields via plasma interactions during the radiation era, originally suggested by Harrison [3], is based on conventional physics and does not require any new postulates. The essential ingredient in this mechanism is Thomson scattering between photons and charged particles. Because momentum transfer is more effective between photons and electrons than between photons and protons due to the mass difference, Thomson scattering induces differences in the velocity and the distribution of protons and electrons. These differences induce local electric currents and net charge density, and the electric field in turn generates a magnetic field.

We can follow this argument in a simple Newtonian formalism. The evolution of the magnetic field is described by the induction equation,

\[ \dot{\vec{B}} = -\vec{\nabla} \times \vec{E}. \] (1)

Analysis of the momentum transfer in scattering leads to the generalized Ohm’s law,

\[ \vec{E} = \eta \vec{J} + \vec{S}, \] (2)

where \( \eta \) is the plasma resistivity and \( \vec{S} \) is the contribution from Thomson scattering. Thus we obtain

\[ \dot{\vec{B}} = \eta \nabla^2 \vec{B} - \vec{\nabla} \times \vec{S}, \] (3)

which is a diffusion equation with a source term. Because the resistivity in the early universe is very small, we can neglect the first term on the right hand side. Thus the magnetic field is generated by Thomson scattering between photons and charged particles. Because momentum transfer is more effective between photons and electrons than between photons and protons due to the mass difference, Thomson scattering induces differences in the velocity and the distribution of protons and electrons. These differences induce local electric currents and net charge density, and the electric field in turn generates a magnetic field.

As we will see below, \( \vec{S} \sim n\vec{v} \), where \( n \) is the number density of charged particles and \( \vec{v} \) is the velocity difference between radiation and charged particles. Equation (3) shows that there are two sources for magnetogenesis – vorticity, \( \nabla \times \vec{v} \), and the vector product of density gradient and velocity, \( \vec{\nabla} n \times \vec{v} \). In the standard cosmology, where perturbations are generated from inflation, there are no vector modes at first-order, and therefore the vorticity vanishes at first order. The density-velocity term is a product of first-order scalar perturbations and therefore also vanishes at first order. Thus in the standard model a perturbative analysis of magnetogenesis during the radiation era must start at second order. (Exceptions can arise if there are sources of first-order vector perturbations, such as cosmic strings [4], or fine-tuned anisotropies in collisionless neutrinos [5].)

Matarrese et al. [6] analysed how vorticity and magnetic fields can be generated from second order cosmological perturbations. Subsequent work has also used second-order perturbations [7–11], but has neglected the vorticity and metric vector perturbations, and focused on the density-velocity terms, i.e., the product of first-order scalar terms. (For other work on magnetogenesis during the radiation era, see Refs. [12].) The different approaches lead to estimates which are roughly of the same order [6, 7, 10, 11]:

\[ B \sim 10^{-24} \text{ G at recombination on 1 Mpc scales}. \] (4)

This is a very weak field, but it provides a seed which is
amplified via the dynamo mechanism. It is possible that the dynamo amplification can reach the current observed value of about $10^{-6}$G on galaxy scales [13].

The simplistic Newtonian description given above allows us to identify the key physical effects, but the real situation is much more complicated. The second-order perturbative treatments are a necessary foundation for computing the power spectrum of the magnetic field. However, it is also useful to adopt a covariant approach that directly generalises the Newtonian treatment to cosmology [14, 15]. This allows us to develop a direct physical understanding of the magnetogenesis process, and also to deal with the problem in the fully nonlinear regime.

The greatest complexity arises from the dynamics of momentum transfer via scattering. We use the tight coupling approximation [16], which is based on the fact that the scattering time $\tau$ is much less than the cosmic expansion time $H^{-1}$,

$$\tau \ll 1,$$

so that photons and charged particles are closely bound. In the limit, i.e., at the zero-order of exact tight-coupling, we have $\tau = 0$ and $\nu = 0$, so that all particles share the same velocity and behave as a single fluid, and no magnetogenesis takes place. Beyond the zero-order of tight coupling, there is a nonzero netogenesis takes place. Beyond the zero-order of tight coupling approximation, the dynamo problem in the radiation era is much more complicated. The second-order perturbative treatments are a necessary foundation for computing the power spectrum of the magnetic field.

### II. COSMOLOGICAL ELECTROMAGNETO-DYNAMICS

The Faraday tensor can be split into electric and magnetic fields as measured by a congruence of fundamental observers $u^a$ (with $u_a u^a = -1$):

$$F_{ab} = 2u_{[a} E_{b]} + \varepsilon_{abc} B^c,$$  

where $E_{a} u^a = B_{a} u^a = 0$. The spatial alternating tensor is $\varepsilon_{abc} = \eta_{abcd} u^d$, where $\eta_{abcd}$ is the spacetime alternating tensor, using the convention $\eta_{0123} = \sqrt{-g}$. The tensor indices represent an arbitrary coordinate or tetrad frame; at any event one can choose local inertial coordinates such that $u^a = (1, \tilde{0})$, $E_0 = 0 = B^0$.

The induced metric in the observer’s comoving rest space is

$$h_{ab} = g_{ab} + u_a u_b,$$ 

and it defines a covariant spatial derivative $D_a$. Generalizing the Newtonian case, we define kinematical quantities of the $u^a$ congruence via its covariant derivative:

$$\nabla_b u_a = \frac{1}{3} \theta u_a + \sigma_{ab} + \varepsilon_{abc} u^c - \hat{u}_a u_b.$$  

Here $\theta = \nabla_a u^a$ is the volume expansion rate, $\sigma_{ab} = h_{a}^{\ d} \hat{h}_b \nabla_c u_d$ is the shear, $\omega_a = -\frac{1}{2} \varepsilon_{abc} \hat{u}^b u^c$ is the vorticity, and $\hat{u}^a = u^b \nabla_b u^a$ is the acceleration.

Since we are working mainly in the radiation era, it is useful to choose $u^a$ as the radiation four-velocity in the energy frame, i.e., with no energy flux:

$$u^a = u^a_\gamma,$$  

The four-velocities of charged particles, $I = p, e$, are

$$u_p^a = \gamma_I (u^a + V_I^a), \quad u_e^a V_e^a = 0, \quad \gamma_I = (1 - V_I u_I)^{-1/2},$$  

where we also choose the energy frames, so that $q_e^2 = 0$.

The Maxwell equations $\nabla_a F_{bc} = 0$ and $\nabla_a E^a = j^a$ can be split in a 1+3-covariant way (relative to $u^a$) as [14]

$$D_a B^a = 2 E_{a} \omega^a,$$  

$$D_a E^a = -2 B_{a} u^a + \mu,$$  

$$\dot{B}_a = -\frac{2}{3} \theta B_a + (\sigma_{ab} + \varepsilon_{abc} \omega^c) B^b - \text{curl} E_a - \varepsilon_{abc} \hat{u}^b E^c,$$  

$$\dot{E}_a = -\frac{2}{3} \theta E_a + (\sigma_{ab} + \varepsilon_{abc} \omega^c) E^b + \text{curl} B_a + \varepsilon_{abc} \hat{u}^b B^c - \dot{J}_a.$$  

Here $\mu = -j_a u^a$ is the charge density and $J_a = h_{ab} j^b$ is the current. For the radiation era plasma

$$j^a = j_p^a + j_e^a, \quad j_p^a = \epsilon_I n_I u_I^a,$$  

$$\mu = \epsilon (\gamma_p u_p - \gamma_e n_e V_e),$$  

$$J^a = \epsilon (\gamma_p u_p V_p - \gamma_e n_e V_e).$$
where $\epsilon_t = \pm e$, and $n_I$ are the number densities. We can write
\[ n_I = n (1 + \Delta t), \tag{18} \]
where $n$ is the density of charged particles in the tight coupling limit.

The four-current satisfies local charge conservation, $\nabla_a j^a = 0$, which implies
\[ \dot{\mu} + \theta \mu = -D_a J^a - \dot{u}_a J^a. \tag{19} \]
In order to close the Maxwell equations, we need to specify $j^a$, and this is done via the equations of motion for photons and charged particles.

In the Maxwell equations we use the covariant curl,
\[ \text{curl} S_a := \varepsilon_{abc} D^b S^c, \tag{20} \]
and the overdot is the covariant time derivative along $u^a$, projected into the rest-space:
\[ \dot{S}_a := h_a^b u^b \nabla_a S_b. \tag{21} \]
At any spacetime event one can choose inertial coordinates so that $D_a f = (0, \nabla f)$, $\dot{S}_a = (0, \partial_t \dot{S})$, curl $S_a = (0, \nabla \times \dot{S})$. Two important identities are [15]
\[ D_a \dot{f} = (D_a f) - \dot{f} \dot{u}_a \]
\[ + \left( \frac{1}{3} \theta h_{ab} + \sigma_{ab} - \varepsilon_{abc} \omega^c \right) D^b f, \tag{22} \]
\[ \text{curl} D_a f = -2 \dot{f} \omega_a. \tag{23} \]

The vorticity propagation equation is independent of the field equations, and is given covariantly by [15]
\[ \dot{\omega}_a = -\frac{2}{3} \theta \omega_a + \sigma_{ab} \omega^b - \frac{1}{2} \text{curl} \dot{u}_a. \tag{24} \]

The conservation law for the electromagnetic energy-momentum tensor $T_F^a = F^b F^{bc} - \frac{1}{4} F^{cd} F_{cd} g^{ab}$ follows from the Maxwell equations:
\[ \nabla_b T_F^{ab} = -F^{ab} J_b. \tag{25} \]
Photons and charged particles obey the balance equations
\[ \nabla_b T^a_{\gamma b} = K_a, \quad \nabla_b T^b_{\gamma b} = K_\gamma + e_I n_I F^a \dot{u}_I^b, \tag{26} \]
where the $K^a$ four-vectors are the rates of energy-momentum density transfer to the species. By Eqs. (15) and (25), the conservation of the total energy-momentum, $\nabla_b (T^{ab} + \sum_I T_I^{ab}) = 0$, implies that $K^a + K_\gamma + K_e = 0$.

The photon energy and momentum balance equations in the general nonlinear case are [15]
\[ \dot{\rho}_\gamma + \frac{4}{3} \rho_\gamma \theta = -\sigma_{ab} \pi_{\gamma a} + U_\gamma, \tag{27} \]
\[ \frac{4}{3} \rho_\gamma \dot{u}_a + \frac{4}{3} \rho_\gamma^a = -D_b \pi_{\gamma b} - \dot{u}_b \pi_{\gamma b} + M^a_\gamma, \tag{28} \]
where $\pi_{\gamma a}^{ab}$ is the anisotropic stress, with $\pi_{\gamma a}^{ab} u_a = 0 = \pi_{\gamma a}^{ab} h_{ab}$. Here $U_\gamma = -u_a K^a_\gamma$ and $M^a_\gamma = h_{a b} \pi_{\gamma b}^{ab}$ are the rates of energy and momentum density transfer to photons from Thomson scattering. From now on, we take $\pi_{\gamma a}^{ab} = 0$; the role of photon anisotropic stress in magnetogenesis has been investigated by Takahashi et al. [8, 9, 11]. For electrons and protons, it is reasonable to neglect pressure and anisotropic stresses. Then the energy conservation equations are
\[ u_I^b \nabla_b \rho_I + \theta_I \rho_I = U_I, \tag{29} \]
where $\rho_I = m_I n_I$ and $U_I = -u_a K_{\gamma a}$ is the rate of energy density transfer due to Thomson and Coulomb scattering. The momentum balance equations are
\[ \rho_I u_I^b \nabla_b \rho_I = e_I n_I F^a_{\gamma b} u_I^b + M^a_I, \tag{30} \]
where $M^a_I = h_{a b} K_{\gamma b}$ is the rate of momentum density transfer due to Thomson and Coulomb scattering.

As shown in Maartens et al. [15], the Thomson energy transfer, $U_\gamma$, is nonzero at $O(V_\gamma^2)$, and at this order there are also corrections to the standard Thomson momentum transfer term. Since the peculiar velocity $v_I$ of charged particles relative to radiation is suppressed by Thomson scattering, it is reasonable to neglect terms of $O(V_\gamma^2)$, i.e., to take $U_\gamma = 0 = U_I$ and to use the linear form of the momentum transfer terms:
\[ M^a_\gamma = -\sum_I C_\gamma I (u^b_I - \dot{u}^b_I) h_{a b}, \tag{31} \]
\[ M^a_I = -C_\gamma I (u^b_I - \dot{u}^b_I) h^a_{\gamma b} - C_{I J} (u^b_I - \dot{u}^b_I) h^a_{\gamma b}, \tag{32} \]
where $C_\gamma I$, $C_{I J}$ are the Thomson and Coulomb collision coefficients. At $O(V_I)$ we also have $\gamma_I = 1$ and $h^a_{\gamma b} = h_{a b} + 2u^a v_I^b$. The energy and momentum balance equations reduce to
\[ \dot{\rho}_\gamma + \frac{4}{3} \rho_\gamma \theta = 0, \tag{33} \]
\[ \dot{n}_I + n_I (\theta + D_a V_I^a + \dot{u}_a V_I^a) + V_I^a D_a n_I = 0, \tag{34} \]
\[ \frac{4}{3} \rho_\gamma \dot{u}_a + \frac{4}{3} D^a \rho_\gamma = \sum_I C_\gamma I V_I^a, \tag{35} \]
\[ m_I n_I (u^a + u^b \nabla_b V_I^a + V_I^b \nabla_b u^a) = e_I n_I (E^a + F^a_{\gamma b} V_I^b) - C_\gamma I V_I^a - C_{I J} (V_I^a - V_J^a). \tag{36} \]

The vorticities of charged particles are given at $O(V_I)$ by [15]
\[ \omega_I^a = \omega^a - \frac{1}{2} \text{curl} V_I^a + \frac{1}{2} \varepsilon_{abc} u^b V_I^c + \omega_b V_I^b u^a. \tag{37} \]

Taking the tight coupling limit in Eq. (34) we recover number conservation,
\[ \dot{n} + \theta n = 0, \tag{38} \]
and with Eq. (33), this leads to $u^a \nabla_a \ln(n/\rho_\gamma^{3/4}) = 0$. We define the entropy
\[ s_a := D_a \left[ \ln \left( \frac{n}{\rho_\gamma^{3/4}} \right) \right]. \tag{39} \]
Using the identity (22), we arrive at
\[ s_a + \frac{1}{3} \theta s_a + (\sigma_{ba} + \varepsilon_{bac} \omega^c) s^b = 0 . \] (40)
In what follows we assume the adiabatic condition \( s_a = 0 \), i.e.,
\[ \frac{D_a n}{n} = \frac{3}{4} \frac{D_a \rho_{\gamma}}{\rho_{\gamma}} , \] (41)
which is consistent with Eq. (40).

### III. Tight Coupling Limit

In the exact tight coupling limit, the velocities of photons, protons, and electrons are equal, \( V_{\gamma}^a = 0 \), and the momentum transfer terms vanish. Intuitively we expect that vorticity and magnetic fields cannot be generated from zero. In fact, we can prove a stronger result, that vorticity is zero at all nonlinear orders. Equation (36) reduces to \( m_{\beta} \dot{u}^a = e E^a \), which implies \( \dot{E}_a = 0 = u_a \). Then the photon momentum balance equation (35) reduces to \( D_a \rho_{\gamma} = 0 \), and since \( \dot{\rho}_{\gamma} \neq 0 \) by Eq. (33), the identity (23) implies
\[ \omega_a = 0 . \] (42)
The induction equation (13) reduces to
\[ \dot{B}_a = -\frac{2}{3} \theta B_a + \sigma_{ab} B^b , \] (43)
so there is no source term and no magnetogenesis. Equations (42) and (43) hold in the fully nonlinear regime.

Next we consider what happens if exact tight coupling is weakened by neglecting scattering terms, and neglecting the velocity difference between protons and photons, i.e., \( V_{\gamma}^a = 0 \), but allowing \( V_{\rho}^{\gamma} \neq 0 \). This is effectively the assumption made in Ref. [6], and here we reconsider the problem in the covariant formalism. The proton equation of motion (36) becomes \( m_{\rho} \dot{u}^a = e E^a \), and taking the curl gives
\[ \text{curl} \, \dot{u}_a = \frac{e}{m_{\rho}} \text{curl} \, E_a . \] (44)
The curl of the photon momentum equation (35), using the photon energy equation (33) and the identity (23), gives
\[ \text{curl} \, \dot{u}_a = \frac{2}{3} \theta \omega_a . \] (45)
Using these equations, the induction equation (13) and the vorticity propagation equation (24) become
\[ \dot{B}_a = -\frac{2}{3} \theta B_a + (\sigma_{ab} + \varepsilon_{abc} \omega^c) B^b + \frac{2 m_{\rho}}{3 e} \theta \omega_a , \] (46)
\[ \dot{\omega}_a = -\frac{1}{3} \theta \omega_a + \sigma_{ab} \omega^b . \] (47)
Thus vorticity and the magnetic field are conserved and no magnetogenesis is possible. This gives a no-go theorem:

- (i) the scattering terms are neglected and the proton-photon velocity difference is neglected,
- (ii) anisotropic stress is neglected,
- (iii) the initial magnetic field and vorticity are zero, and
- (iv) the energy-momentum conservation equations hold, then vorticity and magnetic fields cannot be generated, at any perturbative order.

The same result holds if we assume the electron velocity equals the radiation velocity but instead \( V_{\gamma}^a \neq 0 \). It is not clear how the results of [6] relate to our no-go result.

### IV. Tight Coupling Approximation

Nonlinear magnetogenesis is ruled out in the tight coupling limit of zero collision time \( \tau = 0 \). Beyond the zero-order of tight coupling, there is a nonzero \( \tau \) and a nonzero velocity difference \( v \), which is governed by the momentum balance equations (35) and (36). Schematically, these are of the form
\[ \dot{v} = v + A , \] (48)
where \( A \) represents terms other than scattering terms. Since \( \dot{v} \sim H v \), we have \( \dot{v} \ll v/\tau \). We expand in terms of the tight coupling parameter \( \tau H \):
\[ v = v(1) + v(2) + \cdots , \quad A = A(0) + A(1) + \cdots , \] (49)
and we use TCA(\( n \)) to denote \( n \)-th order in the tight coupling approximation. Then
\[ \text{TCA}(1): \quad 0 = \frac{v(1)}{\tau} + A(0) , \] (50)
\[ \text{TCA}(2): \quad \dot{v}(1) = \frac{v(2)}{\tau} + A(1) . \] (51)
The TCA is complicated by the presence of Coulomb scattering, so that strictly we need to perform TCA expansions in both Thomson and Coulomb small parameters. However, as we will argue, it is reasonable to neglect the Coulomb collision time, i.e., to assume tight coupling of protons and electrons.

The collision coefficients in Eqs. (35) and (36) are
\[ C_{\gamma e} = \frac{4}{3} \sigma T \rho_{\gamma} n_e , \] (52)
\[ C_{\gamma p} = \beta^2 n_p C_{\gamma e} , \] (53)
\[ C_{pe} = e^2 n_e n_p \eta , \] (54)
where \( \sigma T \) is the Thomson cross-section and
\[ \beta := \frac{m_e}{m_p} . \] (55)

Here \( \eta \) is the resistivity of the cosmic plasma,
\[ \eta = \frac{4 \pi e^2}{m_e} \left( \frac{m_e}{T} \right)^{3/2} \ln \Lambda \sim 10^{-13} \left( \frac{T}{eV} \right)^{-3/2} . \] (56)
where $\ln \Lambda$ is the Coulomb logarithm, $\Lambda \sim T^{3/2}e^{-3}n^{-1/2}$ and $n \sim 10^{-10}T^3$. The C’s define key timescales, together with the Hubble timescale:

\[
\tau_{\gamma e} := \frac{m_e n_e}{C_{\gamma e}} \sim 10^5 \left( \frac{T}{eV} \right)^{-4} \text{s}, \quad (57)
\]

\[
\tau_{\gamma p} := \frac{m_p n_p}{C_{\gamma p}} = \beta^{-3} \tau_{\gamma e} \sim 10^{15} \left( \frac{T}{eV} \right)^{-4} \text{s}, \quad (58)
\]

\[
\tau_{\gamma e} := \frac{m_e n_e}{C_{\gamma p e}} \sim 10^{-4} \left( \frac{T}{eV} \right)^{-3/2} \text{s}, \quad (59)
\]

\[
H^{-1} \sim 10^{12} \left( \frac{T}{eV} \right)^{-2} \text{s}. \quad (60)
\]

Thus

\[
H\tau_{\gamma p} \sim 10^3 \left( \frac{T}{eV} \right)^{-2}, \quad (61)
\]

\[
H\tau_{\gamma e} \sim 10^{-7} \left( \frac{T}{eV} \right)^{-2}, \quad (62)
\]

\[
H\tau_{\gamma e} \sim 10^{-16} \left( \frac{T}{eV} \right)^{1/2}. \quad (63)
\]

As one can see, Thomson scattering between photons and electrons and Coulomb scattering between protons and electrons are very effective on cosmological timescales, so that they are tightly coupled before recombination. Although Thomson scattering between photons and protons is less effective at low temperatures, protons also closely follow photons through their Coulomb coupling to electrons.

From Eqs. (62) and (63), we see that,

\[
\frac{\tau_{\gamma e}}{\tau_{\gamma p}} \sim 10^9 \left( \frac{T}{eV} \right)^{-5/2}. \quad (64)
\]

Therefore, at low temperatures, $T \lesssim 1$ keV, Coulomb scattering is more effective than Thomson scattering, so that protons and electrons are more tightly coupled than photons and charged particles are. This suggests that we can safely neglect Coulomb scattering, i.e., take $\tau_{\gamma p} = 0 = V_p^{a} - V_e^{a} = \Delta_p - \Delta_e$. In this approximation, we use the centre of mass velocity $V^a$ and number density deviation $\Delta$:

\[
(m_p n_p + m_e n_e)V^a = m_p n_p V_p^{a} + m_e n_e V_e^{a}, \quad (65)
\]

\[
(m_p + m_e)\Delta = m_p \Delta_p + m_e \Delta_e. \quad (66)
\]

The peculiar velocities decompose as

\[
V_i^a = V^a + v_i^a \approx V^a, \quad (67)
\]

where $v_i^a$ are the deviations of proton and electron velocity from their centre of mass velocity, with $m_p n_p v_p^{a} + m_e n_e v_e^{a} = 0$. The number density deviations decompose as

\[
\Delta_I = \Delta + \delta_I \approx \Delta, \quad (68)
\]

where $\delta_I$ are the deviations of proton and electron number density from their centre of mass density, with $m_p \delta_p + m_e \delta_e = 0$.

The approximations in Eqs. (67) and (68) are based on

\[
H\tau_{\gamma e} \sim |\Delta| \sim |V^a| \gg H\tau_{\gamma p} \sim |\delta_I| \sim |v_i^a|. \quad (69)
\]

In fact, protons and electrons are coupled not only by Coulomb scattering but also by the electric field, so that $|\delta_I|, |v_i^a|$ are further suppressed by a factor $[17],

\[
H\eta \sim 10^{-27} \left( \frac{T}{eV} \right)^{1/2}. \quad (70)
\]

Thus we can safely apply the approximation at any temperatures we consider here, $m_e \gtrsim T \gtrsim T_{rec}$, and we can solve equations perturbatively with respect to the small parameter $H\tau_{\gamma e}$.

In Eqs. (34)–(36), we can set $V_i^a = V^a$. The momentum equations become,

\[
\frac{4}{3} \rho_\gamma \dot{u}^a + \frac{1}{3} D^a \rho_\gamma = \left( \frac{3}{T} \right)^{\beta} (m_p n_p + m_e n_e) V^a, \quad (71)
\]

\[
m_I \left( \dot{u}^a + u^a \nabla b V^a + V^a \nabla b u^a \right) = e_I (E^a + F_a b V^b) - \frac{m_I}{\tau_I} V^a, \quad (72)
\]

where we have defined

\[
\tau := \tau_{\gamma e}, \quad \tau_I := \tau_{\gamma I} = (\beta^{-3} \tau, \tau). \quad (73)
\]

The charged particle conservation equation (34) becomes, at $O(V)$,

\[
\dot{\Delta} + D_a V^a + u_a V^a + V^a D_a n = 0. \quad (74)
\]

A. First order – TCA(1)

At first-order TCA, we follow Eq. (50) and keep only the first-order $V^a$ and the zero-order of other terms.

By summing the photon and charged particle equations (71) and (72), we obtain

\[
\dot{u}^a = - \frac{1}{4 \rho_{\gamma} (1 + R)} D^a \rho_{\gamma}, \quad (75)
\]

where

\[
R := \frac{3 \rho_b}{4 \rho_{\gamma}}, \quad \rho_b := (m_p + m_e)n. \quad (76)
\]

The charged particle equations (72) imply

\[
0 = (1 + \beta) \frac{e}{m_e} E^a + \frac{1 - \beta^3}{\tau} V^a_{(1)}. \quad (77)
\]

The photon and charged particle equations (71) and (72) also lead to

\[
0 = \frac{1}{4 \rho_{\gamma}} D^a \rho_{\gamma} - \frac{1 + \beta^2 (1 + R)}{1 + \beta} \frac{V^a_{(1)}}{\tau}. \quad (78)
\]
It follows that
\[ E^a = -\frac{m_e}{e} (1 - \beta^3) \frac{1}{(1 + \beta)^2} V^a. \]
Using Eqs. (71) and (79) in the induction equation (13), we find the evolution equation for the magnetic field at TCA(1):
\[ \dot{B}_a = -\frac{2}{3} \theta B_a + (\sigma_{ab} + \varepsilon_{abc} \omega^c) B^b - \left[ \frac{m_p}{e} (1 - \beta^3) \frac{1}{(1 + \beta)^2} \frac{\rho_\gamma}{2 \rho_\gamma} \right] \omega_a, \]
where we used the adiabatic condition (41). The magnetic field is sourced by the vorticity of photons – however, from Eqs. (41) and (75), we see that \( \text{curl} \, \dot{u}^a = (\rho_\gamma/2 \rho_\gamma) \omega^a \), and the vorticity propagation equation (24) becomes
\[ \dot{\omega}_a = \left[ 1 + 2 R \frac{\rho_\gamma}{(1 + \beta) \rho_\gamma} \right] \omega_a + \sigma_{ab} \omega^b. \]
This shows that there is no source term for the vorticity. Therefore, no generation of vorticity or magnetic fields is possible at first order in the tight coupling approximation. This results holds at fully nonlinear order of cosmological inhomogeneity.

Note also that at TCA(1), terms of the form \( \vec{n} \times \dot{\vec{v}} \) do not arise. These terms come in at TCA(2).

**B. Second order – TCA(2)**

Now we proceed to the second-order TCA version of the nonlinear evolution equations for the magnetic field and vorticity.

The charged particle equations (72) imply
\[ E^a + F^a_b V^b = -\frac{m_e}{e} (1 - \beta^3) \frac{1}{(1 + \beta)} V^a. \]
Substituting into the induction equation (13), we have
\[ \dot{B}_a = -\frac{2}{3} \theta B_a + (\sigma_{ab} + \varepsilon_{abc} \omega^c) B^b + \left[ \frac{m_e}{e} (1 - \beta^3) \frac{1}{(1 + \beta)} \frac{1}{\tau} \right] \text{curl} \, V^a - \frac{3}{4} \varepsilon_{abc} V^a_b D_c \rho_\gamma \]
where \( F^a_b V^b \) in Eq. (82) was dropped because it is always negligible compared to \( \dot{B}^a \).

We need to solve for \( V^a \) at TCA(2). The photon and charged particle equations (71) and (72) give
\[ u^b \nabla_b V^a + V^b \nabla_b u^a - \frac{1}{4} \dot{\Delta} \rho_\gamma \]
\[ = -\frac{\beta (1 + \beta^2) (1 + R + R \Delta)}{(1 + \beta) \tau} V^a. \]
This can be split into TCA(1) and TCA(2) equations:
\[ -\frac{1}{4} \dot{\Delta} \rho_\gamma = -\frac{\beta (1 + \beta^2) (1 + R)}{(1 + \beta) \tau} V^a, \]
\[ \frac{\beta (1 + \beta^2)}{(1 + \beta) \tau} \nabla_b V^a_b + \nabla_b u^a \]
\[ = -\frac{\beta (1 + \beta^2)}{(1 + \beta) \tau} \left[ (1 + R) V^a_1 + R \Delta_1 V^a_1 \right]. \]
where the first is equivalent to Eq. (78). We can solve these equations for \( V^a \):
\[ V^a_1 = \frac{\beta (1 + \beta)}{4(1 + R)^2 (1 + R) \tau} \rho_\gamma, \]
\[ V^a_2 = -\frac{\beta R}{4(1 + R)^2 (1 + R) \tau} \Delta_1 \rho_\gamma - \frac{\beta (1 + \beta)}{4(1 + R)^2 (1 + \beta^2) \tau} \varepsilon_{abc} \nabla_b V^a_1 + \varepsilon_{abc} \nabla_b u^a. \]

The last term on the right of Eq. (88) is determined in terms of \( D^a \rho_\gamma \) from Eq. (87), and the vorticity occurs explicitly since \( V^a_1 \nabla_b u^a + D^a \rho_\gamma [13 \delta^b_a + \sigma^a_b + \varepsilon_{abc} \omega^c] \).

Now we can compute the crucial term in Eq. (83):
\[ \text{curl} \, V^a - \frac{3}{4} \varepsilon_{abc} V^a_b D_c \rho_\gamma \]
\[ = -\frac{\beta}{2(1 + R)^2 (1 + \beta^2) \rho_\gamma} \omega_a - \frac{\beta R}{4(1 + R)^2 (1 + \beta^2) \tau} \varepsilon_{abc} D_b \Delta_1 \rho_\gamma. \]

Finally the evolution equation (83) for the magnetic field can be written, up to TCA(2), as
\[ \dot{B}^a = -\frac{2}{3} \theta B^a + \sigma^a_b B^b \]
\[ -\left[ \frac{m_p}{e} \frac{R}{4(1 + R)^2 (1 + \beta^2)} \right] \varepsilon_{abc} D_b \Delta_1 \rho_\gamma \]
\[ -\left[ \frac{m_p}{e} \frac{1}{2(1 + R)(1 + \beta^2) \rho_\gamma} \right] \omega^a. \]

The vorticity evolution equation (24) can be rewritten, using Eqs. (71), (87) and (88), as
\[ \dot{\omega}_a = \left[ \frac{(1 + 2 R) \rho_\gamma}{4(1 + R) \rho_\gamma} \right] \omega_a + \sigma^a_b \omega^b \]
\[ = \left[ \frac{R}{8(1 + R)^2} \right] \varepsilon_{abc} D_b \Delta_1 \rho_\gamma. \]

The evolution of the baryonic number density deviation is governed by Eq. (74), which becomes, up to TCA(2),
\[ \dot{\Delta} = \frac{3}{4} \left[ V^a_1 + V^a_2 \right] \frac{D_a \rho_\gamma}{\rho_\gamma} - D_a \left[ V^a_1 + V^a_2 \right] - \dot{u}_a \left[ V^a_1 + V^a_2 \right], \]
where $V^a_\nu$, are given by Eqs. (87) and (88).

Equations (90), (91) and (92), for given $D^a \rho_\gamma/\rho_\gamma$, form a complete set of equations which describe the evolution of $\Delta$, $B^a$ and $\omega^a$. We can see that both magnetic field and vorticity are generated at the second order in the tight coupling approximation.

V. SUMMARY

In this paper we have derived the evolution equations for cosmological magnetic fields and vorticity using the 1+3-covariant formalism. The covariant approach allows us to construct a set of equations describing the fully nonlinear evolution of cosmic inhomogeneities. We have performed a tight coupling expansion for Thomson and Coulomb interactions to make the key physical processes transparent. The present analysis is complementary to previous studies based on cosmological perturbation theory.

First, we have shown a no-go theorem: magnetic fields and vorticity cannot be generated in the tight coupling approximation. We have found that magnetic fields and vorticity are not generated at first order in the tight coupling approximation [TCA(1)]. The second order tight coupling approximation [TCA(2)] is necessary for generating both of them, and we have derived a closed set of nonlinear evolution equations at TCA(2).

It is worth noting that we have not invoked the Einstein equations, so that our result does not rely on any specific theory of gravity.

The anisotropic stress of photons is neglected in the present analysis. However, as reported by Ichiki et al. [9, 11], this is important for magnetogenesis on small scales ($\lesssim 1$ Mpc) and in the earlier universe. It is expected that the anisotropic stress is important also for the generation of vorticity on the same scales and in the same era. We will discuss the effect of the anisotropic stresses in the near future.

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