Physical properties of the Schur complement of local covariance matrices

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General properties of global covariance matrices representing bipartite Gaussian states can be decomposed into properties of local covariance matrices and their Schur complements. We demonstrate that given a bipartite Gaussian state $\rho_{12}$ described by a $4 \times 4$ covariance matrix $\Sigma$, the Schur complement of a local covariance submatrix $\Sigma_1$ of it can be interpreted as a new covariance matrix representing a Gaussian operator of party 1 conditioned to local parity measurements on party 2. The connection with a partial parity measurement over a bipartite quantum state and the determination of the reduced Wigner function is given and an operational process of parity measurement is developed. Generalization of this procedure to an n-partite Gaussian state is given and it is demonstrated that the $n-1$ system state conditioned to a partial parity projection is given by a covariance matrix such as its $2 \times 2$ block elements are Schur complements of special local matrices.

INTRODUCTION

Several aspects of the quantum information theory rely on the ability to prepare and manipulate the quantum state of a given physical system. Obviously this ability is related to the available physical operations on the actual state encoding. For quantum protocols whose state encoding is spanned by a two-dimensional Hilbert space (qubits) one has to be able to completely describe the actual physical operations acting directly on the physical system codifying state space. Thus any physical operation (including measurements) is described through completely positive maps. For continuous variable encoding as given by Gaussian states [27] physical operations given by positive maps can or cannot keep the Gaussian character of the system state. Those operations that map every Gaussian input state into a Gaussian output state are called Gaussian operations [1]. For unitary operations acting on a single system Gaussian state, for example, the positive map is completely described by a local $Sp(2,R)$ symplectic (canonical) transformation over the system’s covariance matrix. Conversely, as a consequence of the Stone-von Neumann theorem, to every symplectic transformation over the covariance matrix there exists a unique unitary operation acting on the corresponding state space. Those operations correspond to the transformations induced by linear (active and passive) optical devices (beam-splitters, phase-shifters, and squeezers). It is thus not surprising that continuous variable quantum information protocols have been implemented with a Gaussian state encoding and Gaussian operations [2,3].

The whole set of unitary operations does not describe the most general transformation on quantum state systems. A general evolution map of a quantum system including coupling to ancillas (or reservoir) and measurement is given by quantum operations [4]. In this formalism the map

$$\rho \rightarrow \frac{\varepsilon(\rho)}{Tr[\varepsilon(\rho)]}, \quad (1)$$

connects the initial (input) to the final (output) state. The quantum operation $\varepsilon$ is a linear, trace decreasing map that preserves positivity. A quantum operation $\varepsilon$ satisfying complete positivity is written as $\varepsilon(\rho) = \sum_j A_j \rho A_j^\dagger$, where $A_j$ is a set of system operators, which must satisfy $\sum_j A_j^\dagger A_j \leq I$. Generally speaking, “interactions” with an ancilla (or reservoir) satisfy the trace-preserving condition $\sum_j A_j^\dagger A_j = I$, while quantum measurements satisfy $\sum_j A_j^\dagger A_j < 1$. Those operations which are irreversible are sometimes addressed as quantum channels. Gaussian channels have the additional feature to keep the Gaussian nature of the system(s) quantum state.

Several authors have described Gaussian channels and operations through [1, 3, 4, 8, 9]. Regardless the specific interest in each one of those references, one common interesting feature observed is that the Schur complement [10] of square matrices representing Gaussian states covariances embodies a manifestation of a physical operation when considering partial projections and trace operations onto Gaussian states [6, 7]. However the process that yields the Schur complement of a local covariance matrix of an input bipartite Gaussian state has not been discussed yet. As we show in this paper, this is only possible when non-positive operations are allowed. The purpose of the present work is to derive this process for an arbitrary input bipartite Gaussian state and generalize it for the case of an n-partite Gaussian state. Particularly we show that the Schur complement of one of the local covariance matrices of a bipartite Gaussian state appears as the covariance matrix describing a Gaussian operator, which embodies a parity measurement over the other mode of the input state.

We begin by reviewing in Sec. 2 some properties of bipartite Gaussian states [11] and we describe a decomposition positivity criteria for a covariance matrix using the Schur complement [10] structure. In Sec. 3 we discuss an example...
of Gaussian channel, which consists of the vacuum state projection on one subsystem of a bipartite input state. In Sec. 4 we develop our central result, namely the covariance matrix that represents the Schur complement of one of the input subsystem covariance matrix. In Sec. 5 we describe the parity measurement process and the physical properties behind the mathematical quantity as given in Sec. 4. In Sec. 6 we provide a generalization of this local parity measurement for a Gaussian state with $n$-modes in terms of an output covariance matrix. Finally Sec. 7 concludes the paper.

**BIPARTITE GAUSSIAN STATES**

Any bipartite quantum state, $\rho_{12}$, is Gaussian if we can write it as

$$\rho_{12} = \int dz \ e^{z^\dagger E a} e^{-\frac{1}{2} z^\dagger V z}.$$  \hspace{1cm} (2)

Alternatively, if its symmetric characteristic function is given by $\chi(z) = Tr[D(z)\rho] = e^{-\frac{1}{2} z^\dagger V z}$, where $D(z) = e^{-z^\dagger E a}$ is a displacement operator in the parameter four-vector $z$ space, with $z^\dagger = (z_1^*, z_1, z_2^*, z_1^*)$, $a^\dagger = (a_1^\dagger, a_1, a_2^\dagger, a_1)$, being $a_1(a_1^\dagger)$ and $a_2(a_2^\dagger)$ the annihilation (creation) operators for party 1 and 2, respectively. Here,

$$E = \left( \begin{array}{cc} Z & 0 \\ 0 & Z \end{array} \right), \quad Z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$  \hspace{1cm} (3)

and $V$ is the Hermitian $4 \times 4$ covariance matrix with elements $V_{ij} = (-1)^{i+j}(\{a_1, a_2^\dagger\})/2$,

$$V = \left( \begin{array}{cc} V_1 & C \\ C^\dagger & V_2 \end{array} \right) = \left( \begin{array}{cccc} n_1 + \frac{1}{2} & m_1 & m_s & m_c \\ m_1^* & n_1 + \frac{1}{2} & m_s^* & m_c^* \\ m_s^* & m_s & n_2 + \frac{1}{2} & m_2 \\ m_c^* & m_c & m_2 & n_2 + \frac{1}{2} \end{array} \right),$$  \hspace{1cm} (4)

where $V_1$ and $V_2$ are $2 \times 2$ Hermitian matrices containing only local elements while $C$ is the correlation between the two parties. Furthermore any covariance matrix must be positive semidefinite ($V \geq 0$) and for the operator (2) to represent a physical state the inequality

$$V + \frac{1}{2} E \geq 0,$$  \hspace{1cm} (5)

must hold, which is nothing but the fundamental uncertainty principle.

Those general positivity criteria can all be decomposed into properties of the block matrices $V_1, V_2$ and $C$. A convenient decomposition procedure is given by properties of positivity of this block matrices. In this way for the covariance matrix (4), $V \geq 0$ if and only if

$$V_2 \geq 0$$  \hspace{1cm} (6)

and the Schur complement of $V_2$, given by

$$S(V_2) = V_1 - CV_2^{-1}C^\dagger \geq 0.$$  \hspace{1cm} (7)

Similar properties have to be satisfied for the generalized uncertainty principle. The emergence of the Schur complement structure and its importance can be fully appreciated through the well known determinant decomposition property. Given $V$ as described above,

$$\det V = \det V_1 \det(V_2 - C^\dagger V_1^{-1}C) = \det V_2 \det(V_1 - CV_2^{-1}C^\dagger).$$  \hspace{1cm} (8)

Thus, it is immediate that any operation that yields $V_1$ and $S(V_1)$ (or $V_2$ and $S(V_2)$ by symmetry) can be used to qualify the global covariance matrix $V$. 


GAUSSIAN CHANNELS

Let us consider an example of a Gaussian operation over a bipartite Gaussian state, which was thoroughly considered in Ref. [1, 6, 7]. For that, we perform a vacuum state projection over the mode 2 of a bipartite Gaussian state $\rho_{12}$:

$$\sigma^{(0)}_1 = Tr_2\{|0\rangle_2\langle 0|\rho_{12}\}, \quad (9)$$

the mode 1 is conditioned by this local projection measurement in such a manner that the covariance matrix of this mode is given by

$$\Gamma^{(0)}_1 = V_1 - C\left(V_2 + \frac{1}{2}I\right)^{-1}C^\dagger. \quad (10)$$

Proof: Expanding $\rho_{12}$ in terms of coherent states, i.e.

$$\rho_{12} = \int d^2 \beta_1 d^2 \beta_2 P(\beta_1, \beta_2)|\beta_1, \beta_2\rangle\langle \beta_1, \beta_2|, \quad (11)$$

where $P(\beta_1, \beta_2)$ is the Glauber P function of the system [13], replacing this form in (9), we have

$$\sigma^{(0)}_1 = Tr_2\left\{\int d^2 \beta_1 d^2 \beta_2 P(\beta_1, \beta_2)|\beta_1, \beta_2\rangle\langle \beta_1, \beta_2|\right\}$$

$$= \int d^2 \beta_1|\beta_1\rangle\langle \beta_1| Tr_2\left\{\int d^2 \beta_2 e^{-|\beta_2|^2}P(\beta_1, \beta_2)\right\}$$

$$= \int d^2 \beta_1|\beta_1\rangle\langle \beta_1| \int d^2 \beta_2 e^{-|\beta_2|^2}e^{-z^\dagger a_2}e^{-z a_2}|\beta_1, \beta_2\rangle\langle \beta_1, \beta_2| e^{z^\dagger a_2}\langle \alpha_2|$$

$$= Tr_2\left\{\int d^2 z_2 e^{-|z_2|^2}e^{z^\dagger a_2}e^{-\frac{1}{2}|z_2|^2}e^{-z^\dagger a_2} \int d^2 \beta_1 d^2 \beta_2 P(\beta_1, \beta_2)|\beta_1, \beta_2\rangle\langle \beta_1, \beta_2|\right\}$$

$$= Tr_2\left\{\int dz_2 e^{-\frac{1}{4}|z_2|^2}e^{z^\dagger a_2}e^{-\frac{1}{2}|z_2|^2}e^{-z^\dagger a_2}\rho_{12}\right\}, \quad (12)$$

Now replacing the Gaussian form for $\rho_{12}$ and evaluating the trace, we have that

$$\sigma^{(0)}_1 = \int dz_1 e^{-\frac{1}{2}|z_1|^2}z_1 \int dz_2 e^{-\frac{1}{2}|z_2|^2}V^{(0)}z_2$$

$$= \frac{1}{\sqrt{\det (V_2 + \frac{1}{2}I)}} \int dz_1 e^{-\frac{1}{2}|z_1|^2}z_1 \int dz_2 e^{-\frac{1}{2}|z_2|^2}\left[V_1 - C(V_2 + \frac{1}{2}I)^{-1}C^\dagger\right]z_1,$$  

where we notice that the resulting state has also a Gaussian form with covariance matrix given by (10).

In this case, the covariance matrix of the reduced state is the Schur complement of the matrix

$$V^{(0)} = \begin{pmatrix} V_1 & C \\ C^\dagger & V_2 + \frac{1}{2}I \end{pmatrix} \quad (14)$$

in relation to $V_1$. Note that $\frac{1}{2}I$ represents the covariance matrix of a vacuum state.

EMERGENCE OF THE SCHUR COMPLEMENT STRUCTURE

As we noted the vacuum projection yields a Schur complement of a $4 \times 4$ matrix. However this matrix is not the covariance matrix of the input bipartite Gaussian state since the vacuum state covariance matrix must be inserted.
Therefore, we need a process that is able to yield the exact Schur complement of one of the local covariance matrices. In this section we describe the mathematical operation which is able to realize this task and in the next section we describe the physical operation behind it.

**Theorem 1**: Given a bipartite Gaussian state $\rho_{12}$, the covariance matrix $\Gamma_1$ describing the Gaussian operator $\sigma_1$ of mode 1 conditioned to a parity projection over the mode 2,

$$\sigma_1 = Tr_2 \left\{ e^{i\pi a_1^2} \rho_{12} \right\},$$

(15)
is given by the Schur complement of the covariance matrix $V$ of the input bipartite state $\rho_{12}$ in relation to $V_2$:

$$\Gamma_1 = V_1 - CV_2^{-1}C^\dagger.$$

(16)

**Proof**: We can write the parity operator as an integral over the displacement operator [20, 22], in this case the equation (15) can be rewritten as

$$\sigma_1 = \frac{1}{2} Tr_2 \left\{ \int ds_2 e^{-s_1^2} Z a_2 \rho_{12} \right\}.$$

(17)

As $\rho_{12}$ represent a Gaussian state we can write it in the form (2) and replacing this Gaussian form in (17) we have

$$\sigma_1 = \frac{1}{2} Tr_2 \left\{ \int ds_2 e^{-s_1^2} Z a_2 \int dze^{z_e} e^{-\frac{i}{2} z_e^2 V z} \right\}.$$

(18)

but

$$Tr_2 \left\{ e^{(z_e - s_2)} Z a_2 \right\} = \delta^{(2)}(z_2 - s_2),$$

(19)

therefore

$$\sigma_1 = \frac{1}{2} \int ds_2 \int dz_1 \int dze^{z_1^2} \delta^{(2)}(z_2 - s_2) e^{z_1^2} Z a_1 e^{-\frac{i}{2} z_1^2 V z}.$$

(20)

where we notice that the resulting state has also a Gaussian form with covariance matrix given by (16).

**EXPECTED PARITY AND PROBABILITY DISTRIBUTION FUNCTIONS**

Parity operation properties have received considerable attention recently, in connection to experimentally accessible measures, for description of quasiprobability distribution functions for single and entangled systems and for quantum information and computation proposals [14, 15, 16, 17, 18]. Moreover the average parity of a given quantum state is related with its Wigner function [19] at the origin of the phase space and nowadays it can be experimentally determined for radiation fields through photocounting experiments [20, 21], or in microwave cavity quantum electrodynamics experiments [22, 23].

Below we give a description of the parity measurement process. Given a prior quantum state $\rho$, the post-selected state conditioned to a parity $(p = 1, -1)$ measurement is given by

$$\rho_p = \frac{N_p \rho}{Tr \{N_p \rho\}},$$

(21)
with probability \( P_p = Tr \{ N_p \rho \} \), where the operation \( N_p \), \( p = 1, -1 \) is given by

\[
N_p = \sum_{n \in p} |n\rangle\langle n| \cdot |n\rangle\langle n|.
\]

For \( p = 1 \) the sum runs over even natural numbers, while if \( p = -1 \) it runs over odd natural numbers. The probability for the occurrence of the two events write independently as

\[
P_1 = Tr \{ N_1 \rho \} = \sum_{n_{\text{even}}} \langle n|\rho|n\rangle,
\]

\[
P_{-1} = Tr \{ N_{-1} \rho \} = \sum_{n_{\text{odd}}} \langle n|\rho|n\rangle,
\]

and it can be evidenced that \( P_1 + P_{-1} = 1 \), as it should be. The average parity, \( \bar{p} \), is then simply given by

\[
\bar{p} = \sum_{p} p P_p = \sum_{n_{\text{even}}} \langle n|\rho|n\rangle - \sum_{n_{\text{odd}}} \langle n|\rho|n\rangle = Tr \left\{ e^{i\pi a^\dagger a} \rho \right\}.
\]

The average parity was recognized independently by Grossmann [25] and by Royer [26] to be proportional to the Wigner distribution function at the origin of the phase space:

\[
W(0) = 2 \bar{p},
\]

The values of this function in other points of the phase-space can be achieved by performing displacements over the input state [24, 25, 26], such that

\[
W(\alpha) = \sum_{n=0}^{\infty} (-1)^n \langle D(\alpha)|n\rangle\langle n|D^\dagger(\alpha)\rangle.
\]

This approach can be extended for bipartite systems as given by the density operator \( \rho_{12} \). The joint post-selected state \( \rho^{(p)}_{12} \) conditioned to a parity \( (p = 1, -1) \) measurement over the is given by

\[
\rho^{(p)}_{12} = \frac{N_p \rho_{12}}{Tr \{ N_p \rho_{12} \}}.
\]

with probability \( P_p = Tr_{12} \{ N_p \rho_{12} \} \), and the operation \( N_p \), now reads as

\[
N_p = I_1 \otimes \sum_{n \in p} |n\rangle\langle n|_2 \cdot |n\rangle\langle n|_2.
\]

Now the probability for the occurrence of the two events write independently as

\[
P_1 = Tr_2 \{ N_1 \rho_2 \} = \sum_{n_{\text{even}}} \langle n|\rho_2|n\rangle,
\]

\[
P_{-1} = Tr_2 \{ N_{-1} \rho_2 \} = \sum_{n_{\text{odd}}} \langle n|\rho_2|n\rangle,
\]

where \( \rho_2 = Tr_1 \{ \rho_{12} \} \). The average parity of mode 2 is related to its Wigner function as

\[
W_2(0) = 2 \bar{p} = 2 Tr_{12} \left\{ e^{i\pi a_1^\dagger a_2} \rho_{12} \right\},
\]

In view of this, the significance of the operator \( \sigma_1 \) is clearly identified as the difference between the states of mode 1 conditioned to projections on the even Fock subspace and on the odd Fock subspace of mode 2:

\[
\sigma_1 = \sum_{n_{\text{even}}} \langle n|\rho_{12}|n\rangle_2 - \sum_{n_{\text{odd}}} \langle n|\rho_{12}|n\rangle_2.
\]

Finally it is immediate to check that the Wigner function of mode 2 can be inferred by

\[
W_2(0) = 2 Tr_1 \{ \sigma_1 \}.
\]
Moreover, we note that the Wigner function of a mode in the origin, and thus we have

$$W_2(0) = \frac{1}{\sqrt{(n_2 + \frac{1}{2}) S(n_2 + \frac{1}{2})}},$$

(35)

where

$$S(n_2 + \frac{1}{2}) = n_2 + \frac{1}{2} - \frac{|m_2|^2}{n_2 + \frac{1}{2}}.$$  

(36)

Moreover, we note that the Wigner function of a mode in the origin, and thus $$S(n_2 + \frac{1}{2})$$ is directly related with the

$$I_2 = (n_2 + \frac{1}{2})^2 - |m_2|^2$$ element of the four invariant set of the $$Sp(2, R) \otimes Sp(2, R)$$ group [8]: $$I_1 = \det V_1, I_2 = \det V_2, I_3 = \det C$$ and $$I_4 = Tr[V_1ZCZV_2ZC^\dagger Z].$$

Another way to write $$\sigma_1$$ is by expanding it in a coherent states basis,

$$\rho_{12} = \int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2) \langle \beta_1, \beta_2 \rangle \langle \beta_1, \beta_2 |$$

(37)

where $$P(\beta_1, \beta_2)$$ is the Glauber $$P$$ function of the joint system $$\rho_{12}$$. Replacing (37) in (15) and using (32) we have

$$\sigma_1 = \int d^2\beta_1 |\beta_1\rangle \langle \beta_1| \int d^2\beta_2 W_c(0) P(\beta_1, \beta_2),$$

(38)

where $$W_c(0) = e^{-2|\beta_2|^2},$$ is the Wigner function of the coherent state $$|\beta_2\rangle$$ (that spans the base of mode 2) at the origin of phase space. Thus by defining

$$\mathcal{P}(\beta_1) \equiv \int d^2\beta_2 W_c(0) P(\beta_1, \beta_2)$$

(39)

we have

$$\sigma_1 = \int d^2\beta_1 |\beta_1\rangle \langle \beta_1| \mathcal{P}(\beta_1).$$

(40)

When there is only a reduction of the system without association to a measurement in one subsystem, the $$P$$ function of the reduced state can be obtained by integrating the bipartite state $$P$$ function, $$P(\beta_1, \beta_2)$$, over the traced mode variable. But in (37), the $$P$$ function is associated with a weight in the form of a Gaussian function, where it shows that the values of the $$P$$ function of $$\sigma_1$$ over the variables nearly to the origin of the phase space are more important. The influence in mode 1 by a parity measurement in mode 2 can also be noted if we compare the covariance matrix [10] with the respective matrix $$V_1$$ of the reduced operator $$\rho_1 = Tr_2(\rho_{12})$$, indicating that the parity measurement insert global properties in the local terms of mode 1.

**PARITY MEASUREMENT IN A N-MODE GAUSSIAN STATE**

Now we can generalize the result of theorem 1 for the case of a n-mode Gaussian state.

**Theorem 2:** Parity measurement in the m mode of a Gaussian state of n-modes with covariance matrix $$2n \times 2n$$ given by

$$V_{2n \times 2n} = \begin{pmatrix}
V_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\
C_{12}^\dagger & V_{22} & C_{23} & \cdots & \vdots \\
C_{13}^\dagger & C_{23}^\dagger & V_{33} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & C_{(n-1)n} \\
C_{1n}^\dagger & \cdots & C_{(n-1)n}^\dagger & V_{nn}
\end{pmatrix},$$

(41)
associate the state of \( n - 1 \) resulting modes, in such a manner that the resulting covariance matrix \( 2(n - 1) \times 2(n - 1) \) is formed by \( 2 \times 2 \) block matrices, located in line \( i \) and column \( j \), given by

\[
\Gamma_{ij} = M_{ij} - M_{im}M^{-1}_{mm}M_{im}^t,
\]

such that for \( i = j \), \( M_{ii} = V_j \) is the covariance matrix of the reduced operator for the subsystem \( i \), and for \( i \neq j \), \( M_{ij} = C_{ij} \) are the matrices that represent the correlations between the \( n \) modes of the global system, noting that in this case \( M_{ji} = M_{ij}^t \).

**Proof:** This proof is made by induction. As we have already derived how a parity measurement in one mode of a bipartite Gaussian state affect the other mode in terms of the covariance matrix \( (16) \), it is possible to derive a similar relation for states with \( 3 \) and \( 4 \) modes and we can verify that there is a standard form between the influence of a parity measurement with the resulting reduced covariance matrices, allowing to make a generalization in the case of a \( n \)-mode state. For a tripartite Gaussian state, \( \rho_{123} \), with covariance matrix given by

\[
V_{123} = \begin{pmatrix}
V_1 & C_{12} & C_{13} \\
C_{12}^t & V_2 & C_{23} \\
C_{13}^t & C_{23}^t & V_3
\end{pmatrix},
\]

the resulting bipartite Gaussian state will be conditioned to a parity measurement in the mode \( 3 \) of the global system, \( \sigma_{12} = Tr\{e^{i\pi a^\dagger a_3} \rho_{123}\} \), such that the respective covariance matrix is

\[
\Gamma_{12} = \begin{pmatrix}
V_1 - C_{13}V_3^{-1}C_{13}^t & C_{12} - C_{13}V_3^{-1}C_{13}^t \\
C_{12}^t - C_{13}V_3^{-1}C_{13}^t & V_2 - C_{23}V_3^{-1}C_{23}^t
\end{pmatrix}.
\]

Note that each block element of the matrix above are a Schur decomposition of another matrix \( 4 \times 4 \) and can be obtained by the relation \( (42) \).

With an analogous calculus for the case of a \( 4 \)-mode state, the reduced covariance matrix conditioned to a parity measurement in mode \( 4 \) is given by

\[
\Gamma_{123} = \begin{pmatrix}
V_1 - C_{14}V_4^{-1}C_{14}^t & C_{12} - C_{14}V_4^{-1}C_{24}^t & C_{13} - C_{14}V_4^{-1}C_{34}^t \\
C_{12}^t - C_{14}V_4^{-1}C_{24}^t & V_2 - C_{24}V_4^{-1}C_{24}^t & C_{23} - C_{24}V_4^{-1}C_{34}^t \\
C_{13}^t - C_{34}V_4^{-1}C_{14}^t & C_{23} - C_{34}V_4^{-1}C_{24}^t & V_3 - C_{34}V_4^{-1}C_{34}^t
\end{pmatrix}.
\]

In the same way of the case of a measurement over the states with \( 2 \) and \( 3 \) modes, we noted that the block elements of the matrix given by \( (45) \) can also be described by \( (12) \).

**CONCLUSION**

In this work, we have investigated what operation over a bipartite Gaussian state delivers the Schur complement form of an input local covariance matrix. We discovered that parity measurements in one subsystem of a global bipartite system influence the other subsystem in such a manner that the latter is also Gaussian and that there is an Gaussian operator whose covariance matrix has the form of a Schur complement of one of the local covariance matrices of the input state. This operator consists by the difference between the reduced state of one subsystem conditioned to even and odd projections on the other subsystem. As the parity measurement and the Wigner function are strictly related, it is possible to associate the Schur complement of a local covariance matrix with the process to achieve the Wigner function of one mode and, due to the invariance of the Gaussian properties by displacements, with the achievement of only one point of this function. At the origin, the Wigner function of an one mode Gaussian state is related with one element of the four invariant set of the \( Sp(2, R) \otimes Sp(2, R) \) group and so is the Schur complement of a local covariance matrix. Moreover we have generalized for a \( n \)-partite Gaussian state verifying that after a parity measurement in one mode, the \( n - 1 \) system state has a covariance matrix with \( 2 \times 2 \) block elements in a form of Schur complements of special block matrices. We believe that our findings have both conceptual and practical implications for the development of continuous variable protocols with Gaussian states \( [12] \).

We are pleased to thank G. Rigolin for valuable discussions. This work is partially supported by FAPESP and by CNPq.
[27] A Gaussian state is the one that can be described by a covariance matrix containing only second order moments