Stability of toroidal magnetic fields in rotating stellar radiation zones

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ABSTRACT

Aims. The questions of how strong magnetic fields can be stored in rotating stellar radiative zones without being subjected to pinch-type instabilities and how much radial mixing is produced if the fields are unstable are addressed.

Methods. Linear equations are derived for weak disturbances of magnetic and velocity fields which are global in horizontal dimensions but short-scaled in radius. The equations are solved to evaluate the stability of toroidal field patterns with one or two latitudinal belts under the influence of a rigid basic rotation. Hydrodynamic stability of latitudinal differential rotation is also considered.

Results. The magnetic instability is essentially three-dimensional. It does not exist in a 2D formulation with strictly horizontal disturbances on decoupled spherical shells. Only stable (magnetically modified) $r$-modes are found in this case. The instability recovers in 3D. The most rapidly growing modes for the Sun have radial scales smaller than 1 Mm. The finite thermal conductivity makes a strong destabilizing effect. The marginal field strength for the onset of the instability in the upper part of the solar radiative zone is about 600 G. The toroidal field can only slightly exceed this critical value for otherwise the radial mixing produced by the instability would be too strong to be compatible with the observed lithium abundance. Also the threshold for hydrodynamic instability of differential rotation which exists in 2D is lowered in 3D. When radial displacements are included, the value of 28\% for critical shear is reduced to 21\%.

Key words. instabilities – magnetohydrodynamics (MHD) – stars: interiors – stars: magnetic fields – Sun: magnetic field

1. Introduction

The old problem of hydromagnetic stability of stellar radiative zones attains a renewed interest in relation with Ap stars magnetism (Braithwaite & Spruit 2004), the solar tachocline (Gilman 2005) and transport processes in stably stratified stellar interiors (Barnes et al. 1999).

Which fields the stellar radiative cores can possess is rather uncertain. The resistive decay in radiation zones is so slow that primordial fields can be stored there (Cowling 1945). Whether the fields of $10^5$ G which can influence g-modes of solar oscillations (Rashba et al. 2006) or still stronger fields which can cause neutrino oscillations (Burgess et al. 2003) can indeed survive inside the Sun mainly depends on their stability.

Among several instabilities to which the fields can be subjected (Acheson 1978), the current-driven (pinch-type) instability of toroidal fields (Tayler 1973) is probably the most relevant one because it proceeds via almost horizontal displacements. The radial motions in radiative cores are strongly suppressed by buoyancy. Watson (1981) estimated the ratio of radial ($u_r$) to horizontal ($u_\theta$) velocities of slow (subsonic) motions in rotating stars as $u_r/u_\theta \sim \Omega^2/N^2$, where $\Omega$ is the basic angular velocity and $N$ the buoyancy (Brunt-Väisälä) frequency. This ratio is small in radiative cores (Fig. 1). If the radial velocities are completely neglected, the stability analysis can be done in 2D approximation with decoupled spherical shells (Watson 1981; Gilman & Fox 1997). There is, however, some radial motion still excited. How much radial mixing the Tayler (1973) instability produces is not well-known so far. As the radial mixing is relevant to the transport of light elements, a theory of the mixing compared with the observed abundances can help to restrict the amplitudes of the internal magnetic fields (Barnes et al. 1999). In the present paper the vertical mixing produced by the Tayler instability is estimated and then used to evaluate the upper limit on the magnetic field amplitude.

In this paper, the equations governing linear stability of toroidal magnetic fields in differentially rotating radiation zone are derived. They are solved for two latitudinal profiles of the toroidal field with one and with two belts in latitude but only for rigid rotation. The unstable modes are expected to have the longest possible horizontal scales (Spruit 1999). Accordingly, our equations are global in horizontal dimensions. They are, however, local in radius, i.e. the radial scales are assumed short, $kr \gg 1$ ($k$ is radial wave number). The computations confirm that the most rapidly growing modes have $kr \sim 10^3$ but they are global in latitude. The derived equations reproduce the 2D approximation as a special limit. It is shown with an exactly solvable case that for rigid rotation the Tayler instability is missing in the 2D approximation. It recovers, however, in 3D case. Finite diffusion is found important for the instability. The minimum field producing the instability is strongly reduced by allowance for finite thermal conductivity. This field amplitude is about 600 G for the upper part of the solar radiation zone. Considering the light elements transport by the Tayler instability we find that the field strength can be only slightly above the
marginal value. Otherwise, the mixing would not be compatible with the observed lithium abundance.

2. The model

2.1. Background state and basic assumptions

The stability of rotating radiation zone of a star containing magnetic field is considered. The field is assumed axisymmetric and purely toroidal. Hence, it can be written as

\[ B = e_\phi r \sin \theta \sqrt{\mu \rho} \Omega_\Lambda(r, \theta) \]  

in terms of the Alfvén angular frequency \( \Omega_\Lambda \). In this equation, \( \rho \) is the density, \( r, \theta, \phi \) are the usual spherical coordinates and \( e_\phi \) it the azimuthal unit vector. Equation (1) automatically ensures that the toroidal field component vanishes – as it must – at the rotation axis. Centrifugal and magnetic forces are assumed small compared to gravity, \( g/r \gg \Omega^2 \) and \( g/r \gg \Omega^2 \). Deviation of the fluid stratification from spherical symmetry can thus be neglected.

The stabilizing effect of a subadiabatic stratification of the radiative core is characterized by the buoyancy frequency,

\[ N^2 = \frac{g}{C_p} \frac{\partial s}{\partial r}, \]  

where \( s = C_v \ln(P/\rho^\gamma) \) is the specific entropy of ideal gas. The frequency \( N \) is very large in the radiative cores of not too rapidly rotating stars like the Sun (\( N \gg \Omega, N \gg \Omega_\Lambda \), see Fig. 1).

The larger is \( N \) the more the radial displacements are opposed by the buoyancy force. Radial velocities should therefore be small. They are often neglected in stability analysis what might be dangerous as certain instabilities may even be suppressed by the neglect (we shall see later how this indeed happens for the Tayler instability).

The consequences of the neglect of the radial velocity perturbations, \( u'_r \), can be seen from the expression for (divergence-free) velocity, \( u' \), in spherical geometry in terms of the two scalar potentials for the poloidal, \( P_\theta \), and toroidal, \( T_\phi \), flows

\[ u' = \frac{e_r}{r^2} \hat{\mathbf{L}} P_\theta - \frac{e_\theta}{r} \left( \frac{1}{\sin \theta} \frac{\partial T_\phi}{\partial \phi} + \frac{\partial^2 P_\theta}{\partial \phi^2} \right), \]

\[ \hat{\mathbf{L}} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \]  

and

\[ \mathbf{B}' = \frac{e_r}{r^2} \hat{\mathbf{L}} P_m - \frac{e_\theta}{r} \left( \frac{1}{\sin \theta} \frac{\partial T_m}{\partial \phi} + \frac{\partial^2 P_m}{\partial \phi^2} \right) + \frac{e_\phi}{r} \left( \frac{1}{\sin \theta} \frac{\partial T_m}{\partial \theta} - \frac{\partial^2 P_m}{\partial \theta^2} \right), \]  

\[ \hat{\mathbf{L}} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \]  

(cf. Chandrasekhar [1961]). Here and in the following disturbances are marked by dashes. For zero radial velocity, the whole class of disturbances corresponding to the poloidal flows vanishes. Then the remaining toroidal flow can produce only toroidal magnetic disturbances so that in the expression for the magnetic field perturbations \( B' \),

\[ B' = \frac{e_r}{r^2} \hat{\mathbf{L}} P_m - \frac{e_\theta}{r} \left( \frac{1}{\sin \theta} \frac{\partial T_m}{\partial \phi} + \frac{\partial^2 P_m}{\partial \phi^2} \right) + \frac{e_\phi}{r} \left( \frac{1}{\sin \theta} \frac{\partial T_m}{\partial \theta} - \frac{\partial^2 P_m}{\partial \theta^2} \right), \]  

the poloidal field potential becomes zero, \( P_m = 0 \). Reducing the class of disturbances may switch-off some instabilities.

It can be seen from Eq. (3) that the horizontal part of the poloidal flow can remain unchanged when the radial velocity is reduced and the radial scale of the flow is reduced proportionally. The disturbances can thus avoid the stabilizing effect of the stratification by decreasing their radial scale.

Our stability analysis is local in the radial dimension, i.e. we use Fourier modes exp(\( ikr \)) with \( kr \gg 1 \). It will be confirmed that the most unstable modes do indeed prefer short radial scales. The analysis remains, however, global in horizontal dimensions.

Instabilities of toroidal fields or differential rotation proceed via not compressive disturbances. Characteristic growth rates of the instabilities are small compared to p-modes frequencies. In the short-wave approximation the velocity field can be assumed divergence-free, \( \nabla \cdot \mathbf{u}' = 0 \). Note that even a slow motion in a stratified fluid can be divergent if its spatial scale in radial direction is not small compared to scale height. In the short-wave approximation (in radius) the divergency can, however, be neglected.

Our next assumption concerns the pressure. More precisely, local thermal disturbances occur at constant pressure so that \( \rho'/\rho = -T'/T \) or \( s' = -C_p \rho'/\rho \). This assumption is again justified by the incompressible nature of the perturbations. Another interpretation of this assumption is given by Acheson [1978]. Acheson assumed zero disturbances of total (including magnetic) pressure to involve magnetic buoyancy instability. In our derivations, assumptions on constant total or only gas pressure are identical because the effect of magnetic buoyancy appears in higher order in \( (kr)^{-1} \) compared to the terms kept in the equations of the next section.

2.2. Equations

We start from the linearized equations for small velocity perturbations, i.e.

\[ \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{u} + \frac{1}{\mu_0 \rho} \nabla (\mathbf{B} \cdot \mathbf{B}') = \frac{1}{\rho} \nabla P + \nu \Delta \mathbf{u}' \]  

(5)

magnetic field,

\[ \frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}') + \mathbf{u}' \times \mathbf{B} - \eta \nabla \times \mathbf{B}', \]  

(6)
The basic flow is a rotation with nonuniform angular velocity, \( \Omega = \Omega(\rho, \theta) \), and the mean magnetic field is the toroidal one of the form (1). Perturbations of velocity and magnetic field are expressed in terms of scalar potentials after (3) and (4). The identities

\[
\begin{align*}
    r (r \cdot \nabla \times B') &= \hat{L} T_m, \\
    r (r \cdot B') &= \hat{L} P_m, \\
    r (r \cdot \nabla \times u') &= \hat{L} T_v, \\
    r^2 (r \cdot \nabla \times \nabla \times u') &= \left( \hat{L} + r^2 \frac{\partial^2}{\partial r^2} \right) \hat{L} P_v
\end{align*}
\]

are used to reformulate the equations in terms of the potentials. The radial component of Eq. (6) then gives the equation for the poloidal magnetic field and the radial components of curled equations (5) and (6) give the equations for toroidal flow and toroidal magnetic field, respectively (9).

The perturbations are considered as Fourier modes in time, azimuth and radius in the form of \( \exp(i(\omega t + m \phi + kr)) \). For an instability, the eigenvalue \( \omega \) should possess a positive imaginary part. Only the highest-order terms in \( kr \) for the same variable were kept in the same equation.

When deriving the poloidal flow equation, the pressure term was transformed as follows

\[
r \cdot \nabla \nabla \left( \frac{1}{\rho} \nabla P \right) = -r \cdot \nabla \left( \frac{1}{\rho^2} (\nabla \rho) \times (\nabla P) \right) = \frac{r}{C_p} \cdot \nabla \times (g \times \nabla s') = \frac{g}{r C_p} \hat{L} s'.
\]

In order to use normalized variables the time is measured in units of \( \Omega_0^{-1} \) (\( \Omega_0 \) is a characteristic angular velocity), the velocities are scaled in units of \( r \Omega_0 \), the normalized frequency (\( \hat{\omega} \)) is measured in units of \( \Omega_0 \), and the other normalized variables are

\[
A = \frac{k}{\Omega_0 r^2 \sqrt{\mu_0 \rho}}, \quad B = \frac{1}{\Omega_0 r^2 \sqrt{\mu_0 \rho}} T_m, \quad V = \frac{k}{\Omega_0 r^2} P_v, \\
W = \frac{1}{\Omega_0 r^2} T_v, \quad S = \frac{ikr}{C_p r^2 \Omega_0} \hat{s}', \quad \hat{\Omega} = \frac{\Omega}{\Omega_0}, \quad \hat{\Lambda} = \frac{\Omega_0}{\Omega_0}.
\]

Introducing the factors \( kr \) in the normalizations of poloidal potentials makes them equal by order of magnitude to the toroidal potentials, while in Eqs. (3) and (4) it was \( P_v \ll r T_v, P_m \ll r T_m \).

The equation for the poloidal flow then reads

\[
\hat{\omega} (\hat{L} V) = -\hat{\Lambda}^2 (\hat{L} S) - \frac{\epsilon_v}{\lambda^2} (\hat{L} V)
\]

\[
- 2 \hat{\mu} \hat{\Lambda} \hat{L} W - 2 \left( 1 - \mu^2 \right) \frac{\partial (\hat{\mu} \hat{\Lambda})}{\partial \mu} \frac{\partial W}{\partial \mu} - 2 m \frac{\partial \hat{\Lambda} W}{\partial \mu}
\]

\[
+ 2 \hat{\mu} \hat{\Lambda} \hat{L} B + 2 \left( 1 - \mu^2 \right) \frac{\partial (\hat{\mu} \hat{\Lambda})}{\partial \mu} \frac{\partial B}{\partial \mu} + 2 m^2 \frac{\partial \hat{\Lambda} B}{\partial \mu}
\]

\[
m \hat{\Lambda} \hat{L} A + 2 m \frac{\partial (\hat{\mu} \hat{\Lambda})}{\partial \mu} \frac{\partial A}{\partial \mu} - 2 m \left( 1 - \mu^2 \right) \frac{\partial \hat{\Lambda} A}{\partial \mu}
\]

\[
+ m \hat{\Lambda} \hat{L} A + 2 m \frac{\partial (\hat{\mu} \hat{\Lambda})}{\partial \mu} \frac{\partial A}{\partial \mu} + 2 m \left( 1 - \mu^2 \right) \frac{\partial \hat{\Lambda} V}{\partial \mu},
\]

where \( \mu = \cos \theta \), and

\[
\hat{\lambda} = \frac{N}{\Omega_0 kr}
\]

(10) can be understood as special normalization for radial wavelength. The first term on the right-hand side describes the stabilizing effect of the stratification. It vanishes for small \( \hat{\lambda} \). Apart from this stabilizing buoyancy term, the wavelength is only present in diffusive terms. The second term of the RHS includes the action of finite viscosity,

\[
e_v = \frac{v N^2}{\Omega_0^2 r^2}.
\]

Similarly, we use below

\[
e_\eta = \frac{\eta N^2}{\Omega_0^2 r^2}, \quad e_\chi = \frac{\chi N^2}{\Omega_0^2 r^2}
\]

for the diffusive parameters \( \eta \) and \( \chi \). The second and the following lines of Eq. (9) describe the influences of the basic rotation and the toroidal field. Only latitudinal derivatives of \( \Omega \) and \( \Lambda \) appear. All radial derivatives are absorbed by disturbances which vary on much shorter radial scales than \( \Omega \) or \( \Lambda \). The complete system of five equations also includes the equation for toroidal field

\[
\hat{\omega} (\hat{L} W) = -\frac{\epsilon_v}{\lambda^2} (\hat{L} W) + m \hat{\Lambda} (\hat{L} W) - m \hat{\Lambda} \hat{L} (\hat{L} B)
\]

\[
- m \hat{\Lambda} \hat{L} \hat{B} + m \frac{\partial \hat{\Lambda}}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \hat{\Omega} \hat{A} + m \frac{\partial \hat{\Omega} \hat{A}}{\partial \mu} \right]
\]

\[
+ \frac{\partial \hat{\Lambda}}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \hat{\Omega} \hat{V} + m \frac{\partial \hat{\Lambda} \hat{V}}{\partial \mu} \right],
\]

the equation for toroidal magnetic field

\[
\hat{\omega} (\hat{L} B) = -\frac{\epsilon_\eta}{\lambda^2} (\hat{L} B) + m \hat{\Lambda} (\hat{L} B) - m \hat{\Lambda} \hat{L} (\hat{L} \Lambda)
\]

\[
- m \frac{\partial \hat{\Lambda}}{\partial \mu} \hat{A} + m \frac{\partial \hat{\Lambda} \hat{A}}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \hat{\Omega} \hat{A} + m \frac{\partial \hat{\Omega} \hat{A}}{\partial \mu} \right]
\]

\[
+ m \frac{\partial \hat{\Lambda} \hat{V}}{\partial \mu} \hat{V} + m \frac{\partial \hat{\Lambda} \hat{V}}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \hat{\Omega} \hat{V} + m \frac{\partial \hat{\Lambda} \hat{V}}{\partial \mu} \right],
\]

(14) the poloidal field equation

\[
\hat{\omega} (\hat{L} A) = -\frac{\epsilon_\chi}{\lambda^2} (\hat{L} A) + m \hat{\Lambda} (\hat{L} A) - m \hat{\Lambda} \hat{L} (\hat{L} V)
\]

(15) and the entropy equation,

\[
\hat{\omega} S = -\frac{\epsilon_\chi}{\lambda^2} S + m \hat{\Lambda} S + \hat{L} V.
\]

(16)

In the simplest case of \( \hat{\Omega} = \hat{\Lambda} = 0 \) and vanishing diffusivities the only nontrivial solution of the above equations are short (in radius) gravity waves with

\[
\omega = \frac{N}{kr} \sqrt{l(l+1)}, \quad l = 1, 2, ...
\]

(17)

The instabilities we shall find are thus due to either magnetic field or nonuniform rotation. It should be kept in mind that the above equations are only valid for \( kr \gg 1 \). We shall see that the maximum growth rates do indeed belong to the short radial scales.
2.3. 2D approximation ($\lambda \gg 1$)

Generally, the ratio of $N/\Omega_0$ in radiative zones is so large (Fig. 1) that $\lambda$ can also be large in spite of $kr \gg 1$. Equation (9) in the leading order in $\lambda$ then gives $S = 0$. Then Eqs. (15) and (16) successively yield $V = 0$ and $A = 0$. Diffusive terms can also be neglected. The equation system reduces to two coupled equations for toroidal magnetic field and toroidal flow (Gilman & Fox [1997]).

\[
\dot{\omega} (\hat{L} L) = m \hat{\Omega} (\hat{L} W) - m \hat{\Omega}_\lambda (\hat{L} B) - m W \frac{\partial^2}{\partial \mu^2} \left((1 - \mu^2) \hat{\Omega}_\lambda \right) + m B \frac{\partial^2}{\partial \mu^2} \left((1 - \mu^2) \hat{\Omega}_\lambda \right)
\]

\[
\dot{\omega} (\hat{L} B) = m \hat{\Omega} (\hat{L} B) - m \hat{L} (\hat{\Omega} B, W).
\]

(18)

Here the wave number $k$ drops out and the equations describe 2D disturbances within decoupled spherical shells.

The particular case where $\Omega$ and $\Omega_\lambda$ are both constant simplifies equations (15) strongly. The equations have constant coefficients in this case and can be solved analytically. The solution in terms of Legendre polynomials $W, B \sim P_m^l(\mu)$ leads to the eigenfrequencies

\[
\frac{\dot{\omega}}{m} = \frac{1}{k(l+1)} \pm \sqrt{\frac{\hat{\Omega}_\lambda}{l(l+1)}} - \frac{1}{l^2(l+1)^2}
\]

(19)

of magnetically modified $r$-modes (Longuet-Higgins [1964]; Papaloizou & Pringle [1978]) describing stable horizontal patterns drifting in longitude. We shall see in Section 3.1 that the case of constant $\Omega$ and $\Omega_\lambda$ shows Tayler instability with 3D approach. 2D approximation, however, misses the instability. Though the result is obtained for particular case of constant $\Omega_\lambda$, it is most probably valid in general. 2D incompressive distortions on spherical surfaces do not change area encircled by toroidal magnetic field lines. The circular lines of background field have minimum length for given encircled area. Any distortion increases the length of closed field lines thus increasing magnetic energy. There is no possibility to feed an instability by magnetic energy release. Only with differential rotation 2D instabilities are possible (Gilman & Fox [1997]).

3. Results and discussion

We proceed by discussing numerical solutions of the perturbation equations for special profiles of $\Omega_\lambda$. In the present paper the interaction of toroidal magnetic fields and the basic rotation is formulated only for rigid rotation. The work with differential rotation is much more complicated, but it is in progress. In the Appendix as a first announcement the hydrodynamic stability of latitudinal differential rotation in 3D is discussed.

For the toroidal field two simple geometries are considered. First, the quantity $\Omega_\lambda$ is taken constant so that the toroidal field has only one belt symmetric with respect to the equator. Second, two magnetic belts are considered with equatorial antisymmetry, i.e. with a node of $B_\phi$ at the equator.

3.1. Fields with equatorial symmetry

Constant $\Omega$ and $\Omega_\lambda$ give the simplest realization of the Tayler instability. The instability criterion for nonaxisymmetric disturbances (Goossens et al. [1981]) reads

\[
\frac{\partial}{\partial \mu} \ln \left( B_\phi^2 \right) < \frac{2 \mu^2 - m^2}{\mu(1 - \mu)}, \quad \text{for } \mu \geq 0.
\]

It is satisfied with constant $\Omega_\lambda$ for $m = 1$ and close to the poles.

From Section 2.3 we know that the instability can appear only in 3D formulation when radial displacements are allowed. For this case the equations (9), (13) - (16) have been solved numerically.

Consider first ideal fluids with zero diffusivities, $\chi = \eta = \nu = 0$. Figure 2 gives the resulting stability map. Standard notations, $S_m$ and $A_m$, are used for the symmetry types of the global modes with azimuthal wave number $m$ symmetric and antisymmetric to the equator. However, the disturbances of different fields of the same mode possess different symmetries. Rather arbitrarily, our symmetry notations correspond to the symmetry of the potential $W$ of toroidal field. We did not find instabilities for $m \neq 1$.

The smallest field strength producing instability corresponds to shortest radial scales, $l \to 0$. The radial velocity is also zero in this limit. Note that putting $u' \to 0$ regardless of which value the vertical wavelength, $\lambda$, has returns to the 2D case without any instability. The point here is that for $u' \to 0$ the entire poloidal flow vanishes. The horizontal part of the flow is proportional to $u' r / \lambda$. A horizontal poloidal flow is necessary for the pinch-type instability and the flow remains finite when radial velocity and radial scale both approach zero keeping their ratio constant. The largest growth rates correspond to this limit. In this sense, the instability indeed proceeds via "almost horizontal" displacements. The tendency for the instability to prefer infinitely short radial scales shows that finite diffusivities should be included.

Figure 2 does not show any instability for weak fields with $\Omega_\lambda < \Omega$. With other words, the basic rotation makes a strong stabilizing effect. This may be a special property of our model where the ratio $\Omega_\lambda / \Omega$ is uniform (Pitts & Tayler [1985]). We indeed find that the threshold field for $\Omega_\lambda \sim \cos \theta$ is about ten times smaller compared to constant $\Omega_\lambda$. A much stronger destabilization, however, is produced by finite diffusion.

From now on the values

\[
e_\nu = 2 \cdot 10^{-10}, \quad e_\eta = 4 \cdot 10^{-8}, \quad e_\chi = 10^{-4}.
\]

Fig. 2. The stability map for constant $\Omega_\lambda$ and zero diffusivities. There is no instability for weak fields with $\Omega_\lambda < \Omega$. The threshold field strength for the instability increases with increasing vertical wavelength $\lambda$. 
which are characteristic for the upper part of the solar radiative core, are used for the diffusion parameters (11). The relations \( \chi \gg \eta \gg \nu \) of Eq. (21) are quite typical of stellar radiation zones.

For ideal fluids the Tayler instability operates with extremely small radial scales. When there is no stabilizing stratification, however, finite vertical scales are preferred (Arlt et al. 2007). The thermal conductivity decreases the stabilizing effect of the stratification and reduces strongly the critical field strengths for the instability. The characteristic minima in Fig. 3 correspond to small, \( \hat{\lambda} \lesssim 1 \), but finite vertical scales.

The growth rates for weak fields (\( \Omega_\lambda < \Omega \)) where the instability exists due to finite diffusion, are very small. For strong fields (\( \Omega_\lambda > \Omega \)) the basic rotation is not important and \( \Omega_\lambda \) scales the growth rate. The dependence of Fig. 4 does indeed approach the \( \sigma \sim \Omega_\lambda \) relation (\( \sigma \) is the growth rate) in the strong-field limit. The growth rate drops by almost four orders of magnitude when \( \hat{\Omega}_\lambda \) is reduced below 1. In the weak-field regime it is \( \sigma \sim \hat{\Omega}_\lambda / \Omega \) (Spruit 1999).

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there is a range of unstable wavelengths. The maximum growth rates appears, however, at wavelengths $\lesssim 1$ Mm (Fig. 7).

The flow field of any instability mixes chemical species also in radial direction. Such an instability can thus be relevant to the radial transport of the light elements (Barnes et al. 1999). The magnetic field amplitude. All for $\lambda$ modes.

The flow field of any instability mixes chemical species also in radial direction. Such an instability can thus be relevant to the radial transport of the light elements (Barnes et al. 1999). The effective diffusivity, $D_T \approx u' \ell$ ($u'$ and $\ell$ are rms velocity and correlation length in radial direction) can roughly be estimated from our linear computations assuming that $\sigma \approx \ell/u'$ and $\ell \approx \lambda/2$. With Eq. (23) this yields

$$D_T \approx 7 \cdot 10^3 \sigma \, \text{cm}^2\text{s}^{-1},$$

where $\sigma$ is the normalized growth rate given in Fig. 8. For the range of $0.01 < b < 0.2$ where the plot is closely approximated by the parabolic law $\sigma \approx 0.1 b^2$, Eq. (25) can be rewritten in terms of $B_0$ (Eq. (23)) as

$$D_T \approx 7 \cdot 10^4 \left( \frac{B_0}{1 \, \text{K}} \right)^2 \, \text{cm}^2\text{s}^{-1}.$$

As known, diffusivities exceeding $10^3 \text{cm}^2\text{s}^{-1}$ in the upper radiative core are not compatible with the observed solar lithium abundance. Hence, the toroidal field amplitude can only slightly exceed the marginal value of about 600 G. In our (simplified) formulation, the observed solar lithium abundance seem to exclude any concept of hydromagnetic dynamos driven by Taylor instability in the upper radiation zone of the Sun. We should not forget, however, that the superrotation ($\partial \Omega/\partial r > 0$) at the bottom of the convection zone in the equatorial region acts $\text{stabilizing}$ so that the critical field amplitudes for Tayler instability may be higher than the computed 600 G.

4. Summary

Linear stability of toroidal magnetic field in rotating stellar radiation zones is analyzed assuming that the vertical scale of the fluctuations is short compared to the local radius (‘short-wave approximation’). The analysis is global in horizontal dimensions. Stability computations confirm that the most rapidly growing perturbations have short radial scales: $kr \sim 10^3$.

We have shown that pinch-type instability of toroidal field require nonvanishing radial displacements. The instability does not appear in 2D approximation with zero radial velocities. The maximum amplitude of stable toroidal magnetic fields for the Sun which we have found is about 600 G. This value results only for rigid rotation. It will most probably increase if the stabilizing influence of the positive radial gradient of $\Omega$ in the equatorial region of the tachocline is included into the model.

The field strength in the upper part of the solar radiative interior can only marginally exceed the resulting critical values. Otherwise the instability would produce too strong radial mixing of light elements. After our results all the axisymmetric hydromagnetic models of the solar tachocline (Rüdiger & Kitchatinov 1997; Garaud 2002; Sule et al. 2005; Kitchatinov & Rüdiger 2006) have stable toroidal fields. On the contrary, the strong fields $\sim 10^5$ G which are able to modify $g$-modes or even stronger fields which may influence solar neutrinos are strongly unstable with e-folding times shorter than one rotation period.

3D computations of joint instabilities of toroidal fields and differential rotation (Gilman & Fox 1997; Cally 2003; Rüdiger et al. 2007) can be a perspective for further work. Another tempting extension is the inclusion of the poloidal field. The field can be important in view of the very short vertical scales of the unstable modes.

In the Appendix we present a calculation with the same equations for the hydrodynamic instability of latitudinal differential rotation. This instability can already be found in 2D approx-
imations (Watson 1981) but it is substantially modified in the 3D theory.

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References


Appendix A: The Watson problem in 3D

Latitudinal differential rotation can be unstable even without magnetic field if the shear $\partial \Omega / \partial \theta$ is sufficiently large (Watson [1981] Dziewonski & Kosovichev [1987]). The instability may reduce the differential rotation to its critical value (Garaud [2001]) which can be relevant to the theory of the solar tachocline. The critical relative value of 28% for differential rotation found by Watson resulted from a 2D theory (cf. Section 2.3). The value has also appeared in a 3D numerical probe of marginal stability of a shell rotating fast enough with the rotation law

$$\Omega = \Omega_0 \left(1 - a \cos^2 \theta \right), \quad (A.1)$$

but for not stratified material (Arlt, Sule & Rüdiger 2007). The critical shear increases to much higher values, however, if the real rotation law (including its radial variation) of the solar tachocline is adopted.

Equations (9), (13), and (16) of the Section 2.2 (in their hydromagnetic version) can also be applied to extend the Watson approach by allowance for radial displacements. Note that the Reynolds number $Re = \Omega r^2 / \nu$ can be written as

$$Re = \frac{N^2 / \Omega^2}{\nu}, \quad (A.2)$$

which with Eq. [21] gives a very large value, $O(10^{15})$.

For positive and sufficiently large $a$, the modes A1 and S2 become unstable. Figure [A.1] shows the dependence of the critical values of $a$ on the normalized wavelength $\lambda$ [10]. For large enough radial wavelengths the 28%-value of the 2D theory is reproduced. It is reduced, however, to $a = 0.21$ in 3D calculations.

We see that short rather than long radial scales are preferred. The minimum $a$ appears for $\lambda = 0.6$, so that the characteristic wavelength of $\lambda \approx 6 \text{ Mm}$ results after Eq. (24) for the solar tachocline.