On Entropy Function for Supersymmetric Black Rings

Rong-Gen Cai

Institute of Theoretical Physics
Chinese Academy of Sciences
P.O.Box 2735, Beijing 100080, China
cairg@itp.ac.cn

Da-Wei Pang

Institute of Theoretical Physics
Chinese Academy of Sciences
P.O.Box 2735, Beijing 100080, China
and
Graduate University of the Chinese Academy of Sciences
YuQuan Road 19A, Beijing 100049, China
pangdw@itp.ac.cn

Abstract: The entropy of five-dimensional black rings, which are solutions of $U(1)^3$ minimal supergravity, is calculated via the entropy function formalism. We find that after dimensional reduction down to four-dimensional spacetime, where the near horizon geometry becomes $AdS_2 \times S^2$, the entropy function works very well and gives the correct entropy. We also compute the higher order corrections due to the Gauss-Bonnet term.

Keywords: Black Holes in String Theory; Black Holes.
1. Introduction

The attractor mechanism has played an important role in understanding black hole physics in string theory and has been studied extensively in the past decade. It was initiated in the context of $N = 2$ extremal black holes [1] and generalized to more general cases, such as supersymmetric black holes with higher order corrections [2] and non-supersymmetric attractors [3], [4], [5].

Recently, based on Wald’s entropy formula [6], A.Sen proposed an effective method for calculating the entropy of D-dimensional black holes with near horizon geometry $AdS_2 \times S^{D-2}$, which is named as “entropy function” method [7]. It states that the entropy of such kind of black holes can be obtained by extremizing the “entropy function” with respect to various moduli, where the entropy function is defined as integrating the Lagrangian over the horizon coordinates and taking the Legendre transformation with respect to the electric charges. This method has been applied to many specific examples, such as extremal black holes in higher dimensions, rotating black holes and non-supersymmetric black holes. For recent developments, see [8]. It is also an useful way to calculate the higher order corrections to the entropy. In particular, more recently we have shown that for some nonextremal black holes in string theory, the entropy function method also works quite well [9].

It is well known that for black holes in four-dimensional asymptotically flat spacetime, there exists only one horizon topology $S^2$. But for black holes in five-dimensional spacetime, the horizon topology is not unique. A black hole solution with horizon topology $S^1 \times S^2$, named as black ring, was presented firstly in [10]. Several important developments are listed in [11], [12], [13], [14], [15], where various
solutions, the microscopic entropy and relations to other topics are discussed. For reviews, see [10].

The near horizon geometry of black rings turns out to be $AdS_3 \times S^2$. It becomes $AdS_2 \times S^2$ after dimensional reduction so that one expects the entropy function formalism also works. Such attempts have been discussed in [17] and an interesting paper appeared very recently [18], where the entropy function for five-dimensional extremal black holes and black rings is constructed by making use of the $4D - 5D$ lift.

Inspired by [18], we construct the entropy function for black rings in $U(1)^3$ supergravity. Since the five-dimensional Lagrangian is not gauge invariant, we have to make a dimensional reduction down to four-dimensional spacetime to ensure that the reduced action is gauge invariant. We find that the entropy function can reproduce both the Bekenstein-Hawking entropy and the near horizon geometry, while the correct attractor values of the moduli fields can also be obtained by extremizing the entropy function.

The rest of the paper is organized as follows. In section 2 we review the supersymmetric black ring solutions in the $U(1)^N$ theory and specialize to the case of $N = 3$. After dimensional reduction to four-dimensional spacetime, the entropy function for $U(1)^3$ black rings is carried out in section 3. The higher order corrections to the entropy are discussed in section 4. We summarize the results and discuss some related topics in section 5.

2. Supersymmetric Black Rings in $U(1)^3$ Theory

In this section, we review some salient properties of supersymmetric black ring in the $U(1)^3$ theory, which are needed in the following calculations. For more details, see Sec. II and Appendix B of [12].

Consider the case of minimal supergravity coupled to $N - 1$ Abelian vector multiplets with scalars taking values in a symmetric space. The action for such a theory is

$$S = \frac{1}{16\pi G_5} \int (R \ast 1 - G_{IJ} dX^I \wedge \ast dX^J - G_{IJ} F^I \wedge \ast F^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge F^K), \quad (2.1)$$

where $I, J, K = 1, \cdots, N$ and the constants $C_{IJK}$ are symmetric in $(IJK)$. The $N - 1$ dimensional scalar manifold is conveniently parameterized by the $N$ scalars $X^I$, which obey the constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1. \quad (2.2)$$
The matrix $G_{IJ}$ is defined by

$$G_{IJ} \equiv \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K,$$  \hspace{2cm} (2.3) $$

where $X_I \equiv \frac{1}{6} C_{IJK} X^J X^K$ such that $X_I X^I = 1$.

The supersymmetric black ring in $U(1)^3$ theory can be viewed as an eleven-dimensional supertube carrying three charges and three dipoles after dimensional reduction down to $D = 5$ on $T^6$. The configuration can be summarized as follows:

$$
\begin{align*}
Q1 & \quad M2 : 1 2 - - - - - , \\
Q2 & \quad M2 : - - 3 4 - - - , \\
Q3 & \quad M2 : - - - - 5 6 - , \\
q1 & \quad m5 : - - 3 4 5 6 \psi , \\
q2 & \quad m5 : 1 2 - - 5 6 \psi , \\
q3 & \quad m5 : 1 2 3 4 - - \psi .
\end{align*}
(2.4)$$

Such a configuration can be taken as a solution of $D = 11$ supergravity with the effective action

$$S_{11} = \frac{1}{16\pi G_{11}} \int \left( R_{11} \ast_{11} 1 - \frac{1}{2} F \wedge \ast_{11} F \\
- \frac{1}{6} F \wedge F \wedge A \right).$$  \hspace{2cm} (2.5) $$

The eleven-dimensional solution describing this system takes the form

$$
\begin{align*}
ds^2_{11} &= ds^2_5 + X^1(dz_1^2 + dz_2^2) + X^2(dz_3^2 + dz_4^2) \\
&\quad + X^3(dz_5^2 + dz_6^2), \\
A &= A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 \\
&\quad + A^3 \wedge dz_5 \wedge dz_6,
\end{align*}
(2.6)$$

where $z_i$ denote the coordinates along the 123456-directions and $A$ is the three-form potential.

Note that if we reduce the eleven-dimensional action to five-dimensional space-time on $T^6$ using the ansatz 2.6, we will obtain precisely the action 2.1 with $N = 3$, $C_{IJK} = 1$ if $(IJK)$ is a permutation of $(123)$ and $C_{IJK} = 0$ otherwise, and

$$G_{IJ} = \frac{1}{2} \text{diag}[(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}].$$  \hspace{2cm} (2.7) $$

The resulting five-dimensional black ring solution is characterized by the metric $ds^2_5$, three scalars $X_i$, and three one-forms $A^i$, with field strengths $F^i = dA^i$, which are
given by

\[ ds_5^2 = -(H_1 H_2 H_3)^{-2/3}(dt + \omega)^2 + (H_1 H_2 H_3)^{1/3}d\mathbf{x}_4^2, \]

\[ A^i = H_i^{-1}(dt + \omega) - \frac{q_i}{2}[(1 + y)d\psi + (1 + x)d\phi], \]

\[ X^i = H_i^{-1}(H_1 H_2 H_3)^{1/3}, \]

(2.8)

where

\[ d\mathbf{x}_4^2 = \frac{R^2}{(x-y)^2}\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 \]

\[ \quad + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2, \]

(2.9)

\[ H_1 = 1 + \frac{Q_1 - q_2 q_3}{2R^2}(x-y) - \frac{q_2 q_3}{4R^2}(x^2 - y^2), \]

\[ H_2 = 1 + \frac{Q_2 - q_1 q_3}{2R^2}(x-y) - \frac{q_1 q_3}{4R^2}(x^2 - y^2), \]

\[ H_3 = 1 + \frac{Q_3 - q_1 q_2}{2R^2}(x-y) - \frac{q_1 q_2}{4R^2}(x^2 - y^2), \]

(2.10)

and \( \omega = \omega_\phi d\phi + \omega_\psi d\psi \) with

\[ \omega_\phi = -\frac{1}{8R^2}(1 - x^2)[q_1 Q_1 + q_2 Q_2 + q_3 Q_3 \]

\[ \quad - q_1 q_2 q_3(3 + x + y)], \]

\[ \omega_\psi = \frac{1}{2}(q_1 + q_2 + q_3)(1 + y) - \frac{1}{8R^2}(y^2 - 1) \]

\[ \quad \times [q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3(3 + x + y)]. \]

(2.11)

Note that the six-torus \( T^6 \) has constant volume because \( X^1 X^2 X^3 = 1 \).

The horizon locates at \( y = -\infty \) and in order to obtain the near horizon geometry, we have to take rather complicated coordinate transformations, which are discussed extensively in Appendix D of \cite{12} and here we will not repeat any more. The resulting near horizon metric is

\[ ds^2 = \frac{4L}{Q}d\tilde{r}d\tilde{\psi} + L^2d\tilde{\psi}^2 + \frac{Q^2}{4}\frac{d\tilde{r}^2}{\tilde{r}^2} \]

\[ \quad + \frac{Q^2}{4}(d\theta^2 + \sin^2 \theta d\phi^2), \]

(2.12)

where

\[ L \equiv \frac{1}{2Q^2} \left[ 4 \sum_{i<j} Q_i q_i Q_j q_j - \sum_i Q_i^2 q_i - 4R^2 Q^3 \sum_i q_i \right], \]

\[ Q_1 = Q_1 - q_2 q_3, \quad Q_2 = Q_2 - q_1 q_3, \quad Q_3 = Q_3 - q_1 q_2, \]

\[ Q \equiv (q_1 q_2 q_3)^{1/3}. \]

(2.13)
Finally, let $\tilde{t} = Q^2 \tau/4$ and $e^0 = Q/2L$, the near horizon metric becomes
\begin{equation}
 ds^2 = \frac{Q^2}{4} (-\tilde{r}^2 d\tau^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}) + L^2 (d\psi + e^0 \tilde{r} d\tau)^2 \\
 + \frac{Q^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.14}
\end{equation}

The Bekenstein-Hawking entropy is
\begin{equation}
 S_{BH} = \frac{A_5}{4G_5} = \frac{2\pi^2 LQ^2}{4G_5}. \tag{2.15}
\end{equation}

3. The Entropy Function Analysis in Four-dimensional Spacetime

In this section, we calculate the entropy of supersymmetric black rings in $U(1)^3$ supergravity via the entropy function formalism, following [18]. Since the five-dimensional effective action contains a Chern-Simons term, the Lagrangian is not gauge invariant and we have to reduce it to four dimensions so that the entropy function can be applied.

Take the five-dimensional configuration as follows:
\begin{align*}
 ds_5^2 &= g_{\mu\nu} dx^\mu dx^\nu + w^2 (d\psi + e^0 r dt)^2, \\
 A^{(5)} &= e^I r dt + p^I \cos \theta d\phi + a^I (d\psi + e^0 r dt). \tag{3.1}
\end{align*}

We do dimensional reduction on the coordinate $\psi$ and obtain the four-dimensional effective action
\begin{equation}
 S = \frac{1}{16\pi G_4} \int d^4 x \{ \sqrt{-\det gw} [R - f_{ij} F_i^{\mu\nu} F_j^{\mu\nu}] \\
 - \tilde{f}_{ij} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}^{i} F_{\mu\nu}^{j} \}, \\
 i,j = (0, I), \quad I = 1, 2, 3. \tag{3.2}
\end{equation}

where
\begin{equation}
 f_{00} = \frac{1}{4} w^2 + \frac{1}{8} G_{IJ} a^I a^J, \quad f_{IJ} = \frac{1}{8} G_{IJ} \\
 f_{0I} = f_{I0} = \frac{1}{8} G_{IJ} a^J \tag{3.3}
\end{equation}

and
\begin{equation}
 \tilde{f}_{00} = \frac{1}{6} C_{KLM} a^K a^L a^M, \quad \tilde{f}_{IJ} = \frac{1}{2} C_{IJK} a^K, \\
 \tilde{f}_{0I} = \tilde{f}_{I0} = \frac{1}{4} C_{IKL} a^K a^L. \tag{3.4}
\end{equation}
Note that $G_4 = G_5/ \int d\psi$, i.e., $G_4 = G_5/2\pi$ and we have omitted the terms involving the derivatives of the scalar fields $X^I$ because they are set to be constants in the following calculations. The $\psi$ components of the five-dimensional gauge fields become axions in four dimensions.

Assume the resulting four-dimensional solution takes the following field configuration

$$ds^2 = v_1(-r^2 dt^2 + \frac{dr^2}{r^2}) + v_2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$F_{rt}^0 = e^0 = Q/2L, \quad F_{rt}^I = e^I, \quad F^I = -q^I \sin \theta,$$

$$X^I = u^I, \quad I = 1, 2, 3.$$  \tag{3.5}

Defining a function $f$ as

$$f = \int d\theta d\phi \sqrt{-\det gL},$$  \tag{3.6}

the entropy function $F$ can be obtained by taking the Legendre transformation of $f$ with respect to $e^0$ and $e^I$

$$F \equiv e^0 \frac{\partial f}{\partial e^0} + e^I \frac{\partial f}{\partial e^I} - f.$$  \tag{3.7}

Substituting the above field configuration into $F$, we find that

$$F = \frac{1}{4G_4} \left\{ v_1 v_2 w \left[ \frac{2}{v_1} - \frac{2}{v_2} \right] + \frac{2(e^0)^2}{v_1^2} \left( \frac{1}{4} w^2 + \frac{1}{16} (u^1)^2 a_1^2 \right) + \frac{e^0}{4v_1} ((u^1)^{-2} a_1 e^1 + (u^2)^{-2} a_2 e^2 + (u^3)^{-2} a_3 e^3) \right.$$  

$$\left. + \frac{1}{8v_1^2} ((u^1)^{-2}(e^1)^2 + (u^2)^{-2}(e^2)^2 + (u^3)^{-2}(e^3)^2) \right.$$  

$$\left. + \frac{1}{8v_2^2} ((u^1)^{-2}(q^1)^2 + (u^2)^{-2}(q^2)^2 + (u^3)^{-2}(q^3)^2) \right\}$$  \tag{3.8}

Extremizing $F$ with respect to $a^I$,

$$\frac{\partial F}{\partial a^I} = 0, \quad \Rightarrow \quad a^I = -\frac{e^I}{e^0},$$  \tag{3.9}

so that the entropy function can be simplified significantly,

$$F = \frac{1}{4G_4} \left\{ w(2v_2 - 2v_1) + \frac{1}{2v_1} (e^0)^2 w^3 + \frac{v_1}{8v_2} w(u^1)^{-2} (q^1)^2 \right.$$  

$$\left. + \frac{v_1}{8v_2} w(u^2)^{-2} (q^2)^2 + \frac{v_1}{8v_2} w(u^3)^{-2} (q^3)^2 \right\}.$$  \tag{3.10}

To proceed further, the electric field strength $e^0$ has to be replaced by the “conjugate” electric charge $Q_0$, which is defined as

$$Q_0 \equiv \frac{\partial f}{\partial e^0} - \tilde{f}_0 q^I = \frac{v_2}{v_1} (\frac{1}{4} w^3 e^0).$$  \tag{3.11}
Then we have
\[
F = \frac{1}{4G_4} \left[ w(2v_2 - 2v_1) + 8\frac{v_1}{v_2}(Q_0)^2 w^{-3} + \frac{v_1}{8v_2} w(u^1)^{-2}(q^1)^2 \\
+ \frac{v_1}{8v_2} w(u^2)^{-2}(q^2)^2 + \frac{v_1}{8v_2} w(u^3)^{-2}(q^3)^2 \right]. \tag{3.12}
\]

Note that due to $X^1X^2X^3 = 1$, the $u^I$'s are not independent. One has to be careful with this fact when extremizing $F$ with respect to $u^I$.

We can obtain the values of the rest moduli fields by solving the following equations
\[
\frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial v_2} = \frac{\partial F}{\partial w} = 0, \quad \frac{\partial F}{\partial u^1} = \frac{\partial F}{\partial u^2} = 0. \tag{3.13}
\]
The solutions turn out to be
\[
v_1 = \frac{Q^2}{4}, \quad v_2 = \frac{Q^2}{4}, \quad w = L, \\
u^1 = \frac{q^1}{Q}, \quad u^2 = \frac{q^2}{Q}, \quad u^3 = \frac{q^3}{Q}, \tag{3.14}
\]
where we have taken $Q_0 = QL^2/8$, which can be realized by carefully defining the electric charges.

Substituting the solution (3.14) back into $F$, we obtain
\[
F = \frac{1}{4G_4} \frac{1}{2} LQ^2 = \frac{\pi LQ^2}{4G_5}, \tag{3.15}
\]
and
\[
S_{BH} = 2\pi F = \frac{2\pi^2 LQ^2}{4G_5} = S_{BH}, \tag{3.16}
\]
which reproduces the Bekenstein-Hawking entropy (2.15) precisely. Note that the attractor behavior of supersymmetric black rings has been discussed in [19], where it states that the scalar fields $X^I$ take the attractor values $X^I = q^I/q$ with $q \equiv (\frac{1}{6} C_{IJK} q^I q^J q^K)^{1/3}$. We can see that when $I = 1, 2, 3$, these are exactly equal to those values given in (3.14).

4. Higher Order Corrections

We calculate the higher order corrections to the entropy of black rings in this section. In principle, one should obtain the effective action in the following way: First reducing the eleven-dimensional action with higher order corrections on $T^6$ and then reducing the five-dimensional action on $\psi$. However, as pointed out in [20], the five-dimensional effective action contains a term proportional to the Gauss-Bonnet term. Such a correction due to the Gauss-Bonnet term has been considered in [21]. In our case, we should first make a dimensional reduction for this term to four dimensions.
However, we assume that adding the Gauss-Bonnet term will not change the geometry (3.1) very much, and the moduli field $w$ remains a constant. In this case, we can take the correction to the effective action in four dimensions as \[20, 21\]

$$
\Delta L = \frac{K w c_{2I}}{16\pi G_4} \cdot X^I (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2), \tag{4.1}
$$

where $K = 1/48$, $c_{2I}$ are the components of the second Chern class of $T^6$. One can easily obtain the corrections to the entropy function by substituting the field configuration (3.5) into (4.1)

$$
\Delta L = - \frac{K w v_1^{-1} v_2^{-1} c_{2I} \cdot q^I}{2\pi G_4} Q, \quad \Delta f = - \frac{2K w c_{2I} \cdot q^I}{G_4} Q, \quad \Delta F = \frac{2K w c_{2I} \cdot q^I}{G_4} Q, \tag{4.2}
$$

where we have used the fact that the scalar fields get "attracted" to the values $X^I = q^I/Q$. Note that the equations involving the electric and magnetic field strengths and charges remain invariant because the corrections are irrelevant to them. Following the same procedure presented above, we obtain

$$
v_1 = v_2 = \frac{Q^2}{4} + \frac{4K c_{2I} q^I}{3Q}, \quad w^4 = \frac{3Q^3 L^4}{64K c_{2I} q^I + 3Q^3}, \tag{4.3}
$$

while the values of other moduli fields remain unchanged.

The corrections to the entropy can be obtained by substituting the solution (4.3) back into $F$, which is a much complicated result. However, if we take the large $Q$ limit, $v_1$, $v_2$ and $w$ return to the original values. So it is interesting to study the large $Q$ behavior of the corrections. The corrections to the entropy function can be expressed as

$$
\delta F = \frac{2KL c_{2I} \cdot q^I}{G_4} Q, \tag{4.4}
$$

where we have approximated $w \approx L$. So the corrections to the entropy can be given as

$$
\delta S_{BH} = 2\pi \delta F = \frac{4\pi KL c_{2I} \cdot q^I}{G_4} Q. \tag{4.5}
$$

We have noticed that the corrections to the black ring entropy, including both macroscopic and microscopic corrections, have been discussed extensively in \[21\] and \[24\]. It is shown that the microscopic entropy correction at the first leading order can be given as

$$
\Delta S_{BR} = \frac{\pi}{6} c_2 \cdot q \sqrt{\frac{q_0}{Q}}, \tag{4.6}
$$

where

$$
q_0 = -J_\psi + \frac{1}{12} D^{AB} Q_A Q_B + \frac{c_L}{24}, \quad c_L = Q, \tag{4.7}
$$
and $D^{AB}$ is the inverse of $D_{AB} \equiv 1/6 C_{ABC} q^C$. When taking the large $Q$ limit, (4.6) can be approximated as

$$\Delta S_{BR} \sim c_2 \cdot q.$$  \hspace{1cm} (4.8)

Thus in the limit of large charge, our result (4.3) gives out the microscopic entropy correction (4.8) by difference with a numerical factor.

5. Summary and Discussion

In this paper, we have constructed the entropy function for supersymmetric black rings in $U(1)^3$ theory. We have found that after dimensional reduction down to four-dimensional spacetime, the effective action is gauge invariant and the near horizon geometry turns out to be $AdS_2 \times S^2$, so the entropy function formalism can be applied. We have reproduced the Bekenstein-Hawking entropy precisely and have obtained the correct attractor values of the scalar fields via the entropy function. The higher order corrections to the entropy have also been discussed and we have found that our result is equivalent to the microscopic corrections by a numerical factor in the large charge limit. Note that when we set $Q_1 = Q_2 = Q_3$, $q_1 = q_2 = q_3$, the black ring becomes the solution presented in [11], which implies that the entropy function formalism can also be applied to those cases.

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