Papapetrou Energy-Momentum Tensor for Chern-Simons Modified Gravity

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(Dated: February 7, 2007)

We construct a conserved, symmetric energy-momentum (pseudo-)tensor for Chern-Simons modified gravity, thus demonstrating that the theory is Lorentz invariant. The tensor is discussed in relation to other gravitational energy-momentum tensors and analyzed for the Schwarzschild, Reissner-Nordstrom, and FRW solutions. To our knowledge this is the first confirmation that the Reissner-Nordstrom and FRW metrics are solutions of the modified theory.

MIT-CTP 3810

I. INTRODUCTION

The possibility of modifying a four dimensional theory with a three dimensional Chern-Simons (CS) term was first investigated in [1], where such a term was added to electrodynamics. There, it was found that the extra term created a birefringence of the vacuum, leading to plane waves traveling with two polarizations whose velocities differ from c (Lorentz violation) and from each other (parity violation).

In ensuing work [2], a similar modification of General Relativity (GR) was proposed. In order to carry out such a construction for gravity, one must decide how to embed a three dimensional CS term into four dimensional GR. This is done with the aid of an embedding coordinate, $v_\mu$. In contrast to CS electrodynamics, there is no birefringence of the vacuum, though there are parity violating effects that cause gravitational wave polarizations to carry different intensities. Moreover, it was argued that the theory allows the construction of a symmetric and conserved two-index object, which could serve as an energy-momentum (pseudo-)tensor. For these reasons it was suggested that the apparent Lorentz violation of the theory is “dynamically suppressed.”

In this paper we use the Noether/Belinfante procedure to construct a symmetric, conventionally conserved energy momentum tensor for CS modified gravity. The existence of such a tensor signals the absence of Lorentz violation in the theory. The methods are similar to those used in the construction of the so-called Papapetrou energy-momentum tensor for GR in [3] and [4]. We find that while the constructed tensor initially appears not to be conserved, a subsidiary condition on solutions of the theory forces the tensor’s non-vanishing divergence to zero.

II. A BRIEF REVIEW OF CS MODIFIED GRAVITY

This section is a brief review of [2], where four dimensional CS modified gravity was examined. The Lagrangian density of the theory is

$$L = \frac{1}{16\pi G} (\sqrt{-g} R + \frac{1}{4} \theta(x)^* R R),$$

(1)

where $\theta(x)$ is a prescribed, external field that breaks diffeomorphism symmetry, $^* RR \equiv ^* R^\sigma_{\tau \mu\nu} R^\tau_{\sigma \mu\nu}$, and $^* R^\sigma_{\tau \mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R^\sigma_{\tau \rho\beta}$. One generally takes $\theta(x) = v_\tau x^\sigma$, and timelike $v_\mu = (\frac{1}{\rho}, 0, 0, 0)$, with $\frac{1}{\rho}$ constant. This ensures the persistence of some familiar GR solutions and also maintains the close analogy with 3 dimensional CS theories. We note that $^* RR = 2 \partial_\mu K^\mu$ is a total derivative, where

$$K^\mu = 2\epsilon^{\mu\alpha\beta\gamma} [\frac{1}{2} \Gamma^\sigma_{\alpha\tau} \partial_\beta \Gamma^\tau_{\gamma\sigma} + \frac{1}{3} \Gamma^\sigma_{\alpha\tau} \Gamma^\eta_{\beta\gamma} \Gamma^\alpha_{\eta\sigma}],$$

(2)

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1 The two indexed objects of this paper do not correctly transform as tensors and for this reason are referred to as pseudotensors. See the discussion in Section III for more on this issue. Henceforth, all references to gravitational energy-momentum tensors should be understood to be references to pseudotensors.
and \( \Gamma^\gamma_{\alpha\beta} \) is the Christoffel connection. Upon integrating the Lagrangian by parts, it may be rewritten
\[
\mathcal{L}' = \mathcal{L}_{EH} - \frac{1}{32\pi G} (v_\sigma K^\sigma),
\]
where \( \mathcal{L}_{EH} \) is the Einstein-Hilbert Lagrangian. Thus the translation non-invariance of (1) is confined to a surface term in the action. By varying the Lagrangian (plus matter degrees of freedom) with respect to \( g \), one finds the equations of motion
\[
G^{\mu\nu} + C^{\mu\nu} = 8\pi G T^{\mu\nu}.
\]
(4)

\( G^{\mu\nu} \) is the usual Einstein tensor, \( T^{\mu\nu} \) is the energy-momentum tensor for matter and \( C^{\mu\nu} \) is the following four dimensional analogue of the Cotton tensor,
\[
C^{\mu\nu} = -\frac{1}{\sqrt{-g}} \left[ v_\sigma (\epsilon^{\sigma\mu\alpha\beta} \nabla_\alpha R^\beta_\nu + \epsilon^{\sigma\nu\alpha\beta} \nabla_\alpha R^\beta_\mu) - v_\alpha \Gamma^\alpha_{\sigma\tau} \left( \ast R^{\tau\mu\sigma\nu} + \ast R^{\tau\nu\sigma\mu} \right) \right].
\]
(5)

Taking the divergence of this equation gives
\[
\nabla_\mu C^{\mu\nu} = \frac{1}{8\sqrt{-g}} v^{\nu} \ast RR.
\]
However, via the Bianchi identity, \( \nabla_\mu G^{\mu\nu} = 0 \) and for diffeomorphism invariant matter terms, \( \nabla_\mu T^{\mu\nu} = 0 \). Therefore we have a consistency condition for solutions to (4):
\[
\ast RR = 0.
\]
(6)

CS modified gravity theories have been studied as models for parity violation [5] and leptogenesis [6], [7] in the early universe. CS models have also been used as effective theories where the CS term is radiatively generated via fermions coupling to gravity in a parity violating way [8].

A. Solutions

The Schwarzschild, Reissner-Nordstrom, and all FRW metrics have vanishing \( C^{\mu\nu} \) in their usual coordinatizations and hence are solutions of CS modified gravity. However, the most general black hole solution, the Kerr metric, has non-vanishing \( C^{\mu\nu} \) and is not a solution of CS modified gravity. This can be seen easily by noting \( \ast RR \not= 0 \) for the Kerr metric. The discovery of an appropriate generalization of the Kerr metric is an outstanding problem. The only non-GR (\( C^{\mu\nu} \not= 0 \)) solutions yet discovered are gravitational waves [2]. Unlike their GR counterparts, parity violating effects cause the two CS modified wave polarizations to travel with different intensities.

Though CS modified gravity is not invariant under general diffeomorphisms, we may identify a smaller equivalence class of coordinate transformations. In [2] it is shown that constant shifts in time and arbitrary space reparametrizations are symmetries of the CS modified action. Thus we may view solutions related by these coordinate transformations as identical.

III. A WORD ON GRAVITATIONAL ENERGY-MOMENTUM TENSORS

The issue of ordinarily conserved energy-momentum tensors for gravity has been controversial since the birth of GR. Einstein’s own “tensor” was non-symmetric and not a tensor (almost all, including the type derived in this paper are coordinate dependent “pseudotensors”), drawing criticism from leading physicists of the day (these criticisms are nicely reviewed in [9]). The problem with a local definition of gravitational energy-momentum is that there always exists a coordinate system where the energy and momentum densities vanish at a point, viz. Riemannian normal coordinates. In GR, local energy momentum can be “gauged” away. Since Einstein’s pseudotensor, various other pseudotensors have appeared in the literature including those of Tolman [10], Landau and Lifshitz [11], Papapetrou [3], [4], Weinberg [12] and Møller [13]. None but Møller’s are coordinate invariant. Also, many involve an auxiliary
Minkowski metric $\eta = \text{diag}(-1, 1, 1, 1)$, and all but Møller’s give physically sensible results only when restricted to “quasi-Cartesian” coordinate systems. (“Quasi-Cartesian” is defined sometimes as $ds^2 \rightarrow -dt^2 + dx^2 + dy^2 + dz^2$ asymptotically or, less restrictively, that all four coordinates be non-compact. This definition is still a point of debate and is of fundamental importance when one tries to apply these pseudotensors to cosmological models.)

There are other problems. Aguirregabiria, et al. [14] have shown that the Einstein, Tolman, Landau and Lifshitz, Papapetrou and Weinberg (ETLLPW) pseudotensors are identical for any Kerr-Schild metric. Many standard solutions can be put in Kerr-Schild form, including the Schwarzschild, Reissner-Nordstrom, Kerr, and Kerr-Newman metrics. However, Virbhadra later showed [15] that ETLLPW each give different results for the energy contained in a sphere of radius $r$ when applied to the most general non-static, spherically symmetric metric in “Schwarzschild Cartesian coordinates” $(r, \theta, \phi) \rightarrow (x, y, z)$ in the usual way). Furthermore, the Einstein pseudotensor is the only one of ETLLPW whose result for the energy contained in a sphere of radius $r$ agrees for the Schwarzschild metric when compared in Kerr-Schild coordinates and Schwarzschild Cartesian coordinates.

For a short time, it seemed that these problems might be solved by using the concept of quasi-local energy momentum: energy and momentum associated to closed, spacelike 2-surfaces surrounding a region [16]. In this way, some of the issues that plague local, pointwise definitions of gravitational energy-momentum are circumvented. However, Bergqvist has investigated seven different definitions of quasi-local mass [17]. Computing them on cross sections of the event horizon in a Kerr spacetime and spheres in a Reissner-Nordstrom spacetime, he found that no two of the seven definitions give the same result.

Despite these problems, though, many authors have given compelling physical arguments for the existence of truly localizable gravitational energy-momentum [18, 19]. These details remain largely unresolved in GR and all other metric theories of gravity. They have been famously confusing for a long time. N. Rosen calculated the Einstein and Landau-Lifshitz pseudotensors for cylindrical gravitational waves [20]. He erroneously used cylindrical coordinates and found that the waves carry zero energy and momentum. These results had many, including Einstein, briefly convinced that gravitational waves did not exist and were merely a coordinate artifact.

At the very least, it is widely agreed that while these issues are unclear locally, all pseudotensors give correct results when applied at infinity for asymptotically flat spacetimes in quasi-Cartesian coordinates.

From now on we shall not concern ourselves with these, admittedly troubling, details. We shall restrict the calculations to infinity, though it should be understood that there might be some local sense in which our energy-momentum tensor is valid, perhaps when restricted to Kerr-Schild metrics, for example.

The energy-momentum tensor that we derive for CS modified gravity is closely analogous to the Papapetrou tensor of GR. Like the Papapetrou tensor, it is nicely derived by a Noether argument followed by a Belinfante symmetrization.

IV. THE BELINFANTE PROCEDURE FOR LORENTZ INVARIANT THEORIES

We assume a Lorentz invariant Lagrangian of some field (possibly non-scalar, Lorentz indices are suppressed) $\phi$, and show how to construct a symmetric, conserved energy-momentum tensor. We consider the possibility that the Lagrangian involves second derivatives, $L = L(\phi, \partial \phi, \partial^2 \phi)$. In terms of the quantities, $\pi^{\mu \nu} \equiv \frac{\partial L}{\partial (\partial_\mu \phi)}$, $\pi^\mu \equiv \frac{\partial L}{\partial \partial^\mu \phi}$, and $\theta_{\alpha \mu}^{\mu} \equiv \frac{\partial \pi^\alpha}{\partial (\partial_\mu \partial^\alpha \phi)}$, the (non-symmetric) canonical tensor derived via Noether’s theorem is

$$\theta_{\alpha \mu}^{\mu} = \pi^\nu \partial^\mu \phi + \pi^{\mu \nu} \partial_\nu \phi - \partial_\nu \pi^{\mu \nu} \partial^\nu \phi - \eta^{\mu \alpha} L$$

(7)

and the equations of motion are

$$\partial_\mu \pi^{\mu} = \pi + \partial_\mu \partial_\nu \pi^{\mu \nu}.$$  

(8)

We seek to decompose the tensor as

$$\theta_{\alpha \mu}^{\mu} = \theta_{\alpha \mu}^{\mu} + \partial_\nu X^{[\nu \mu] \alpha},$$

(9)

with $\theta_{\alpha \mu}^{\mu}$ symmetric and $\partial_\nu X^{[\nu \mu] \alpha}$ a manifestly conserved, so-called “superpotential.” Then $\theta_{\alpha \mu}^{\mu}$ will be our conserved, symmetric Belinfante improved energy-momentum tensor. For completeness, the Belinfante calculation is carried out in Appendix A. The result for $\theta_{\alpha \mu}^{\mu}$ is

2 We use the conventions $T^{[ab]} \equiv \frac{1}{2}(T^{ab} - T^{ba})$ and $T^{(ab)} \equiv \frac{1}{2}(T^{ab} + T^{ba})$. 


The antisymmetric part of $\theta$ is a conserved energy-momentum tensor even while using (1). Such a tensor will be given by $\partial_{\nu} \pi^{\nu} - \partial_{\nu} \pi^{\nu}$. Invariance seems to be broken by Lagrangian (1) but is retained in (3). Therefore, it should still be possible to find it may be regained explicitly, say, by an integration by parts. This is the case in CS modified gravity where translation canonical energy-momentum tensor via Noether’s theorem. However, it is possible that this invariance is hidden and may also break translational invariance, in which case it will not even be possible to derive a symmetric, conserved energy-momentum tensor if the right hand side of (14) does not vanish. In Appendix B, we discuss using the Belinfante procedure for theories that explicitly break Lorentz invariance, i.e. Lagrangians of the form

$$L = L_{inv.} + \Delta L(\phi, \partial \phi, \partial \partial \phi, x),$$

where $L_{inv.}$ is Lorentz invariant, and $\Delta L$ is not. We allow for $\Delta L$ to be explicitly dependent on $x$ (as it is for CS modified gravity).

In general, $\Delta L$ may also break translational invariance, in which case it will not even be possible to derive a canonical energy-momentum tensor via Noether’s theorem. However, it is possible that this invariance is hidden and may be regained explicitly, say, by an integration by parts. This is the case in CS modified gravity where translation invariance seems to be broken by Lagrangian (1) but is retained in (3). Therefore, it should still be possible to find a conserved energy-momentum tensor even while using (1). Such a tensor will be given by $\theta^{\mu \alpha} = \theta^{\mu \alpha} + M^{\mu \alpha}$, where $\theta^{\mu \alpha}$ is the usual formula (11), and $M^{\mu \alpha}$ are additional terms. One may again use the Belinfante relation to massage the antisymmetric part of $\theta^{\mu \alpha}$ into symmetric parts plus superpotentials. The algebra is exactly the same as before (though with the three additional terms from the adapted Belinfante relation, see Appendix B). Defining $\theta^{\mu \alpha}$ by (10), we get

$$\theta^{\mu \alpha} = \theta^{\mu \alpha} + \partial_{\nu} X^{[\nu \rho]} + M^{\mu \alpha} + (\delta^{\mu \alpha} \Delta L - l^{\mu \alpha} \Delta L + l^{\mu \alpha} \Delta L)$$

and

$$\partial_{\mu} \theta^{\mu \alpha} = \partial_{\mu} (-M^{\mu \rho} + l^{\mu \alpha} \Delta L - \delta^{\mu \alpha} \Delta L - l^{\mu \alpha} \Delta L),$$

where $l^{\alpha \beta} = x^{\alpha} \partial^{\beta} - x^{\beta} \partial^{\alpha}$ is the usual angular momentum operator, $l^{\alpha \beta}$ is the angular momentum operator applied to the explicitly $x$ dependent parts of $\Delta L$, and $\delta^{\alpha \beta} \Delta L$ is an infinitesimal Lorentz transformation on $\Delta L$. The net effect of the last two terms of (13) is to differentiate only the fields in $\Delta L$. We note that in a Lorentz non-invariant theory, it may not be possible to find a symmetric, conserved energy-momentum tensor if the right hand side of (14) does not vanish.

V. THE BELINFANTE PROCEDURE FOR LORENTZ NON-INVARIANT THEORIES

In Appendix B, we discuss using the Belinfante procedure for theories that explicitly break Lorentz invariance, i.e. Lagrangians of the form

$$L = L_{inv.} + \Delta L(\phi, \partial \phi, \partial \partial \phi, x),$$

where $L_{inv.}$ is Lorentz invariant, and $\Delta L$ is not. We allow for $\Delta L$ to be explicitly dependent on $x$ (as it is for CS modified gravity).

In general, $\Delta L$ may also break translational invariance, in which case it will not even be possible to derive a canonical energy-momentum tensor via Noether’s theorem. However, it is possible that this invariance is hidden and may be regained explicitly, say, by an integration by parts. This is the case in CS modified gravity where translation invariance seems to be broken by Lagrangian (1) but is retained in (3). Therefore, it should still be possible to find a conserved energy-momentum tensor even while using (1). Such a tensor will be given by $\theta^{\mu \alpha} = \theta^{\mu \alpha} + M^{\mu \alpha}$, where $\theta^{\mu \alpha}$ is the usual formula (11), and $M^{\mu \alpha}$ are additional terms. One may again use the Belinfante relation to massage the antisymmetric part of $\theta^{\mu \alpha}$ into symmetric parts plus superpotentials. The algebra is exactly the same as before (though with the three additional terms from the adapted Belinfante relation, see Appendix B). Defining $\theta^{\mu \alpha}$ by (10), we get

$$\theta^{\mu \alpha} = \theta^{\mu \alpha} + \partial_{\nu} X^{[\nu \rho]} + M^{\mu \alpha} + (\delta^{\mu \alpha} \Delta L - l^{\mu \alpha} \Delta L + l^{\mu \alpha} \Delta L)$$

and

$$\partial_{\mu} \theta^{\mu \alpha} = \partial_{\mu} (-M^{\mu \rho} + l^{\mu \alpha} \Delta L - \delta^{\mu \alpha} \Delta L - l^{\mu \alpha} \Delta L),$$

where $l^{\alpha \beta} = x^{\alpha} \partial^{\beta} - x^{\beta} \partial^{\alpha}$ is the usual angular momentum operator, $l^{\alpha \beta}$ is the angular momentum operator applied to the explicitly $x$ dependent parts of $\Delta L$, and $\delta^{\alpha \beta} \Delta L$ is an infinitesimal Lorentz transformation on $\Delta L$. The net effect of the last two terms of (13) is to differentiate only the fields in $\Delta L$. We note that in a Lorentz non-invariant theory, it may not be possible to find a symmetric, conserved energy-momentum tensor if the right hand side of (14) does not vanish.

VI. ENERGY-MOMENTUM TENSOR FOR CS MODIFIED GRAVITY

In CS modified gravity, we have the Lagrangian $L = L_{EH} + \Delta L$, where $L_{EH}$ is the usual, Lorentz-invariant, Einstein-Hilbert term and $\Delta L = \frac{1}{4} (v_{\nu} x^{\nu})^{*} RR$. We use the abbreviated notation $L = L(\phi, \partial \phi, \partial \partial \phi)$, where $\phi$ is understood to be the spacetime metric with indices suppressed. Though we no longer have manifest translational invariance, we can still construct a conserved (non-symmetric) energy-momentum tensor because the translation non-invariant part of the Lagrangian leads to a surface term. Under some infinitesimal transformation

$$\delta L = \pi \delta \phi + \pi^{\mu} \partial_{\mu} (\delta \phi) + \pi^{\mu \nu} \partial_{\mu} \partial_{\nu} (\delta \phi).$$

Using the equations of motion, it follows that
\[ \delta L = \partial_{\mu}[\pi^\mu \delta \phi + \pi^{\mu\nu} \partial_{\nu} \delta \phi - \partial_{\nu} \pi^{\mu\nu} \delta \phi]. \]  

(16)

For an infinitesimal translation, \( \delta \phi = \partial^\mu \phi \), and equation (15) gives

\[
\begin{align*}
\delta L &= \pi^\alpha \partial_{\alpha} \phi + \pi^{\mu\nu} \partial_{\nu} \partial_{\mu} \phi \\
&= \partial_{\alpha} L - \frac{dL}{dx^\alpha} \\
&= \partial_{\alpha} L - \frac{\nu^\alpha}{4} RR \\
&= \partial_{\mu}[\eta^{\mu\nu} L - \frac{\nu^\alpha}{2} K^\nu].
\end{align*}
\]

(17)

Equating (16) and (17),

\[
\partial_{\mu}[\pi^\mu \partial_{\alpha} \phi + \pi^{\mu\nu} \partial_{\nu} \partial_{\mu} \phi - \partial_{\nu} \pi^{\mu\nu} \partial_{\alpha} \phi - \eta^{\alpha\mu} L + \frac{\nu^\alpha}{2} K^\mu] = 0,
\]

(18)

and so we have a conserved energy-momentum tensor

\[
\theta^{\mu\alpha} = \pi^\mu \partial_{\alpha} \phi + \pi^{\mu\nu} \partial_{\nu} \partial_{\alpha} \phi - \partial_{\nu} \pi^{\mu\nu} \partial_{\alpha} \phi - \eta^{\alpha\mu} L + \frac{\nu^\alpha}{2} K^\mu.
\]

(19)

We label this as \( \theta^{\mu\alpha}_B = \theta^{\mu\alpha}_C + \frac{\omega^\mu}{2} K^\mu \), where \( \theta^{\mu\alpha}_C \) is the usual formula (7) for the energy-momentum tensor for translationally invariant Lagrangians and \( \frac{\omega^\mu}{2} K^\mu \) is the \( \Lambda^{\mu\alpha} \) of the previous section. We can then use the methods of the previous section to massage \( \theta^{\mu\alpha}_C \) such that

\[
\theta^{\mu\alpha}_C = \theta^{\mu\alpha}_B + \partial_{\nu} X^{[\nu|\alpha]} + (\delta^{\mu\alpha} \Delta L - l^{\mu\alpha} \Delta L + l_x^{\mu\alpha} \Delta L),
\]

(20)

where \( \theta^{\mu\alpha}_B \) is as in equation (10). It should be noted that the \( \pi \)'s present in this equation in \( \theta^{\mu\alpha}_B \) and \( \partial_{\nu} X^{[\nu|\alpha]} \) are derivatives of the full, Lorenz non-invariant Lagrangian. Because of the dynamical consistency condition (6)

\[
\Delta L = \frac{1}{\pi^2 \alpha} \theta(x)^* RR = 0,
\]

and so \( \delta^{\mu\alpha} \Delta L - l^{\mu\alpha} \Delta L + l_x^{\mu\alpha} \Delta L = 0. \) Therefore we have

\[
\theta^{\mu\alpha} = \theta^{\mu\alpha}_B + \partial_{\nu} X^{[\nu|\alpha]} + \frac{\nu^\alpha}{2} K^\mu
\]

(21)

and

\[
0 = \partial_{\mu} \theta^{\mu\alpha} = \partial_{\mu} \theta^{\mu\alpha}_B + \frac{\nu^\alpha}{2} \partial_{\mu} K^\mu.
\]

(22)

By (8) \( \partial_{\mu} K^\mu = 0 \), and so \( \theta^{\mu\alpha}_B \) is a conserved, symmetric, energy-momentum tensor. This Papapetrou tensor is given by replacing \( \phi \) in equation (10) by the spacetime metric, \( g_{ab} \):

\[
\theta^{\mu\alpha}_B = \pi^{\{ab\}}(\mu, \alpha) g_{ab} - 2 \partial_{\nu} \pi^{\{ab\} \nu}(\mu, \alpha) g_{ab} - \eta^{\mu\alpha} L + (\partial_{\sigma} \pi^{\{ab\} \sigma}(\mu, \alpha) g_{ab} + \partial_{\sigma} \pi^{\{ab\} \sigma}(\mu, \alpha) g_{ab})
\]

\[
+ \partial_{\nu}(\pi^{\{ab\} \alpha \nu} \partial_{\mu} g_{ab}) - \partial_{\nu}(\partial_{\alpha} \pi^{\{ab\} \beta}(\mu, \alpha) g_{ab}) - \pi^{\{ab\} \beta}(\mu, \alpha) \partial_{\sigma} g_{ab},
\]

(23)

where \( \pi^{\{ab\} \mu} = \frac{\partial_{\mu}}{\partial(\partial_{\mu} g_{ab})} \) and similarly for \( \pi^{(ab) \mu \nu} \). The gravitational spin matrices are

\[
\Sigma^{\alpha \beta} g_{ab} = (\eta^{\alpha \sigma} \delta^\mu_\beta - \eta^{\nu \sigma} \delta^\mu_\alpha) g_{ab} + (\eta^{\alpha \sigma} \delta^\nu_\beta - \eta^{\nu \sigma} \delta^\alpha_\beta) g_{ab}
\]

(24)

By linearity \( \pi^{(ab) \mu} = \pi^{(ab) \mu \nu}_E + \pi^{(ab) \mu \nu}_C \), where

\[
\pi^{(ab) \mu \nu}_E = \frac{\partial(\frac{1}{16 \pi G} \sqrt{-g} R)}{\partial(\partial_{\mu} g_{ab})} \quad \text{and} \quad \pi^{(ab) \mu \nu}_C = \frac{\partial(\frac{1}{16 \pi G} \theta(x)^* RR)}{\partial(\partial_{\mu} g_{ab})},
\]
and similarly for $\pi^{(ab)}_{\mu\nu}$. It is straightforward (though lengthy) to calculate that

$$\pi_{CS}^{(ab)\nu} = \frac{\theta(x)}{32\pi G} \left( -\Gamma^\nu_{\tau\mu} R^{\tau\sigma\mu\alpha} - \Gamma^\nu_{\tau\mu} R^{\tau\sigma\mu\alpha} + \Gamma^\nu_{\tau\mu} R^{\nu\tau\mu\alpha} + \Gamma^\nu_{\tau\mu} R^{\nu\tau\mu\alpha} + \Gamma^a_{\tau\mu} R^{\nu\tau\mu\alpha} + \Gamma^b_{\tau\mu} R^{\tau\sigma\mu\alpha} \right)$$

(25)

and

$$\pi_{CS}^{(ab)\mu\nu} = \frac{\theta(x)}{64\pi G} \left( R^{\nu\rho\mu\alpha} + * R^{\nu\rho\mu\alpha} + * R^{\nu\rho\mu\alpha} \right).$$

(26)

VII. COMPARISON WITH THE WEINBERG TENSOR

We briefly digress on another method for computing an energy-momentum tensor in CS modified gravity that was investigated in [2]. The vacuum equations of motion are

$$G_{\mu\nu} + C_{\mu\nu} = 0.$$  (27)

Now, take a quasi-Cartesian coordinate system with $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. Expanding the above equation in powers of $h$,

$$G^{(1)}_{\mu\nu} + C^{(1)}_{\mu\nu} = 8\pi G t_{\mu\nu},$$  (28)

where

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G} \left[ G_{\mu\nu} + C_{\mu\nu} - G^{(1)}_{\mu\nu} - C^{(1)}_{\mu\nu} \right]$$  (29)

and the superscripts denote the order in $h$. The tensor $t_{\mu\nu}$ has most of the properties we might want from a gravitational energy-momentum tensor: it is symmetric, ordinarily conserved (because of the linear Bianchi identity and the linear version of (6)), and quadratic in $h$ (though it is not, as usual, coordinate invariant). In GR, the energy-momentum tensor derived as (29) is referred to as the Weinberg tensor (see [12]). To compute the total energy or momentum of a gravitational system, we may integrate the left hand side of (28), which is also sometimes referred to as the Weinberg tensor.

We now demonstrate that our CS modified Papapetrou tensor gives the same result for the total energy, momentum, and angular momentum of a spacetime as (29). Taking an asymptotically flat spacetime with $h = O\left(\frac{1}{r}\right)$, we can expand our expression for the energy-momentum tensor, equation (28), to lowest order. The only finite contributions to the total energy, momentum, and angular momentum of a spacetime will be the lowest order terms in $\frac{1}{r}$. These are $O\left(\frac{1}{r}\right)$ for energy and linear momentum and $O\left(\frac{1}{r^2}\right)$ for angular momentum. All higher orders will die off at infinity. It is easily derived that

$$\theta_B^{\mu\alpha} (\pi^{(ab)}_{\beta\alpha}, \pi^{(ab)}_{\beta\sigma}, g_{ab}) = \theta_B^{\mu\alpha} (\pi_{EH}^{(ab)}, \pi_{EH}^{(ab)}, g_{ab}) + O\left(\frac{1}{r^3}\right),$$  (30)

$$\theta_B^{\mu\alpha} (\pi_{EH}^{(ab)}, \pi_{EH}^{(ab)}, g_{ab})$$

can be calculated rather straightforwardly, as in [4], using the equations of motion several times to obtain a nice form. The result is

$$\theta_B^{\mu\alpha} (\pi_{EH}^{(ab)}, \pi_{EH}^{(ab)}, g_{ab}) = -\frac{1}{16\pi G} \partial_\gamma \partial_\beta \sqrt{-g} [\eta^{\mu\alpha} g^{\gamma\beta} - \eta^{\gamma\alpha} g^{\mu\beta} + \eta^{\gamma\beta} g^{\mu\alpha} - \eta^{\mu\beta} g^{\gamma\alpha}] + \frac{\sqrt{g}}{16\pi G} [\eta^{\mu\alpha} g^{\beta\gamma} C_{\beta\gamma} + \eta^{\gamma\alpha} g^{\mu\beta} C_{\beta\gamma}].$$  (31)

We can continue the expansion,

$$\theta_B^{\mu\alpha} (\pi^{(ab)}_{\beta\alpha}, \pi^{(ab)}_{\beta\sigma}, g_{ab}) = \theta_B^{\mu\alpha} (\pi_{EH}^{(ab)}, \pi_{EH}^{(ab)}, g_{ab}) + O\left(\frac{1}{r^3}\right)$$

$$= -\frac{1}{16\pi G} \partial_\gamma \partial_\beta \sqrt{-g} [\eta^{\mu\alpha} g^{\gamma\beta} - \eta^{\gamma\alpha} g^{\mu\beta} + \eta^{\gamma\beta} g^{\mu\alpha} - \eta^{\mu\beta} g^{\gamma\alpha}].$$
The energy-momentum tensor \( T_{\mu\nu} \) is now explicitly diffeomorphism and Lorentz invariant because some sample spaces. We might now consider Lagrangian (1) with physical merit, their results are directly applicable to CS modified gravity. That ELLPW all give zero total energy for any finite volume of flat FRW models. If such calculations turn out to reasonable physical results that agree with other coordinate invariant analyses [21]. In [22], [23] and [24] it is shown is debatable in this case, since the spacetimes are not asymptotically flat. Nevertheless, such analysis seems to give appropriate physical merit and angular momentum black hole solutions remains an open problem. The most general \( (k = \{0, -1, 1\}) \) FRW solution has vanishing \( \pi_{\text{CS}}^{(a)\nu} \) and \( \pi_{\text{CS}}^{(ab)\rho\nu} \) in “Cartesian coordinates”:

\[
dx^2 = dt^2 - a(t)^2 \left[ \frac{kr^2}{1 - kr^2} \left( \frac{x}{r} \right)^2 dx + \frac{y}{r} dy + \frac{z}{r} dz \right]^2 + dx^2 + dy^2 + dz^2,
\]

with \( r^2 = x^2 + y^2 + z^2 \). Therefore, the energy-momentum tensor of FRW models in CS modified gravity is also identical to its value in GR. Beginning with [21] various authors have used the ELLPW pseudotensors in such coordinates to analyze the energy content (\( \theta^{(a)\mu} + T^{(b)\mu\rho} \)) of both open and closed FRW solutions in GR. The merit of these pseudotensors is debatable in this case, since the spacetimes are not asymptotically flat. Nevertheless, such analysis seems to give reasonable physical results that agree with other coordinate invariant analyses [21]. In [22], [23] and [24] it is shown that ELLPW all give zero total energy for any finite volume of flat FRW models. If such calculations turn out to have physical merit, their results are directly applicable to CS modified gravity.

VIII. ENERGY-MOMENTUM OF SOLUTIONS

The energy-momentum tensor [23] was calculated for the Schwarzschild and Reissner-Nordstrom solutions using quasi-Cartesian coordinates ((\( r, \theta, \phi \)) → (\( x, y, z \)) in the usual way). We have shown (in this case, with Maple v7) that for each of these solutions all terms in [23] that involve \( \pi_{\text{CS}}^{(a)\mu} \) and \( \pi_{\text{CS}}^{(ab)\mu\nu} \) are zero. The energy-momentum tensor evaluated for these solutions is thus unchanged from GR. It is comforting to know that in CS modified gravity a black hole with mass \( M \), charge \( Q \) and angular momentum zero is indeed of mass \( M \), charge \( Q \) and angular momentum zero. A generalization of this result to non-zero angular momentum black hole solutions remains an open problem.

The energy-momentum tensor for CS modified gravity and evaluated it on some sample spaces. We might now consider Lagrangian [1] with \( \theta \) as a Lagrange multiplier instead of a prescribed field. This theory is now explicitly diffeomorphism and Lorentz invariant because \( \theta \) now responds to coordinate transformations, and therefore the theory admits a symmetric, conserved energy-momentum tensor. Varying with respect to \( g \) again gives [1] as an equation of motion, while varying with respect to \( \theta \) immediately gives the consistency condition [19] as an equation of motion. By making a coordinate transformation we may set \( \theta(x, t) \propto t \), and we obtain CS modified gravity as a coordinate choice in this new theory. Thus the “Lorentz violation” of CS modified gravity is just a choice of coordinates in the new theory. Viewed in this light, it is not very surprising that CS modified gravity does indeed admit a conserved energy-momentum tensor that signals the absence of Lorentz violation.

IX. CONCLUSIONS

We have constructed a symmetric, conserved energy-momentum tensor for CS modified gravity and evaluated it on some sample spaces. We might now consider Lagrangian [1] with \( \theta \) as a Lagrange multiplier instead of a prescribed field. This theory is now explicitly diffeomorphism and Lorentz invariant because \( \theta \) now responds to coordinate transformations, and therefore the theory admits a symmetric, conserved energy-momentum tensor. Varying with respect to \( g \) again gives [1] as an equation of motion, while varying with respect to \( \theta \) immediately gives the consistency condition [19] as an equation of motion. By making a coordinate transformation we may set \( \theta(x, t) \propto t \), and we obtain CS modified gravity as a coordinate choice in this new theory. Thus the “Lorentz violation” of CS modified gravity is just a choice of coordinates in the new theory. Viewed in this light, it is not very surprising that CS modified gravity does indeed admit a conserved energy-momentum tensor that signals the absence of Lorentz violation.
Acknowledgments

The authors would like to thank R. Jackiw for introducing them to CS modified gravity, providing the inspiration for this project and for many useful related discussions. This work is supported by the U.S. Department of Energy under cooperative research agreement No. DEFG02-05ER41360. A. J. H. acknowledges support by the FQRNT of Canada under their postdoctoral research fellowship program.

APPENDIX A: BELINFANTE PROCEDURE DETAILS

Our aim is to decompose

\[ \theta^{[\mu\alpha]} = \frac{1}{2} (\pi^\mu \partial^\alpha \phi - \pi^\alpha \partial^\mu \phi) + \frac{1}{2} (\pi^{\mu\nu} \partial_\nu \partial^\alpha \phi - \pi^{\alpha\nu} \partial_\nu \partial^\mu \phi) + \frac{1}{2} (\partial_\nu \pi^{\alpha\nu} \partial^\mu \phi - \partial_\nu \pi^{\mu\nu} \partial^\alpha \phi) \]  

such that

\[ \theta^{\mu\alpha} = \theta^{(\mu\alpha)} + \theta^{[\mu\alpha]} = \theta^{(\mu\alpha)} + \partial_\nu X^{[\nu\mu\alpha]} . \]  

To this end, we note that under a Lorentz transformation:

\[ \delta \phi = l \phi + \Sigma \phi \]

\[ l = \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) \]

\[ \Sigma = \omega_{\alpha\beta} \Sigma^{\alpha\beta} , \]

where \( l_{\alpha\beta} \) are the angular momentum matrices and \( \Sigma^{\alpha\beta} \) are the spin matrices. Lorentz indices are suppressed on \( \Sigma \) and \( \phi \). We now note

\[ \delta L = \pi \delta \phi + \pi^\mu \delta \partial_\mu \phi + \pi^{\mu\nu} \delta \partial_\mu \partial_\nu \phi \]

\[ = \pi l \phi + \pi \Sigma \phi + \pi^\mu \partial_\mu[l \phi + \Sigma \phi] + \pi^{\mu\nu} \partial_\mu \partial_\nu[l \phi + \Sigma \phi] \]

\[ = \pi l \phi + \pi^\mu l_\mu \partial_\mu \phi + \pi^{\mu\nu} l_{\mu\nu} \partial_\mu \partial_\nu \phi + \pi \Sigma \phi + \pi^{\mu\nu} \partial_\mu \partial_\nu \phi + \omega_{\alpha\beta} (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) \]

\[ + \pi^{\mu\nu} \partial_\nu \partial_\mu \phi + \frac{2 \omega_{\alpha\beta} (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi)}{\pi^{\alpha\beta}} \]

\[ = l L + \pi^\mu \Sigma l_\mu \partial_\mu \phi + \omega_{\alpha\beta} (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) \]

\[ + \pi^{\mu\nu} \Sigma l_{\mu\nu} \partial_\mu \partial_\nu \phi + \omega_{\alpha\beta} (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) . \]  

(A4)

Since our theory is Lorentz invariant, \( \delta L \) will be a total derivative; in fact, it will almost always be equal to the total derivative \( l L \). We surmise the “Belinfante relation”:

\[ \pi \Sigma^{\alpha\beta} \phi + \pi^\mu \Sigma^{\alpha\beta} \partial_\mu \phi + (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) \]

\[ + \pi^{\mu\nu} \Sigma^{\alpha\beta} \partial_\mu \partial_\nu \phi + 2 (\pi^{\alpha\nu} \partial^\beta \partial_\nu \phi - \pi^{\beta\nu} \partial^\alpha \partial_\nu \phi) = 0. \]  

(A5)

Using (A5) in (A1)

\[ \theta^{[\mu\alpha]} = - \frac{1}{2} \pi^{\mu\nu} \Sigma^{\nu\alpha} \phi - \frac{1}{2} \pi^\nu \Sigma^{\nu\alpha} \partial_\nu \phi - \frac{1}{2} \pi^{\mu\nu} \Sigma^{\nu\alpha} \partial_\nu \phi + \pi^{\nu\sigma} \partial^\alpha \partial_\nu \phi + \pi^{\alpha\nu} \partial^\mu \partial_\nu \phi \]

\[ - \frac{1}{2} \partial_\nu \pi^{\mu\nu} \partial^\alpha \phi + \frac{1}{2} \partial_\nu \pi^{\alpha\nu} \partial^\mu \phi + \frac{1}{2} \partial_\nu \pi^{\nu\sigma} \partial^\alpha \phi - \frac{1}{2} \pi^{\alpha\nu} \partial_\nu \partial^\mu \phi . \]  

(A6)

Using the equations of motion, and collecting terms, we get

\[ \theta^{\mu\alpha} = - \frac{1}{2} \partial_\nu [\pi^{\nu\sigma} \Sigma^{\nu\alpha} \phi] + \frac{1}{2} \partial_\nu \partial_\sigma \pi^{\nu\sigma} \Sigma^{\nu\alpha} \phi - \frac{1}{2} \pi^{\nu\sigma} \Sigma^{\nu\alpha} \partial_\nu \partial_\sigma \phi - \frac{1}{2} \pi^{\nu\sigma} \partial^\nu \partial^\sigma \phi . \]  

(A7)
\[
\frac{1}{2} \pi^\alpha\sigma \partial_\rho \partial_\nu \phi - \frac{1}{2} \partial_\nu \pi^\alpha\sigma \partial^\rho \phi + \frac{1}{2} \partial_\rho \pi^\alpha\nu \partial^\mu \phi \\
= -\frac{1}{2} \partial_\nu [\pi^\nu \Sigma^\mu\alpha \phi] - \frac{1}{2} \partial_\nu [\pi^\mu \partial^\alpha \phi] + \frac{1}{2} \partial_\nu [\pi^\alpha \partial^\mu \phi] \\
+ \frac{1}{2} \partial_\nu [\partial_\rho \pi^\alpha\sigma \Sigma^\mu\alpha \phi - \pi^\nu\sigma \Sigma^\mu\alpha \partial_\sigma \phi].
\] (A7)

We examine each term separately.

\[
-\frac{1}{2} \partial_\nu [\pi^\nu \Sigma^\mu\alpha \phi] = -\frac{1}{2} \partial_\nu [\pi^\mu \Sigma^\nu\alpha \phi - \pi^\nu \Sigma^\mu\alpha \phi - \pi^\alpha \Sigma^\nu\mu \phi] - \frac{1}{2} \partial_\nu [\pi^\nu \Sigma^\mu\nu \phi + \pi^\mu \Sigma^\nu\nu \phi] \\
= -\frac{1}{2} \partial_\nu [\pi^\nu \Sigma^\nu\alpha \phi + \pi^\nu \Sigma^\mu\nu \phi] + \partial_\nu \mathcal{X}_1^{[\nu\mu]_{\alpha}},
\] (A8)

where \(\partial_\nu \mathcal{X}_1^{[\nu\mu]_{\alpha}}\) is a superpotential. Similarly,

\[
-\frac{1}{2} \partial_\nu [\pi^{\mu\nu} \partial^\alpha \phi] = -\frac{1}{2} \partial_\nu [\pi^{\mu\nu} \partial^\alpha \phi + \pi^{\nu\sigma} \partial^\mu \phi - \pi^{\nu\alpha} \partial^\sigma \phi + \frac{1}{2} \partial_\nu [\pi^{\nu\alpha} \partial^\mu \phi - \sigma^{\nu\mu} \partial^\nu \phi] \\
= -\frac{1}{2} \partial_\nu [\pi^{\nu\alpha} \partial^\nu \phi] + \partial_\nu \mathcal{X}_2^{[\nu\mu]_{\alpha}},
\] (A9)

\[
\frac{1}{2} \partial_\nu [\pi^{\alpha\nu} \partial^\mu \phi] = \frac{1}{2} \partial_\nu [\pi^{\alpha\nu} \partial^\mu \phi - \pi^{\alpha\mu} \partial^\nu \phi] + \frac{1}{2} \partial_\nu [\pi^{\alpha\mu} \partial^\nu \phi] \\
= \frac{1}{2} \partial_\nu [\pi^{\alpha\mu} \partial^\nu \phi] + \partial_\nu \mathcal{X}_3^{[\nu\mu]_{\alpha}},
\] (A10)

\[
\frac{1}{2} \partial_\nu [\partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\nu\alpha} \Sigma^{\mu\alpha} \partial_\sigma \phi] = \frac{1}{2} \partial_\nu [\partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\nu\alpha} \Sigma^{\mu\alpha} \partial_\sigma \phi - \partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi \\
+ \pi^{\mu\sigma} \Sigma^{\nu\alpha} \partial_\sigma \phi - \partial_\sigma \pi^{\sigma\nu} \Sigma^{\mu\alpha} \phi + \pi^{\alpha\sigma} \Sigma^{\nu\mu} \partial_\sigma \phi] \\
+ \frac{1}{2} \partial_\nu [\partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\mu\sigma} \Sigma^{\nu\alpha} \partial_\sigma \phi + \partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\alpha\sigma} \Sigma^{\nu\mu} \partial_\sigma \phi] \\
= \frac{1}{2} \partial_\nu [\partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\mu\alpha} \Sigma^{\nu\alpha} \partial_\sigma \phi + \partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\alpha\nu} \Sigma^{\mu\sigma} \partial_\sigma \phi] \\
+ \partial_\nu \mathcal{X}_4^{[\nu\mu]_{\alpha}}.
\] (A11)

Adding in the symmetric parts and simplifying, we finally get

\[
\theta^{\mu\alpha} = \theta_B^{\mu\alpha} + \partial_\nu \mathcal{X}^{[\nu\mu]_{\alpha}},
\] (A12)

where \(\mathcal{X}^{[\nu\mu]_{\alpha}} = \mathcal{X}_1^{[\nu\mu]_{\alpha}} + \mathcal{X}_2^{[\nu\mu]_{\alpha}} + \mathcal{X}_3^{[\nu\mu]_{\alpha}} + \mathcal{X}_4^{[\nu\mu]_{\alpha}}\) and

\[
\theta_B^{\mu\alpha} = \frac{1}{2} (\pi^\mu \partial^\alpha \phi + \pi^\alpha \partial^\mu \phi) - (\partial_\nu \pi^{\mu\nu} \partial^\sigma \phi + \partial_\nu \pi^{\sigma\nu} \partial^\mu \phi) \\
- \frac{1}{2} \partial_\nu (\pi^{\mu\nu} \partial^\alpha \phi + \pi^{\nu\sigma} \partial^\mu \phi) + \partial_\nu (\pi^{\alpha\nu} \partial^\mu \phi) \\
+ \frac{1}{2} \partial_\nu (\partial_\sigma \pi^{\nu\alpha} \Sigma^{\mu\alpha} \phi - \pi^{\nu\sigma} \Sigma^{\mu\alpha} \partial_\sigma \phi) + \frac{1}{2} \partial_\nu \pi^{\nu} \Sigma^{\mu\nu} \partial_\sigma \phi. 
\] (A13)

**APPENDIX B: BELINFANTE RELATION FOR LORENTZ NON-INVARIANT THEORIES**

Suppose we have a Lagrangian of the form
where $\mathcal{L}_{\text{inv.}}$ is Lorentz invariant, and $\Delta\mathcal{L}$ is not. We allow for $\Delta\mathcal{L}$ to be explicitly dependent on $x$ (as it will be for CS modified gravity). Calculation (A2) gets modified as

$$\delta\mathcal{L} = \pi l \phi + \pi^\mu \partial_\mu \phi + \pi^{\mu\nu} \partial_\mu \partial_\nu \pi \Sigma \phi + \pi^{\mu\nu} \Sigma \partial_\mu \partial_\nu \phi + \omega_{\alpha\beta}(\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) + \pi^{\mu\nu} \Sigma \partial_\mu \partial_\nu \phi + 2\omega_{\alpha\beta}(\pi^\alpha \partial^\beta \partial_\sigma \phi - \pi^\beta \partial^\alpha \partial_\sigma \phi)
$$

$$= l\mathcal{L} - l_x \Delta\mathcal{L} + \pi^{\mu\nu} \Sigma \partial_\mu \partial_\nu \phi + \pi \Sigma \phi + \omega_{\alpha\beta}(\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) + \pi^{\mu\nu} \Sigma \partial_\mu \partial_\nu \phi + 2\omega_{\alpha\beta}(\pi^\alpha \partial^\beta \partial_\sigma \phi - \pi^\beta \partial^\alpha \partial_\sigma \phi), \tag{B2}$$

where $l_x$ is the angular momentum operator applied to the explicitly $x$ dependent parts of $\Delta\mathcal{L}$. It should be noticed that the $\pi$’s of equation (B2) are derivatives of the entire Lagrangian, not just the Lorentz invariant part. We expect $\delta\mathcal{L}_{\text{inv.}}$ to transform in the usual way and so

$$\delta\mathcal{L} = \delta\mathcal{L}_{\text{inv.}} + \delta\Delta\mathcal{L} = l\mathcal{L}_{\text{inv.}} + \delta\Delta\mathcal{L} = l(\mathcal{L} - \Delta\mathcal{L}) + \delta\Delta\mathcal{L} = l\mathcal{L} + \delta\Delta\mathcal{L} - l\Delta\mathcal{L}. \tag{B3}$$

Equating, we derive the modified Belinfante relation:

$$\pi \Sigma^{\alpha\beta} \phi + \pi^{\mu\nu} \Sigma^{\alpha\beta} \partial_\mu \phi + (\pi^\alpha \partial^\beta \phi - \pi^\beta \partial^\alpha \phi) + \pi^{\mu\nu} \Sigma^{\mu\nu} \partial_\mu \partial_\nu \phi + 2(\pi^{\mu\nu} \partial^\beta \partial_\mu \phi - \pi^{\mu\nu} \partial^\alpha \partial_\mu \phi) = \delta^{\alpha\beta} \Delta\mathcal{L} - l_x^{\alpha\beta} \Delta\mathcal{L} + l_x^{\alpha\beta} \Delta\mathcal{L}. \tag{B4}$$