Demonstration of the spin-statistics connection in elementary quantum mechanics

J. A. MORGAN

The Aerospace Corporation, P. O. Box 92957
Los Angeles, CA 90009,
United States of America

Abstract

The spin-statistics connection is proved using the methods of elementary quantum mechanics. The effect of exchange and space inversion operators on two-particle states is reviewed. The connection follows directly from successive application of these operations to the wave function for two identical particles in an s-state, evaluated at positions \( \pm x \), but at equal time.
I. INTRODUCTION

The connection between spin and statistics, first conjectured by Pauli, and subsequently proved by Pauli, Burgoyne and Lüders and Zumino has an understandable appeal to students of physics as an example of a phenomenon arising from quantum mechanics and relativity that has palpable consequences in the realm of everyday experience. This paper presents a demonstration of the spin-statistics connection by an elementary argument involving symmetry of two-particle wave functions under the combined operations of exchange and parity. It is intended to be accessible to final-year undergraduate students of quantum mechanics, who will have had exposure to simple angular momentum theory, the Pauli principle, and the concept of parity. The argument makes no explicit appeal to special relativity, although underlying assumptions traceable to Poincaré symmetry may be discerned in what follows.

The proof concerns two-particle states constructed from noninteracting single-particle wave functions possessing definite symmetry under rotations and parity. It assumes particles obey either Bose or Fermi statistics, that is, that multiparticle wave functions are respectively symmetric, or antisymmetric, under exchange of particle identity.

In the following, a (single-particle) state may be described by the ket vector $|\phi\rangle$ or by the wave function $\phi(x,t) = \langle x,t|\phi \rangle$. The dependence upon time will usually not be shown.

II. BACKGROUND

A. Quantum states of higher spin

We start by reciting results from the theory of angular momentum in quantum mechanics that find use in the following. The total angular momentum operators $J_i$ give rise to infinitesimal rotations of a state about the $x_i$ axes. Eigenstates of total angular momentum $\hbar j$ can take on a range of values for the $z$-projection of angular momentum,

$$\langle m|J_z|\phi_j \rangle = m\hbar \langle m|\phi_j \rangle,$$

where the magnetic quantum number $m$ has values in the range

$$-j \leq m \leq j.$$
Coupling of two single-particle states to a state of specified angular momentum is accomplished with a unitary transformation whose matrix elements are Clebsch-Gordan coefficients. The Clebsch-Gordan coefficient coupling two states with total and magnetic angular momentum quantum numbers \((j_a, m_a)\) and \((j_b, m_b)\), respectively, to a state with quantum numbers \((J, M)\) is denoted \(\langle {j_a m_a j_b m_b} | J M \rangle\). Thus,

\[
|JM\rangle = \sum_{-j_a \leq m_a \leq j_a; -j_b \leq m_b \leq j_b} \langle {j_a m_a j_b m_b} | J M \rangle |j_a m_a j_b m_b\rangle.
\]

(3)

B. The exchange operator

The exchange operator \(X\) acting on the state

\[
|\psi\rangle = |\phi(1)\rangle|\phi(2)\rangle
\]

(4)
gives

\[
X|\phi(1)\rangle|\phi(2)\rangle = |\phi(2)\rangle|\phi(1)\rangle
\]

(5)

If the state \(|\psi\rangle\) is either symmetric (bosonic) or antisymmetric (fermionic) under exchange of \(|\phi(1)\rangle\) and \(|\phi(2)\rangle\),

\[
X|\psi\rangle = \pm|\psi\rangle.
\]

(6)

Consider now the inverse to \(X\). Given \(|\psi\rangle\) and another two-particle state \(|\xi\rangle\), their matrix element \(\langle \xi|\psi\rangle\) should be left unchanged by application of \(X\) to both states:

\[
X\langle \xi|X|\psi\rangle = \langle \xi|\psi\rangle,
\]

(7)

which is readily seen to be the same as

\[
\langle \xi|X^\dagger X|\psi\rangle = \langle \xi|\psi\rangle,
\]

(8)

or

\[
X^{-1} = X^\dagger.
\]

(9)

C. The parity operator

The result of the space inversion, or parity, operation on a spinless state \(|\xi_0\rangle\) is

\[
\langle x, t|P|\xi_0\rangle = \langle -x, t|\xi_0\rangle.
\]

(10)
Parity acting on position and momentum variables gives

\[ \mathbf{x} \Rightarrow -\mathbf{x} \]
\[ \mathbf{p} \Rightarrow -\mathbf{p}. \]  

(11)

It follows that (orbital) angular momentum is unaltered by the parity operator:

\[ \mathbf{x} \times \mathbf{p} \equiv \mathbf{L} \Rightarrow \mathbf{L}, \]  

(12)

and that the \( \mathcal{P} \) operation commutes with rotations. In order that \( \mathcal{P} \), which may be regarded as a passive coordinate transformation, not alter the total angular momentum of a wave function possessing both orbital and spin angular momentum degrees of freedom, its effect on components of a state with definite, nonzero spin must likewise be the identity, allowing us to write

\[ \langle \mathbf{x}, m | \mathcal{P} | \xi_j \rangle = \langle -\mathbf{x}, m | \xi_j \rangle. \]  

(13)

With a suitable definition of the origin of spatial coordinates, the effect of the parity operation upon a state containing two identical particles is equivalent to exchanging their positions:

\[ \mathcal{P} | r_m_1; -r_m_2 \rangle = | -r_m_1; r_m_2 \rangle. \]  

(14)

It is, however, distinct from the action of the exchange operator \( \mathcal{X} \), which interchanges spin as well as spatial degrees of freedom.

An eigenstate of parity obeys

\[ \langle \mathbf{x}, t | \mathcal{P} | \psi \rangle = \eta \langle \mathbf{x}, t | \psi \rangle \]  

(15)

As two successive applications of the parity operation give the identity, \( \mathcal{P}^2 = 1 \),

\[ \mathcal{P}^2 = 1, \]  

(16)

which implies

\[ \eta^2 = 1; \]  

(17)

\[ \eta = \pm 1 \]  

(18)

for a state of definite parity. States with \( \eta = +1 \) are symmetric under space inversion (even parity), while states with \( \eta = -1 \) are antisymmetric (odd parity). Parity is a unitary operator, so we also have

\[ \mathcal{P}^\dagger = \mathcal{P}^{-1} = \mathcal{P}, \]  

(19)
analogous to Eq. (9).

If the states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) respectively have parities \( \eta_1 \) and \( \eta_2 \), then the combined state \( |\psi_1\rangle|\psi_2\rangle \) has parity

\[
\eta_{12} = \eta_1 \eta_2. \tag{20}
\]

III. THE CONNECTION BETWEEN SPIN AND STATISTICS

The connection is proved with the aid of a wave function that gives the amplitude for the simultaneous location of one particle at spacetime location \((x, t)\) and another at location \((-x, t)\) in a relative \(s\)-state:

\[
\langle x, -x; 00|\psi\rangle = \sum_m \langle jmj - m|00\rangle \langle xm; -x - m|\psi\rangle. \tag{21}
\]

Consider the effect of exchange and space inversion operations on the wave functions appearing in equation Eq. (21). We have

\[
\mathcal{X}|\psi\rangle = \pm|\psi\rangle \tag{22}
\]
as the particles obey Bose (\(+\)) or Fermi (\(-\)) exchange symmetry,\(^{19}\) and

\[
\mathcal{X}|xm; -x - m\rangle = |-x - m; xm\rangle \tag{23}
\]
so that

\[
\langle xm; -x - m|\psi\rangle = \langle xm; -x - m|\mathcal{X}^{-1}\mathcal{X}|\psi\rangle = \pm \langle -x - m; xm|\psi\rangle. \tag{24}
\]

Next, apply the parity operator to the wave function appearing on the RHS of Eq. (21). The state \(|\psi\rangle\) is composed of products of two identical single-particle wave functions. According to Eq. (20) the parity of such a product must be even,

\[
\mathcal{P}|\psi\rangle = |\psi\rangle \tag{25}
\]
with

\[
\mathcal{P}| -x - m; xm\rangle = |x - m; -xm\rangle \tag{26}
\]
leading to

\[
\langle -x - m; xm|\mathcal{P}^{-1}\mathcal{P}|\psi\rangle = \langle x - m; -xm|\psi\rangle. \tag{27}
\]
Inserting Eq. (27) into Eq. (24) gives

\[ \langle x_m; -x - m | \psi \rangle = \pm \langle x - m; -x m | \psi \rangle. \]  

(28)

Upon substituting Eq. (28) into Eq. (21),

\[ \langle x, -x; 00 | \psi \rangle = \pm \sum_m \langle j m j - m | 00 \rangle \langle x, -m; -x, m | \psi \rangle. \]  

(29)

We may invert the order of summation by replacing \( m \) with \( -m' \) to get

\[ \langle x, -x; 00 | \psi \rangle = \pm \sum_{m'} \langle j - m' j m' | 00 \rangle \langle x m', -x - m' | \psi \rangle. \]  

(30)

At this point it is advantageous to rewrite Eq. (30) in a suggestive way. The Clebsch-Gordan coefficient appearing in Eq. (30) is

\[ \langle j - m' j m' | 00 \rangle = \frac{(-1)^{(j+m')}}{\sqrt{2j+1}}. \]  

(31)

Note that the quantity \( j - m' \) is always an integer, and \( 2j - 2m' \) an even integer. We may write

\[ (-1)^{m'} = (-1)^{m'} (-1)^{2j-2m'} = (-1)^{2j} (-1)^{-m'} \]  

(32)

and conclude

\[ \langle j - m' j m' | 00 \rangle = (-1)^{2j} \langle j m' j - m' | 00 \rangle. \]  

(33)

Employing this relation in Eq. (30) and recalling Eq. (21) gives us

\[ \langle x, -x; 00 | \psi \rangle = \pm (-1)^{2j} \langle x, -x; 00 | \psi \rangle. \]  

(34)

The singlet wave function appearing in Eq. (34) is nonvanishing if the individual wave functions from which it is constructed are themselves nonvanishing. A proof of this assertion appears in the Appendix. If we can assume the matrix element on both sides of Eq. (34) does not vanish, we immediately have

\[ 1 = \pm (-1)^{2j}. \]  

(35)

According to Eq. (35), states \( |x, -x\rangle \) with \( 2j \) even necessarily have Bose exchange symmetry, while those with \( 2j \) odd necessarily have Fermi symmetry. This is the connection between spin and statistics.
Rewriting Eq. (35) slightly as
\[ \pm 1 = (-1)^{2j} \]  
makes clear its connection with a common description of the spin-statistics connection. The operator that rotates a state of definite angular momentum about a specified axis by a given angle is derived in Vol. III, chapter 18 of Ref. 10. In particular, the operator that rotates a state around the $y$ axis by an angle $\pi$ is
\[ R_{y}^{m, m'}(\pi) = (-1)^{j} \delta_{m, -m'} . \]  
It follows immediately that the result of rotating by an angle of $2\pi$ about the $y$ axis is
\[ R_{y}^{m, n}(\pi)R_{y}^{n, m'}(\pi) = (-1)^{2j} \delta_{m, m'} . \]  
Equation (36) is thus equivalent to an oft-stated formulation of the connection: The effect of exchanging the position of two identical particles (LHS) is equivalent to the rotation of one of them by an angle of $2\pi$ around a suitable axis (RHS).

IV. DISCUSSION

The proof just given relies almost entirely on elementary quantum mechanics. Apart from the material appearing in the appendix, it depends on nothing that cannot readily be obtained (or at a minimum, motivated) starting from pertinent discussions in the Feynman Lectures. In addition, the proof makes no overtly relativistic assumptions. This lack is perhaps more apparent than real, however. At certain points the argument rests upon assumptions that flow in a natural and unforced way from requirements of relativistic symmetry, but which arguably enter a nonrelativistic exposition in neither fashion.

An instance is the symmetry of the wave function in Eq. (24), which is a disguised statement of an equal-time commutation relation. Exhibiting the dependence upon $t$, Eq. (24) becomes
\[ \langle (x, t) m; (-x, t) - m|\psi \rangle = \pm \langle (-x, t) - m; (x, t) m|\psi \rangle . \]  
In Eq. (39), the wave functions for the individual particles (1) and (2) located at spatial position $\pm x$ are evaluated at equal time $t$. Put another way,
\[ |x_2 - x_1|^2 - (t_2 - t_1)^2 > 0 \]  
(40)
The statement that a relation holds between two particles separated by nonzero distance at equal time has no unambiguous meaning under admissible changes of coordinates. Equation (40), however, is an invariant statement under arbitrary Lorentz transformations. In proofs of the spin-statistics relation, the exchange symmetry appearing in Eq. (24) is usually stipulated subject to Eq. (40). One says that wave functions of identical particles commute or anticommute outside the light cone.

Moreover, it was assumed that wave functions exist with certain simple, conjoined symmetries with respect to the operations of parity and rotation. The assumed symmetries of wave functions are those of an irreducible representation of the Poincaré group. Thus, elements of the present demonstration that enter a nonrelativistic version of the proof as distinct hypotheses all follow from the single requirement of Poincaré invariance in an explicitly relativistic treatment. Granted this observation, the nonrelativistic view does not appear to be the parsimonious one, whatever pedagogical advantages result from discussing the spin-statistics connection in largely nonrelativistic terms.

V. APPENDIX

We apologize for the fact that we cannot give you an elementary explanation.
-R. F. Feynman, Ref. 10, Vol. III, p. 4-3

In the following it will be convenient to write two-particle wave functions in factored form so that, e. g., the wave function in Eq. (21) is written as

\[ \langle x_m; -x - m | \psi \rangle = \langle x_m | \phi_j(1) \rangle \langle -x - m | \phi_j(2) \rangle. \] (41)

From single-particle wave functions for spin \( j \), which may be assumed to belong to an irreducible representation of the rotation group, form

\[ (\xi_j, \phi_j) \equiv (-1)^{-j} \sum_m \int d^3x \langle jmj - m | 00 \rangle \langle x_m | \xi_j \rangle \langle x_m | \phi_j \rangle^*. \] (42)

This quantity serves as an inner product in the Hilbert space of wave functions on \( \mathbb{R}^3 \). In

\[ (\phi_j, \phi_j) = (-1)^{-j} \sum_m \int d^3x \langle jmj - m | 00 \rangle \langle x_m | \phi_j \rangle \langle x_m | \phi_j \rangle^*. \] (43)

we may write

\[ \langle x_m | \phi_j \rangle = f_j(r) Y_{jm}(\Omega) \] (44)
at radius $r$. Here the function $Y_{jm}(\Omega)$ is a suitable angular momentum eigenfunction that generalizes the properties of spherical harmonics to include half-integral as well as integral angular momenta.\textsuperscript{23,24} It may be defined so as to share with ordinary spherical harmonics $Y_{lm}(\Omega)$ the conjugation property

$$Y^*_{jm} = (-1)^m Y_{-m}. \quad (45)$$

We also have

$$\int d\Omega Y_{jm}^* Y_{jm} = \int d\Omega Y_{jm'} Y_{jm} = \delta_{m'm}. \quad (46)$$

The angular momentum ladder operators $J_{\pm}$ are defined by

$$J_{\pm} = J_x \pm i J_y \quad (47)$$

and have the effect of raising and lowering $m$:

$$\langle m | J_{\pm} | \phi_j \rangle = -i\hbar \sqrt{(j \mp m)(j \pm m + 1)} \langle m \pm 1 | \phi_j \rangle \quad (48)$$

The $J_{\pm}$ are differential operators that act on orbital and spin degrees of freedom only.\textsuperscript{25} This observation means that the $J_{\pm}$ raise and lower $m$ in $Y_{jm}(\Omega)$ and have no effect upon $f_j(r)$. The radial weight $f_j(r)$ can, therefore, have no dependence upon $m$.\textsuperscript{25} Recalling the definition of the Clebsch appearing in Eq. (43) (vide Eq. [31]), we find

$$(\phi_j, \phi_j) = \int r^2 dr f_j(r)f_j^*(r) \geq 0, \quad (49)$$

with equality if $f_j(r)$ vanishes everywhere. Should

$$\sum_m \langle jm j - m | 00 \rangle \langle xm | \phi_j \rangle \langle xm | \phi_j \rangle^* = 0, \forall x \quad (50)$$

then $(\phi_j, \phi_j)$ will vanish. But $(\phi_j, \phi_j) = 0$ iff $(xm | \phi_j)$ vanishes, as well.

Assume $|\zeta_j\rangle$ is a state of a spin $j$ particle such that

$$\langle xm | \zeta_j \rangle \neq 0 \quad (51)$$

From $|\zeta_j\rangle$ form

$$\langle xm | \phi_j \rangle \equiv \langle xm | \zeta_j \rangle^* \pm ( -x - m | \zeta_j \rangle. \quad (52)$$
Then
\[
\sum_m \langle jm \rangle - m |00 \rangle \langle x - m | \phi_j \rangle = \pm \sum_m \langle jm \rangle - m |00 \rangle \langle x | \phi_j \rangle \langle x | \phi_j \rangle^*.
\]
(53)

As a general rule, the wave function \( \langle x | \phi_j \rangle \) will have nonvanishing norm and the RHS of Eq. (53) will differ from zero. But suppose that for one choice of sign in Eq. (52), \( \langle x | \phi_j \rangle \) were to vanish \( \forall x \). In that event \( \langle x | \phi_j \rangle \), and hence Eq. (53), cannot vanish for the other choice. We suppose in the main text that the appropriate choice of sign has been made, if necessary, and that Eq. (21) is therefore nonvanishing on some open set of \( x \).

---

9 Strictly speaking, this assumption is unnecessary, as the restriction to Bose or Fermi exchange symmetry can be proved in quantum mechanics, but the proof of that result is not elementary, vide. M. G. G. Laidlaw and C. M. de Witt, "Feynman Functional Integrals for Systems of Indistinguishable Particles", Phys. Rev. D 3, pp. 1375-1378 (1971)
11 Ref. 10, Vol III, p. 17-9, Eq. 17.29


Ref. 10, Vol III, pp. 4-1–4-2.


Ref. 10, Vol III, pp. 4-2–4-3; 4-12–4-15


Ref. 20, pp. 25-27; 60-61.
