Is the Outer Solar System Chaotic?

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One-sentence summary: Current observational uncertainty in the positions of the Jovian planets precludes deciding whether or not the outer Solar System is chaotic.

100 word technical summary: The existence of chaos in the system of Jovian planets has been in question for the past 15 years. Various investigators have found Lyapunov times ranging from about 5 millions years upwards to infinity, with no clear reason for the discrepancy. In this paper, we resolve the issue. The position of the outer planets is known to only a few parts in 10 million. We show that, within that observational uncertainty, there exist Lyapunov timescales in the full range listed above. Thus, the “true” Lyapunov timescale of the outer Solar System cannot be resolved using current observations.

100 word summary for general public: The orbits of the inner planets (Mercury, Venus, Earth, and Mars) are practically stable in the sense that none of them will collide or be ejected from the Solar System for the next few billion years. However, their orbits are chaotic in the sense that we cannot predict their angular positions within those stable orbits for more than about 20 million years. The picture is less clear for the outer planets (Jupiter, Saturn, Uranus and Neptune). Again their orbits are practically stable, but it is not known for how long we can accurately predict their positions within those orbits.

The existence of chaos among the Jovian planets is a contested issue. There exists apparently unassailable evidence both that the outer Solar System is chaotic\(^1\)\(^2\) and that it is not\(^3\)\(^4\)\(^5\). The discrepancy is particularly disturbing given that computed chaos is sometimes due to numerical artifacts\(^6\)\(^7\). In this paper we discount the possibility of numerical artifacts, and demonstrate that the discrepancy seen between various investigators is real. It is caused by observational uncertainty in the orbital positions of the Jovian planets, which is currently a few parts in 10 million. Within that observational uncertainty, there exist clearly chaotic...
trajectories with complex structure and Lyapunov times ranging from 2 million years to 230 million years, as well as trajectories that show no evidence of chaos over 1Gy timescales. Determining the true Lyapunov time of the outer Solar System will require a more accurate observational determination of the orbits of the Jovian planets. Fully understanding the nature and consequences of the chaos may require further theoretical development.

The Solar System is known to be “practically stable”, in the sense that none of the 9 planets is likely to suffer mutual collisions, or be ejected from the Solar System, over the next several billion years. The motion of Pluto is chaotic with a Lyapunov time of 10–20 million years, while the inner Solar System has a Lyapunov time of about 4–5 million years. Pluto’s chaos is caused by the overlap of two-body resonances. The cause of chaos in the inner Solar System is not yet fully understood, although secular resonances likely play a role.

The existence of chaos amongst the Jovian planets (Jupiter, Saturn, Uranus, and Neptune) is less certain. On one hand, Laskar found that the orbits of the outer planets appeared non-chaotic. This is consistent with the knowledge that there are no two-body resonances among the outer planets, but nonetheless Laskar’s “averaged” integrations could not detect mean-motion resonances, even if they existed. On the other hand, chaos was found in a non-averaged, full integration of the 9 planets by Sussman and Wisdom. However, their computed Lyapunov time varied with simulation parameters for reasons which were (at the time) unknown.

Two obvious but mutually-exclusive explanations can be offered to explain why some investigators find chaos while others do not. First, Sussman and Wisdom’s widely varying Lyapunov time might be the result of numerical artifacts, rather than physical effects. This hypothesis was supported at the time by the lack of an explanation for chaos in the outer Solar System. A second plausible explanation is that since Laskar’s averaged equations do not model planet motion, Sussman and Wisdom were observing real chaos, caused by something other than overlap of two-body resonances.

Unfortunately, both explanations have since been convincingly offered, and neither has been disproved to date. One one hand, chaos can be a numerical artifact, even in an n-body integration. Furthermore, numerous careful integrations of the outer Solar System have been performed specifically to ensure the accuracy and convergence of the numerical result, and these give a clear indication of no chaos. On the other hand, an explanation for the chaos in terms of three-body mean-motion resonances has been offered by Murray and Holman. Furthermore, Guzzo has performed carefully-tested integrations detecting a web of three-body resonances, precisely where Murray and Holman’s theory predicts them to be. Finally, those who have found chaos also appear to perform reasonable convergence tests, making it unlikely that the chaos they have found is a numerical artifact. So we have a quandary: there is apparently unassailable evidence on two sides, demonstrating both that the outer Solar System is chaotic, and that it is not.
The observational uncertainty in the position of the outer planets is a few parts in ten million\(^{19}\). The resolution of the above paradox is simple: within the observational uncertainty, there exist both chaotic and non-chaotic solutions. This is also consistent with Murray and Holman’s theory, since it is known from the study of simpler systems that chaotic “zones” can contain both chaotic and regular-looking trajectories, densely packed amongst each other, with the chaotic ones having widely varying Lyapunov times\(^{20}\).

Although it is impossible with a finite-time integration to demonstrate that a trajectory is not chaotic, we abuse the term “regular” to mean a trajectory that shows no evidence of chaos over timescales ranging from 200My (million years) to 1Gy (10\(^9\) years). Figure 1 plots the divergence between initial conditions (ICs) that initially differ by an infinitesimal amount (1.5mm in the semi-major axis of Uranus). Chaos manifests itself as exponential divergence between such “siblings”, while regularity is manifested by polynomial divergence. Our results were consistent across three very different integration schemes, and all schemes agreed with each other once the timesteps were small enough to demonstrate convergence. To estimate how the uncertainty volume is split between chaotic and regular ICs, we chose 31 systems within the observational uncertainty and integrated each for 200My. We found that 21 of the 31 samples (about 70%) were chaotic, and 10 (about 30%) were regular. When integrated for 1Gy, the percentage of regular trajectories decreased to less than 10%. However, the spectrum of observed Lyapunov times was enormous (Figure 2). Furthermore, the most “authoritative” IC, the only one explicitly fit to all observations in DE405\(^{19}\), shows no evidence of chaos after 1Gy.

**Methods**

We use initial conditions (ICs) for the Sun and all 8 planets (excluding Pluto). The inner planets are deleted, but their effect on the outer planets is crudely accounted for by perturbing the Sun’s position to the centre-of-mass of the inner Solar System, and by augmenting the mass and momentum of the Sun with the masses and momenta of the inner planets; this ensures\(^2\) that the positions of the chaotic zones are shifted by no more than about one part in 10\(^{11}\). The system is then numerically integrated for 10\(^9\) years (1Gy) or until the distance between siblings saturates, using only Newtonian gravity. The masses of all objects are held constant. We also ignore many physical effects which may be important to the detailed motion of the planets\(^{21-5}\). However, we believe that none of these effects will alter the chaotic nature of solutions.

Determining the existence of chaos depends critically on the quality of the numerical integration scheme. We ensure that our results are free of numerical artifacts by performing our integrations using three different algorithms. First, we use the Wisdom-Holman symplectic mapping\(^{12}\) as implemented in the *Mercury 6.2* package\(^{22}\), with timesteps ranging from 400 days down to 2 days. We find that using timesteps of 16 days or more lead to inconsistent results, in that integrating the same IC with different timesteps results in significantly different Lyapunov times\(^\perp\). However, for a particular IC, results converge to reliable Lyapunov
times when the integration timestep is 8 days or less. Our second integration scheme is the *NBI* package, which has been used to demonstrate the non-existence of chaos in the outer Solar System\(^{23,24,25}\). We use NBI’s 14th-order Cowell-Störmer integrator with modifications by the UCLA research group led by William Newman\(^{26,27,28}\) with a timestep of 4 days. With these parameters, NBI is known to produce *exact* solutions to double precision, at each timestep; one cannot do better than this without maintaining machine precision while *increasing* the timestep (which would lower the frequency at which the one-roundoff-per-step occurs).

Our third and highest precision integration scheme is the *Taylor 1.4* package\(^{29}\), which uses a 27th-order Taylor series expansion and a 220-day timestep. Arithmetic in *Taylor 1.4* is performed in 19-digit Intel Extended precision. Like NBI, on each step *Taylor 1.4* produces results which are exact to machine precision, except here the step is 220 days rather than 4 days, and the solution is exact to 19 digits rather than 16. In the case of both *NBI* and *Taylor 1.4*, the numerical error growth is effectively dominated by a single random roundoff error per step. We have verified the “exact to machine precision per step” properties of both *NBI* and *Taylor* by comparison to quadruple-precision integrations which were locally accurate to 30 digits. Once convergence was reached, all integrations had errors that grew (in the absence of chaos) approximately as \(t^{3/2}\), in accordance with Brouwer’s Law\(^{30}\). Our 1Gy integrations using Taylor-1.4 conserved both energy and angular momentum to almost 13 digits. Numerical error caused the center of mass of the system to drift by just 0.45km over 1Gy, which is about 3 orders of magnitude smaller than the current observational error. Thus, the numerical error in our 1Gy experiments is negligible compared to current observational error.

Figure 1 uses ICs for the Sun and 8 planets as listed in the *Mercury 6.2* package\(^{31}\), representing the positions of the planets at JD2451000.5. The semi-major axis of Uranus was increased by \(2 \times 10^{-6}\) AU (bottom figure), and \(4 \times 10^{-6}\) AU (top figure). The percentage of ICs that are chaotic come from 31 eight-planet samples from the latest JPL planetary ephemeris, DE405\(^{32}\). We drew 21 sets of ICs starting at 9.5 Jan 1990 (JD2448235), separated by 30-day intervals. This samples ICs from a span of about 2 years, so it includes samples such that all the inner planets are sampled at widely varying positions in their orbits (before they are thrown into the Sun). We also drew 10 samples starting at the year 1900, separated by 10-year intervals. Figure 2 uses ICs from all of the above sources, and is not intended to represent a uniform sample.

Source code, initial conditions, and outputs are all available from the author upon request.

References


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Figure 1: Divergence between nearby trajectories, integrated with four different numerical integrators. Top figure: a chaotic trajectory with a Lyapunov time of about 12 million years. Bottom figure: a trajectory showing no evidence of chaos over 200My. Both trajectories are within observational uncertainty of the outer planetary positions.
Figure 2: Menagerie of Lyapunov times. All trajectories originate within observational uncertainty, although the regular one (labelled $T^{1.5}$, corresponding to an infinite Lyapunov time) is the most “authoritative” initial condition from JPL’s DE405 ephemeris. Finite ones range from 11My (million years) to 230My.