A cyclic cooling algorithm

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We introduce a scheme to perform the cooling algorithm, first presented by Oscar Boykin et al. in 2002 [1], for an arbitrary number of times on the same set of qubits. We achieve this goal by adding an additional SWAP gate and a bath contact to the algorithm. This way one qubit may repeatedly be cooled without adding additional qubits to the system. By using a product Liouville space to model the bath contact we calculate the density matrix of the system after a given number of applications of the algorithm.

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Algorithmic cooling is a method to obtain highly polarized spins in a spin system without cooling down the environment. It may for example be used for medical magnetic resonance imaging to improve the resolution by cooling down a subset of nuclear spins of a patient without cooling of the patient himself, or for the preparation of the ground state of a quantum computer by means of the computer itself, that means that no external cooling mechanism would have to be attached to the system [2, 3].

The spin to be cooled down (a nuclear spin for example) has to couple weakly to the environment. In addition one uses some rapid relaxing spins to transport energy out of the system. The transportation of energy from the cooled spin to the others is achieved in a strictly nonclassical way by applying a quantum algorithm to the system, therefore the spins are further referred to as qubits.

Recently, Oscar Boykin et al. [1] have developed a quantum algorithm to cool down a single qubit with the aid of two rapidly relaxing auxiliary qubits. Initially the system is prepared in an equilibrium state with all spins at the same inverse temperature $\beta(0)$. By applying several quantum gate operations one spin is cooled down by transferring energy to the others. The algorithm itself consists of a controlled NOT (CNOT) gate and a controlled swap gate (CSWAP) [4]. The CSWAP is a 3 qubit gate which swaps qubit 1 with qubit 3 if qubit 2 is 0, otherwise it does nothing. This results into a reduction of the inverse temperature $\beta(1)$ of qubit 1 of approximately $3/2 \beta(0)$ (cf. [1, 5]). Having applied the algorithm once, the initial state is recovered by two further applications of the algorithm. However, by cooling down two other qubits in the same way using the algorithm as described above allows a second application of the algorithm to the cooled qubit triple with reduced initial inverse temperature $\beta(1)$. Thus, one qubit could be cooled down to the total of $9/4 \beta(0)$. Possessing an unlimited number of qubits the method is, in principle, able to reach arbitrary low temperatures for a single qubit. But, of course, due to an exponential growth of resources [3].

Instead of using an infinite number of qubits to reach an arbitrary low temperature it would be highly desirable to have an algorithm which reaches at least some lower temperature without using more qubits. As will be shown below, this is obtained by starting with a SWAP gate between qubit 1 and 3. Furthermore, after all gates have been applied as before one has to wait until the auxiliary qubits relax back to the initial temperature by coupling to a heat bath (see Fig. 1).

This new algorithm can be applied for an arbitrary number of times to the same set of qubits. We thus call this algorithm cyclical, despite the fact that after one application the system does not return to its initial state. We are then interested in the final inverse temperature $\beta(n)$ of the first qubit after the $n$-th application in dependence of system parameters and relaxation times between subsequent application steps. In principle the complete process could be seen in the context of thermodynamical machines (refrigerators) cooling down a finite “environment” (here only the single spin 1). With this cyclic algorithm it is possible to cool down one third of the accessible qubits. If one was capable of running quantum-gates on any combination of qubits, all but two qubits of the system may be cooled down.

The non-interacting 3 spin system is described by the

![FIG. 1: cyclical cooling algorithm: 3 quantum gates are applied, first a SWAP gate then a CNOT gate and finally a CSWAP gate. Bath contact at inverse temperature $\beta(0)$ is symbolized by the boxes on qubit 2 and 3.](image)
Hamiltonian
\[ \hat{H} = \sum_{\mu=1}^{3} \Delta E_{\mu} \hat{\sigma}_z(\mu) \]

where \( \Delta E_{\mu} \) specifies the respective Zeeman splitting of qbit \( \mu \). Each quantum gate is represented by a unitary transformation \( \hat{U} \). Thus we introduce the transformation operators \( \hat{U}_{\text{SWAP}} \), \( \hat{U}_{\text{CNOT}} \) and \( \hat{U}_{\text{CSWAP}} \), representing the whole algorithm as \( \hat{U} = \hat{U}_{\text{CSWAP}} \hat{U}_{\text{CNOT}} \hat{U}_{\text{SWAP}} \).

The density matrix after the application of the algorithm, but before the next bath contact, is given by
\[ \hat{\rho}_f(n) = \hat{U} \hat{\rho}_i(n) \hat{U}^{-1}, \]

where \( n \) denotes the cycle index. Then the bath contact has to be taken into account in order to calculate the final temperature. Since the qbits are uncoupled each one is thermalized separately. A standard technique to describe such a bath coupling refers to the quantum master equation in Lindblad form \[ 7, 8, 9, 10 \]. For a single spin \( i \) such a bath coupling refers to the quantum master equation
\[ \dot{\hat{\rho}}_i(t) = -i[\hat{H}, \hat{\rho}_i] + W_{\text{rel}} \hat{\rho}_i(t) \]

with the rates \( W_{\text{rel}} = \lambda/(1 + (1/\epsilon)) \) and \( W_{\text{bath}} = \lambda/(1 + \epsilon) \), \( \epsilon = \exp[\Delta E/\beta(0)] \) and the bath coupling strength \( \lambda \). The total Liouville von Neumann equation can be represented by the superoperator \( \mathcal{L} \) acting on operators of the Hilbert space \( \mathcal{H} \). Sorting the entries of an operator \( \hat{O} \) on \( \mathcal{H} \) (for example the density operator) into a \( k^2 \) dimensional vector, we define "ket" and "bra" like vectors \( \hat{O} \rightarrow |\hat{O}\rangle \) and \( \hat{O}| \rightarrow \langle \hat{O}| \) in this super space. Their inner product is defined as \( \langle \hat{A}| \hat{B} \rangle = \text{Tr}\{\hat{A}^\dagger \hat{B}\} \), the trace norm of operators in \( \mathcal{H} \). Operators acting on states \(|\hat{O}\rangle \) in the Liouville space are defined as \( |\hat{A}\rangle \). The superoperator \( \mathcal{D} = |\hat{A}\rangle \langle \hat{B}| \) represents a \( k^2 \times k^2 \) dimensional matrix.

One of the big advantages of this superspace formalism is the possibility of writing down a superoperator projecting an arbitrary state on a solution of \( \mathcal{L} \). Just like in Hilbert space the time evolution operator [the formal solution of \( \mathcal{L} \)] is given by
\[ \rho(t) = e^{\mathcal{L}t} \rho_i(0) = \mathcal{T}(t) \rho_i(0) \]

with the limit
\[ \lim_{t \rightarrow \infty} \mathcal{T}(t) = \mathcal{T}(\infty) \]

defining the complete thermalisation superoperator. Based on equation \( \mathcal{L} \) we find for diagonal density operators the time evolution operator in terms of the Pauli basis
\[ \mathcal{T}(\tau) = |\sigma_0\rangle\langle \sigma_0| + e^{-2\tau \lambda} |\sigma_z\rangle\langle \sigma_z| + (e^{-2\tau \lambda} - 1) \frac{\epsilon - 1}{\epsilon + 1} |\sigma_0\rangle\langle \sigma_0| \]

This superoperator represents the thermalisation process truncated after a time step \( \tau = \Delta t \). In this case the bath has not jet completely thermalised the spin. To extend the thermalising superoperator to more than one spin we use a product Liouville space with the basis \( |\sigma_1 \rangle \otimes |\sigma_2 \rangle \otimes |\sigma_k \rangle = |\sigma_{i,j,k} \rangle \). In this basis the respective thermalising superoperator of qbit 2 and 3 (cf. Fig. 1) reads
\[ \mathcal{T}_{23}(\tau) = I \otimes \mathcal{T}(\tau) \otimes \mathcal{T}(\tau) \]

For the superoperators of the quantum gates we use the corresponding unitary transformation \( \mathcal{U} \) on a general density matrix expanded in Pauli matrices, thus, obtaining the superoperator \( U \rho U^{-1} \rightarrow \mathcal{U}(\rho) \) in the product basis. We thus get the superoperator \( \mathfrak{A} = \mathcal{T}_{2,3}(\tau) \cdot \mathcal{U} \) for the complete algorithm and the state of the system after an arbitrary number \( n \) of cycles
\[ |\hat{\rho}_f(n)\rangle = \mathfrak{A}^n |\hat{\rho}(1)\rangle \]

This could be achieved by finding it’s Jordan decomposition and take it to its \( n \)-th power, getting the density operator in the Liouville space product basis. The result is easily transformed back to the Hilbert space by using the respective Pauli product basis in the Hilbert space \( \hat{\sigma}_{i,j,k} = \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_k \), by inserting the coefficients \( \rho_{i,j,k}(n) \) of \( |\hat{\rho}(n)\rangle \) back into the expansion
\[ \hat{\rho}_f(n) = \sum_{i,j,k} \hat{\rho}_{i,j,k}(n) \cdot \hat{\sigma}_{i,j,k} \]

In the case of complete relaxation of the auxiliary qbits i.e. by taking the limit for \( \tau \rightarrow \infty \) of \( \mathcal{T}_{23}(\tau) \), we have been able to compute for the limit \( n \rightarrow \infty \) of \( \beta(n) \) the asymptotic inverse temperature of the cooled qbit
\[ \beta(\infty) = \frac{\Delta E_2 + \Delta E_3}{\Delta E_1} \beta(0). \]

For the truncated relaxation a complete analytic solution is not available, hence we investigate the algorithm numerically by calculating \( n = 300 \) repetitions (with \( \Delta E_1 = \Delta E_2 = \Delta E_3 = 1 \)). For short bath contact time \( \tau \) the inverse temperature \( \beta(n) \) of the cooled spin is shown in Fig. 2 for long bath contact it evolves like Fig. 3.
For increasing bath contact time the final inverse temperature $\beta(300)$ is depicted in Fig. 4. Multiple application of the algorithm does not always lead to a reduction of the temperature of the cooled spin. For our set of parameters (one quantum gate needs one time-unit to resolve) one has to wait $32$ time-units to achieve approximately the same temperature as already achieved after the first application of the algorithm. This means, that after the first step one has to cool the auxiliary qubits at least back to $0.77$ times the inverse bath temperature $\beta(0)$. Because we have computed with constant relaxation time one would have to keep the relaxation time constant for all repetitions of the algorithm instead of cooling back to $0.77 \beta(0)$ each time. 

Finally we calculate the efficiency $\eta$ of the algorithm. Qubit 1 represents hereby the “second heat bath”, the most elementary bath one can think of. The change of the energy expectation value [13] of qubit 1 represents the heat $Q$

$$Q_n = \Delta \langle E(n) \rangle_1 = \text{Tr}\{\hat{H}_1\hat{\rho}_1(n)\} - \text{Tr}\{\hat{H}_1\hat{\rho}_1(0)\}$$  \hspace{1cm} (13)

transferred by the algorithm, with $\hat{H}_1 = \Delta E_1 \sigma_z \otimes \sigma_0 \otimes \sigma_0$, $\hat{\rho}_1(n)$ and $\hat{\rho}_1(0)$ representing the density operator of the system before and after the $n$-th application of the algorithm without taking the bath contact into account. To compute the work $W$ in the system one has to take a look at qbit 2 and 3. If no work was done on the system the change of the energy expectation of this subsystem, $|\Delta\langle E(n)\rangle_2 \rangle = 2$ would be equal to $|\Delta\langle E(0)\rangle_1 \rangle$ with opposite sign, energy would thus only be moved around within the system, 

$$\Delta \langle E(n) \rangle_{23} = \text{Tr}\{\hat{H}_{23}\hat{\rho}_2(n)\} - \text{Tr}\{\hat{H}_{23}\hat{\rho}_2(0)\}$$  \hspace{1cm} (14)

with $\hat{H}_{23} = \Delta E_2 \sigma_0 \otimes \sigma_z \otimes \sigma_0 + \Delta E_3 \sigma_0 \otimes \sigma_0 \otimes \sigma_z$. If $|\Delta\langle E(n)\rangle_{23}| - |\Delta\langle E(0)\rangle_1|$ was less than zero, work would be extracted from the system, if it was larger than zero work would be done on the system. If $\Delta\langle E(n)\rangle_{23}$ and $\Delta\langle E(n)\rangle_1$ had different signs $|\Delta\langle E(n)\rangle_{23}| - |\Delta\langle E(0)\rangle_1|$ is equivalent to the change of the energy expectation value of the entire system,

$$W_n = \Delta \langle E(n) \rangle = \text{Tr}\{\hat{H}\hat{\rho}_1(n)\} - \text{Tr}\{\hat{H}\hat{\rho}_1(0)\}$$

$$= \text{Tr}\{(\hat{H}_1 + \hat{H}_{23})\hat{\rho}_1(n)\} - \text{Tr}\{(\hat{H}_1 + \hat{H}_{23})\hat{\rho}_1(0)\}$$

$$= \text{Tr}\{\hat{H}_1\hat{\rho}_1(n)\} - \text{Tr}\{\hat{H}_1\hat{\rho}_1(0)\} + \text{Tr}\{\hat{H}_{23}\hat{\rho}_2(n)\} - \text{Tr}\{\hat{H}_{23}\hat{\rho}_2(0)\}$$  \hspace{1cm} (15)

Now we can define the efficiency $\eta_n$ of the algorithm by $\eta_n = -Q_n/W_n$. In case of total relaxation of the auxiliary qubits we have obtained an analytical result for the efficiency $\eta_n$, depicted for the first step for various sets of energy splittings of qbit 2 and 3 in Fig. 5. In the further applications of the algorithm for $n \to \infty$ $\eta_n$ goes asymptotically to zero, and of course always stays below the accounting Carnot efficiency. In the region where the efficiency $\eta_1$ is negative, qbit 1 heats up instead of cooling down.

The cyclic algorithm trades the large amount of resources of the original algorithm against the application.
of a bath contact and one more quantum gate. As the current situation of the quantum computer indicates, that may be easier to implement than the addition of qbits to the system. The final inverse temperature of the cooled qbits is $\beta(\infty) = [(\Delta E_2 + \Delta E_3)/\Delta E_1] \beta(0)$. It just depends on the Zeeman splitting of the qbits involved in the algorithm and may thus be adjusted precisely.

We thank M. Henrich, H. Schmidt, H. Schroeder, J. Teifel, G. Reuther, H. Weimer, P. Vidal for fruitful discussions. Financial support by the DFG is gratefully acknowledged.


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Instead of using an infinite number of qubits to reach an arbitrary low temperature it would be highly desirable to have an algorithm which reaches at least some lower temperature without using more qubits. As will be shown below, this is obtained by starting with a SWAP gate between qbit 1 and 3. Furthermore, after all gates have been applied as before one has to wait until the auxiliary qubits relax back to the initial temperature by coupling to a heat bath (see Fig. 1).

This new algorithm can be applied for an arbitrary number of times to the same set of qubits. We thus call this algorithm cyclical, despite the fact that after one application the system does not return to its initial state. We are then interested in the final inverse temperature $\beta(n)$ of the first qbit after the $n$-th application in dependence of system parameters and relaxation times between subsequent application steps. In principle the complete process could be seen in the context of thermodynamical machines (refrigerators) cooling down a finite “environment” (here only the single spin 1). With this cyclic algorithm it is possible to cool down one third of the accessible qubits. If one was capable of running quantum gates on any combination of qubits, all but two qubits of the system may be cooled down.

The non-interacting 3 spin system is described by the

![diagram](image-url)

**FIG. 1:** cyclical cooling algorithm: 3 quantum gates are applied, first a SWAP gate then a CNOT gate and finally a CSWAP gate. Bathcontact at inverse temperature $\beta(0)$ is symbolized by the boxes on qbit 2 and 3.
where $\Delta E_\mu$ specifies the respective Zeeman splitting of qbit $\mu$. Each quantum gate is represented by a unitary transformation $U$. Thus we introduce the transformation operators $U_{\text{SWAP}}, U_{\text{CNOT}}$ and $U_{\text{CSWAP}}$, representing the whole algorithm as $\hat{U} = U_{\text{CSWAP}}U_{\text{CNOT}}U_{\text{SWAP}}$. The density matrix after the application of the algorithm, but before the next bath contact, is given by

$$\hat{\rho}_f(n) = \hat{U}\hat{\rho}_i(n)\hat{U}^{-1},$$  

where $n$ denotes the cycle index. Then the bath contact has to be taken into account in order to calculate the final temperature. Since the qbits are uncoupled each one is thermalized separately. A standard technique to describe such a bath coupling refers to the quantum master equation in Lindblad form \([7, 8, 9, 10]\). For a single spin $\mu$ the respective Liouville-von-Neumann equation reads

$$\dot{\hat{\rho}}_\mu = -i[\hat{H}, \hat{\rho}_\mu] + W_{1-0}2(\hat{\sigma}_-\hat{\rho}_\mu\hat{\sigma}_+ - \hat{\rho}_\mu\hat{\sigma}_+\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_\mu) + W_{0-1}2(\hat{\sigma}_+\hat{\rho}_\mu\hat{\sigma}_- - \hat{\rho}_\mu\hat{\sigma}_-\hat{\sigma}_+ - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_\mu)$$

$$= \Sigma\hat{\rho}_\mu,$$  

with the rates $W_{1-0} = \lambda/(1 + (1/\varepsilon))$ and $W_{0-1} = \lambda/(1 + \varepsilon)$, $\varepsilon = \exp[\Delta E\beta(0)]$ and the bath coupling strength $\lambda$. The total Liouville von Neumann equation can be represented by the superoperator $\Sigma$ acting on operators of the Hilbert space $\mathcal{H}$. Sorting the entries of an operator $\hat{O}$ on $\mathcal{H}$ (for example the density operator) into a $k^2 \times k^2$ dimensional vector, we define "ket" and "bra" like vectors $\hat{A} \rightarrow |\hat{A}\rangle$ and $\hat{A}^\dagger \rightarrow \langle\hat{A}|$ in this super space. Their inner product is defined as $\langle\hat{A}|\hat{B}\rangle = \text{Tr}\{\hat{A}^\dagger\hat{B}\}$, the trace norm of operators in $\mathcal{H}$. Operators acting on states $|\hat{O}\rangle$ in the Liouville space are defined as \([11]\)

$$\mathcal{D}|\hat{O}\rangle = |\hat{A}\rangle\langle\hat{B}|\hat{O}\rangle = \text{Tr}\{\hat{B}^\dagger\hat{O}\hat{A}\}.$$

One of the big advantages of this superspace formalism is the possibility of writing down a superoperator projecting an arbitrary state on a solution of \([3]\). Just like in Hilbert space the time evolution operator [the formal solution of \([3]\)] is given by

$$\rho(t) = e^{\mathcal{L}t}\rho_i(0) = \mathcal{T}(t)\rho_i(0)$$

with the limit

$$\lim_{t \to \infty} \mathcal{T}(t) = \mathcal{T}(\infty)$$

defining the complete thermalisation superoperator. Based on equation \([3]\) we find for diagonal density operators the time evolution operator in terms of the Pauli basis

$$\mathcal{T}(\tau) = |\sigma_0\rangle\langle\sigma_0| + e^{-2\tau\lambda} |\sigma_z\rangle\langle\sigma_z| + (e^{-2\tau\lambda} - 1)\frac{\varepsilon - 1}{\varepsilon + 1} |\sigma_0\rangle\langle\sigma_0|.$$

This superoperator represents the thermalisation process truncated after a time step $\tau = \Delta t$. In this case the bath has not jet completely thermalised the spin. To extend the thermalising superoperator to more than one spin we use a product Liouville space with the basis $|\sigma_i\rangle \otimes |\sigma_j\rangle \otimes |\sigma_k\rangle = |\sigma_{ijk}\rangle$. In this basis the respective thermalising superoperator of qbit 2 and 3 (cf. Fig. 1) reads

$$\mathcal{T}_{23}(\tau) = 1 \otimes \mathcal{T}(\tau) \otimes \mathcal{T}(\tau).$$

For the superoperators of the quantum gates we use the corresponding unitary transformation \([2]\) on a general density matrix expanded in Pauli matrices, thus, obtaining the superoperator $U\rho U^{-1} \rightarrow \mathcal{U}(\rho)$ in the product basis. We thus get the superoperator $\mathcal{A} = \mathcal{T}_{23}(\tau) \otimes \mathcal{U}$ for the complete algorithm and the state of the system after an arbitrary number $n$ of cycles

$$|\hat{\rho}_f(n)\rangle = \mathcal{A}^n|\hat{\rho}(1)\rangle$$

This could be achieved by finding it’s Jordan decomposition and take it to its $n$-th power, getting the density operator in the Liouville space product basis. The result is easily transformed back to the Hilbert space by using the respective Pauli product basis in the Hilbert space $\sigma_{ijk} = \sigma_i \otimes \sigma_j \otimes \sigma_k$, by inserting the coefficients $\rho_{f,ijk}(n)$ of $|\hat{\rho}_f(n)\rangle$ back into the expansion

$$|\hat{\rho}_f(n)\rangle = \sum_{ijk} \hat{\rho}_{f,ijk}(n) \cdot \sigma_{ijk}$$

In the case of complete relaxation of the auxiliary qbits,i.e. by taking the limit for $\tau \rightarrow \infty$ of $\mathcal{T}_{23}(\tau)$, we have been able to compute for the limit $n \rightarrow \infty$ of \([10]\) the asymptotic inverse temperature of the cooled qbit 1

$$\beta(\infty) = \frac{\Delta E_2 + \Delta E_3}{\Delta E_1} \cdot \beta(0).$$

For the truncated relaxation a complete analytic solution is not available, hence we investigate the algorithm numerically by calculating $n = 300$ repetitions (with $\Delta E_1 = \Delta E_2 = \Delta E_3 = 1$). For short bath contact time $\tau$ the inverse temperature $\beta(n)$ of the cooled spin is shown in Fig. 2 for long bath contact it evolves like Fig. 3.
For increasing bath contact time the final inverse temperature $\beta(300)$ is depicted in Fig. 4. Multiple applications of the algorithm does not always lead to a reduction of the temperature of the cooled spin. For our set of parameters (one quantum gate needs one time-unit to resolve) one has to wait 32 time-units to achieve approximately the same temperature as already achieved after the first application of the algorithm. This means, that after the first step one has to cool the auxiliary qubits at least back to 0.77 times the inverse bath temperature $\beta(0)$. Because we have computed with constant relaxation time one would have to keep the relaxation time constant for all repetitions of the algorithm instead of cooling back to 0.77 $\beta(0)$ each time.

Finally we calculate the efficiency $\eta$ of the algorithm. Qubit 1 represents hereby the “second heat bath”, the most elementary bath one can think of. The change of the energy expectation value $\Delta \langle E(n)/23 \rangle$ of qubit 1 represents the heat $Q$.

$$Q_n = \Delta \langle E(n)/23 \rangle = \text{Tr}\{\hat{H}_{23}\hat{\rho}_{1}(n)\} - \text{Tr}\{\hat{H}_{23}\hat{\rho}_{F}(n)\}$$

transferred by the algorithm, with $\hat{H}_{23} = \Delta E_{23} \sigma_{0} \otimes \sigma_{0} + \Delta E_{0} \sigma_{0} \otimes \sigma_{0} \otimes \sigma_{0}$, $\hat{\rho}_{1}(n)$ and $\hat{\rho}_{F}(n)$ representing the density operator of the system before and after the $n$-th application of the algorithm without taking the bath contact into account.

To compute the work $W$ in the system one has to take a look at qubit 2 and 3. If no work was done on the system the change of the energy expectation of this subsystem, $\Delta \langle E(2)/1 \rangle$ would be equal to $\Delta \langle E(1) \rangle$ with opposite sign, energy would thus only be moved around within the system,

$$\Delta \langle E(n)/23 \rangle = \text{Tr}\{\hat{H}_{23}\hat{\rho}_{1}(n)\} - \text{Tr}\{\hat{H}_{23}\hat{\rho}_{F}(n)\}$$

with $\hat{H}_{23} = \Delta E_{2} \sigma_{0} \otimes \sigma_{0} + \Delta E_{3} \sigma_{0} \otimes \sigma_{0} \otimes \sigma_{0}$. If $|\Delta \langle E(2)/23 \rangle| - |\Delta \langle E(1) \rangle|$ was less than zero, work would be extracted from the system, if it was larger than zero work would be done on the system. If $\Delta \langle E(n)/23 \rangle$ and $\Delta \langle E(n)/1 \rangle$ had different signs $|\Delta \langle E(2)/23 \rangle| - |\Delta \langle E(n)/1 \rangle|$ is equivalent to the change of the energy expectation value of the entire system,

$$W_n = \Delta \langle E(n)/1 \rangle = \text{Tr}\{\hat{H}_{1}\hat{\rho}_{F}(n)\} - \text{Tr}\{\hat{H}_{1}\hat{\rho}_{1}(n)\}$$

$$= \underbrace{\text{Tr}\{\hat{H}_{1}\hat{\rho}_{F}(n)\}}_{\Delta \langle E(1)/1 \rangle} - \underbrace{\text{Tr}\{\hat{H}_{1}\hat{\rho}_{1}(n)\}}_{\Delta \langle E(2)/23 \rangle}$$

$$= \text{Tr}\{\hat{H}_{1}\hat{\rho}_{F}(n)\} - \text{Tr}\{\hat{H}_{1}\hat{\rho}_{1}(n)\}$$

Now we can define the efficiency $\eta_n$ of the algorithm by $\eta_n = -Q_n/W_n$. In case of total relaxation of the auxiliary qubits we have obtained an analytical result for the efficiency $\eta_n$, depicted for the first step for various sets of energy splittings of qubit 2 and 3 in Fig. 5. In the further applications of the algorithm for $n \to \infty$ $\eta_n$ goes asymptotically to zero, and of course always stays below the accounting Carnot efficiency. In the region where the efficiency $\eta_1$ is negative, qubit 1 heats up instead of cooling down.

The cyclic algorithm trades the large amount of resources of the original algorithm against the application
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FIG. 5: The efficiency $\eta_1$ of the first step $n = 1$ of the algorithm


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