Non-linear vector perturbations in a contracting universe

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Abstract. A number of scalar field models proposed as alternatives to the standard inflationary scenario involve contracting phases which precede the universe’s present phase of expansion. An important question concerning such models is whether there are effects which could potentially distinguish them from purely expanding cosmologies. Vector perturbations have recently been considered in this context. At first order such perturbations are not supported by a scalar field. In this paper, therefore, we consider second order vector perturbations. We show that such perturbations are generated by first order scalar mode-mode couplings, and give an explicit expression for them. We compare the magnitude of vector perturbations produced in collapsing models with the corresponding amplitudes produced during inflation, using a number of suitable power–law solutions to model the inflationary and collapsing scenarios. We conclude that the ratios of the magnitudes of these perturbations depend on the details of the collapsing scenario as well as on how the hot big bang is recovered, but for certain cases could be large, growing with the duration of the collapse.

PACS numbers: 04.25.Nx, 98.80.Hw
1. Introduction

Recent years have witnessed tremendous advances in observational cosmology. High-precision data from observations of the Cosmic Microwave Background (CMB) and high redshift surveys have provided strong evidence for a nearly spatially-flat universe with a primordial spectrum of adiabatic, Gaussian and nearly scale-invariant density perturbations, in excellent agreement with the predictions of the simplest inflationary models [1, 2]. Despite these important successes, however, crucial questions remain. Paramount among these is whether the inflationary scenario can be embedded within a fundamental theory. While attempts in this direction continue, a number of alternative scenarios have been put forward motivated by string/M-theory. Among these are the pre-big bang [3, 4] and the ekpyrotic/cyclic scenarios [5, 6, 7, 8, 9]. Despite their differences, an important ingredient shared by these models is the existence of a contracting epoch preceding a poorly understood bounce into the present expansionary phase of the universe. Interestingly, there are now many proposals for realising bouncing cosmologies within the context of string/M-theory [10, 11], braneworld models [12, 13, 14, 15, 16] and recently Loop Quantum Cosmology [17, 18, 19, 20], which add weight to the feasibility of scenarios of this type. Thus, given the successes of inflationary cosmology, on the one hand, and the importance of considering alternative scenarios, on the other, it is important to seek observational signatures that could distinguish between these alternative possibilities.

An important tool in this connection is provided by cosmological perturbation theory. There are in general three types of perturbations, namely scalar, vector and tensor, which decouple at the linear level. A great deal of work has been done on studying the evolution of such perturbations in an expanding universe, both at the linear (for a review see [21]) and nonlinear levels (see e.g. [22, 23, 24]). A number of studies have also been made of the evolution of perturbations in collapsing models. In order to succeed as an alternative to inflation, any model of the early universe must provide an explanation of the primordial scalar perturbations, which are the seeds of structure in the universe. The amplitude and spectrum of these scalar perturbations are highly constrained by CMB observations, and although problems remain, both the pre-big bang and the cyclic/ekpyrotic scenarios have the potential to explain their origin. In this sense scalar perturbations may not be enough to distinguish between these scenarios.

The next step is to consider other perturbation modes. Tensor perturbations are important as they can in principle be observed as a primordial gravitational wave background and can thus provide a way of distinguishing between different scenarios. Indeed, it has been shown that the cyclic/ekpyrotic scenario has very different predictions for tensor perturbations when compared to single field inflationary models, with the cyclic scenario producing a negligible level of gravitational waves with a blue tilted spectrum [25].

Until recently, however, little consideration was given to the evolution of vector perturbations. This is because these perturbations are generally assumed not to
be important in the inflationary scenario for two reasons. First, in an expanding FLRW universe first order metric vector perturbations decay and hence rapidly become insignificant. Secondly, once inflation has begun all matter except for the scalar field driving inflation is rapidly red shifted, with the result that the universe is effectively sourced solely by scalar field matter. This implies that vector perturbations play a minimal role since, as we shall see explicitly in the next section, a scalar field does not support vector perturbations at first order.

In a collapsing scenario, however, the situation is potentially very different, as has been discussed recently in an interesting study [26]. Indeed, the first objection is no longer valid, since during a collapsing phase first order vector perturbations grow and in principal this growth could have important observational consequences for the collapsing scenarios [26]. The second objection is, however, still valid for the collapsing scenarios mentioned above, as these are also sourced solely by scalar field matter. Noting this, the authors of [26] proceeded by considering vector perturbations in presence of pressureless dust. However, the more natural setting to consider perturbations in this context remains that of scalar fields, where first order vector perturbations are absent. In view of this, it is important to ask whether nonlinear vector perturbations can be supported in such regimes, and how they evolve during collapsing phases. This is the aim of the present paper.

We shall study the evolution of second order vector perturbations in a collapsing universe sourced by a scalar field. An important generic feature of nonlinear perturbations is that vector, tensor and scalar modes couple. Consequently, we expect second order vector perturbations to be produced, for example, by first order scalar mode-mode couplings. The analogous production of second order tensor perturbations has recently been studied [27, 28]. It is possible, therefore, that vector perturbations could provide a signature of a collapsing phase which would be absent or highly suppressed in an expanding universe. Indeed, the possibility that second order vector perturbations could act as seeds for large scale cosmic magnetic fields has previously been considered [29, 30], as has their contribution to the polarisation of the CMB [31].

The structure of the paper is as follows. In Section 2 we introduce the flat FLRW model in the presence of a scalar field, and consider perturbations about this background; subsection 2.1 reviews first order scalar and vector perturbations, and an expression for the second order vector perturbations is derived in subsection 2.2. In Section 3 we consider a scalar field self–interacting through an exponential potential, which allows a power–law solution to the cosmological evolution. By considering different power–law regimes we are able to study concrete examples of interest analytically, and calculate and compare the amplitude of vector perturbations in each case in Section 4. Finally, Section 5 contains our discussions and conclusions.

Throughout lower case Latin indices take values 0, 1, 2, 3 and Greek indices 1, 2, 3.
2. Background model and Perturbations

We consider as our background model a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric in the form
\[ ds^2 = g_{ab}^{(0)} dx^a dx^b = a(\tau)^2 (-d\tau^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta) , \]  
where \( \tau \) is the conformal time which is related to the coordinate time \( t \) through \( dt = ad\tau \).

We take the universe to be sourced by a scalar field \( \phi \) with a stress energy tensor given by
\[ T_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} \phi_{,c} \phi_{,c} - g_{ab} V(\phi) , \]
where \( V(\phi) \) is the associated scalar field potential. The background evolution equations are given by the Friedmann equation
\[ H^2 = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) , \]  
where \( H = \dot{a}/a \) is the Hubble parameter and a dot denotes differentiation with respect to the coordinate time \( t \), and the Klein-Gordon equation for the scalar field
\[ \ddot{\phi} + 3H \dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 . \]

Throughout we shall use the formalism developed in [22] in order to give the vector perturbations up to second order in a flat FLRW universe sourced by a scalar field.

In order to derive the perturbation equations we recall that the perturbed FLRW metric can, up to second order, be written in the usual form [22]
\[ g_{00} = -a^2 (1 + 2 (A^{(1)} + A^{(2)})) , \]
\[ g_{0a} = -a^2 (B_{a}^{(1)} + B_{a}^{(2)} ) , \]
\[ g_{\alpha\beta} = a^2 \left( g_{\alpha\beta}^{(0)} + 2 (C_{\alpha\beta}^{(1)} + C_{\alpha\beta}^{(2)}) \right) , \]
where \( g_{\alpha\beta}^{(0)} \) is the background 3-metric and the superscripts \((1)\) and \((2)\) denote first and second order quantities, respectively. The perturbation variables can be decomposed, at each order \( i \), as
\[ B_{\alpha}^{(i)} \equiv \beta_{,\alpha}^{(i)} + B_{\alpha}^{(vi)} , \]
\[ C_{\alpha\beta}^{(i)} \equiv \varphi_{,\alpha\beta}^{(i)} + \gamma_{,\alpha\beta}^{(i)} + C_{\alpha\beta}^{(vi)} + C_{\alpha\beta}^{(ti)} \]
with \( \partial^\alpha B_{\alpha}^{(vi)} = \partial^\alpha C_{\alpha\beta}^{(vi)} = \partial^\alpha C_{\alpha\beta}^{(ti)} = C_{\alpha\beta}^{(vi)} = 0 \). In this splitting, the variables \( \varphi^{(i)} \) and \( \gamma^{(i)} \) represent scalar perturbations, while vector and tensor perturbations are denoted by the superscripts \((v)\) and \((t)\), respectively.

To proceed we require a gauge, which we choose to be the Poisson gauge, defined by \( \beta^{(i)} = \gamma^{(i)} = C_{\alpha}^{(vi)} = 0 \). This gauge is a generalisation of the longitudinal gauge to include vector and tensor modes.

We also note that we do not use the energy frame here. Instead, we follow [22] and use the normal frame, in which the scalar field stress-energy tensor can be identified.
with the fluid stress-energy tensor in such a way that the velocity perturbations are related to the scalar field perturbations by \[22\]

\[
v^{(1)}_{\alpha} = -\frac{1}{a\dot{\phi}}\phi_{,\alpha},
\]

\[
v^{(2)}_{\alpha} = -\frac{1}{a\dot{\phi}}\left(\dot{\phi}_{,\alpha} + \phi_{,\alpha}(\ddot{\phi} + \dot{\phi}\dot{\phi})\right).
\]

We shall decompose the velocity perturbations as

\[
v^{(i)}_{\alpha} = \partial_{\alpha} v^{(i)} + v^{(vi)}_{\alpha},
\]

with \(\partial^\alpha v^{(i)}_{\alpha} = 0\). We shall assume \(C_{\alpha\beta}^{(1)} = 0\) throughout and, in the next section, we shall show that in our scalar field background we must also have \(v^{(v1)}_{\alpha} = B^{(v1)}_{\alpha} = 0\).

2.1. First order perturbations

The first order perturbation equations in the present model have been considered extensively by many authors. An important feature of such perturbations is that the evolution equations for scalar, vector and tensor perturbations decouple and hence can be studied separately.

Confining ourselves to the Poisson gauge, the metric perturbations \(A^{(1)}\) and scalar perturbations \(\phi^{(1)}\) satisfy the relation \(A^{(1)} = -\phi^{(1)}\) and their evolution equation becomes

\[
\ddot{\phi}^{(1)} + \left(\frac{H}{\dot{\phi}} - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\ddot{\phi}^{(1)} - \frac{\nabla^2 \phi^{(1)}}{a^2} + 2\left(\frac{\dot{H}}{\dot{\phi}} - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\phi^{(1)} = 0.
\]

Employing the field equations together with the above fluid–scalar-field identification, to first order, the scalar field perturbation can also be given in terms of \(\phi^{(1)}\) as

\[
4\pi G\dot{\phi}\phi^{(1)} = -(H\phi^{(1)} + \dot{\phi}^{(1)}).
\]

While considering first order scalar perturbations, we take the opportunity to define the curvature perturbations on comoving hypersurfaces \[32\]

\[
\zeta = \frac{2}{3a^2(1 + w)} \left(\frac{\phi^{(1)}}{a'/a^2}\right)'.
\]

This quantity is extremely useful since for an expanding universe it is conserved on super-horizon scales. Thus it is the amplitude and spectrum of these perturbations which are important for the comparison of theoretical predictions with observations.

We recall that the evolution equation for the first order vector perturbations is

\[
\nabla^2 \ddot{B}^{(v1)}_{\alpha} + 2H\nabla^2 B^{(v1)}_{\alpha} = 0,
\]

which admits solutions proportional to \(1/a^2\).

Now, the right hand side of equation \[10\] can be expressed as a gradient of a scalar, hence \(v^{(1)}_{\alpha}\) does not have a pure vector part, implying that a scalar field does not support a pure vector velocity perturbation at first order. We therefore have

\[
v^{(v1)}_{\alpha} = 0,
\]
in this case. Furthermore, the first order metric vector perturbations satisfy the momentum constraint equation

\[
\frac{\nabla^2 B^{(v1)}_\alpha}{2a^2} = -8\pi G (p + p)v^{(v1)}_\alpha .
\]  

(16)

In line with common practice we shall employ Fourier decomposition of the perturbations. Considering Eq. (16) together with Eq. (15) implies that \( B^{(v1)}_\alpha(k) = 0 \) for \( k \neq 0 \), where \( B^{(v1)}_\alpha(k) \) are the Fourier modes of the metric vector perturbations \( B^{(v1)}_\alpha \).

To proceed, therefore, we fix \( B^{(v1)}_\alpha = 0 \), so that at first order there are no metric vector perturbations present.

Finally, to complete our first order analysis we comment on tensor perturbations. Since we are primarily interested in the evolution of second order vector perturbations, our interest in first order tensor perturbations is limited to how their couplings may produce vector perturbations at second order. However, their effects are likely to be subdominant as compared to the scalar terms, so we shall ignore the tensor perturbations at first order. Thus equations (3)-(4) together with equation (12) and the expression (13) define a closed set of evolution equations for the background and the first order scalar perturbations in the Poisson gauge.

2.2. Second order vector perturbations

As was mentioned above, an important feature of perturbations at second and higher orders is that the evolution equations for scalar, vector and tensor modes couple. As a result, even in the absence of vector perturbations at first order, these modes can be generated at the second order by the scalar-scalar mode couplings as we shall see below.

Following [22] (see also [23]), and using the Poisson gauge, the evolution equation for second order vector perturbations is given by

\[
\frac{1}{2a} \nabla^2 (\dot{B}^{(v2)}_\alpha + 2HB^{(v2)}_\alpha) = 8\pi G \nabla_\beta \Pi^\beta_\alpha + \nabla_\beta N^{\beta}_{4\alpha} \\
- \nabla_\alpha \nabla^{-2} \nabla_\gamma (N^\beta_\gamma + 8\pi G \Pi^\beta_\gamma) ,
\]  

(17)

where

\[
\Pi^\alpha_\beta = \frac{1}{a^2} \left( \phi^{(1)}_{\alpha,\beta} \phi^{(1)}_{\gamma} - \frac{1}{3} \delta^{\alpha}_{\beta} \phi^{(1)}_{\gamma} \phi^{(1)}_{\gamma} \right) ,
\]  

(18)

and

\[
N^{\beta}_{4\alpha} = \frac{-1}{2a^2} \left( 2\phi^{(1)}_{\alpha,\gamma} - \frac{2}{3} \delta^{\alpha}_{\beta} \phi^{(1)}_{\gamma} \nabla^2 \phi^{(1)} + \phi^{(1)}_{\gamma,\beta} \phi^{(1)}_{\gamma} - \frac{1}{3} \delta^{\beta}_{\alpha} \phi^{(1)}_{\gamma} \phi^{(1)}_{\gamma} \right) .
\]  

(19)

Taking \( \nabla^2 \) of equation (17) we obtain

\[
\frac{1}{4a} \nabla^2 \nabla^2 (\dot{B}^{(v2)}_\alpha + 2HB^{(v2)}_\alpha) \\
= 4\pi G \left[ \nabla^2 \phi^{(1)}_{\alpha,\beta} \phi^{(1)}_{\gamma} - \phi^{(1)}_{\alpha,\beta} \nabla^2 \phi^{(1)} + \phi^{(1)}_{\gamma,\beta} \phi^{(1)}_{\gamma} \nabla^2 \phi^{(1)} - \phi^{(1)}_{\alpha,\beta} \nabla^2 \phi^{(1)} \right] \\
+ 2\nabla^2 \phi^{(1)}_{\alpha,\beta} \phi^{(1)}_{\gamma} - \phi^{(1)}_{\alpha,\beta} \nabla^2 \phi^{(1)} + \phi^{(1)}_{\gamma,\beta} \nabla^2 \phi^{(1)} .
\]  

(20)
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This equation can be solved by giving $\phi^{(1)}$, which can be extracted from the first order evolution equation (12), together with the zeroth order equations. A non-trivial solution to (17) exists and has the structure

$$B^{(v2)}_\alpha = \frac{C_\alpha(x)}{a^2} + \text{(inhomogeneous terms)}, \quad (21)$$

where $C_\alpha(x)$ are arbitrary spatial functions and by “inhomogeneous terms” we mean the terms generated by the inhomogeneous part of Eq. (20). This solution has to be compatible with the momentum constraint equation, which in the Poisson gauge takes the particularly simple form:

$$\frac{1}{2a} \nabla^2 B^{(v2)}_\alpha = -8\pi G a (\rho + p) (v^{(2)}_\alpha - \partial_\alpha \nabla^{-2} \partial^\beta v^{(2)}_\beta), \quad (22)$$

where the second order velocity perturbations, $v^{(2)}_\alpha$, are given by Eq. (11). We note that, in general, $v^{(2)}_\alpha \neq 0$ and more importantly that $v^{(v2)}_\alpha \neq 0$. This is a crucial difference with respect to the first order case for which $v^{(v1)}_\alpha = 0$ forbids the existence of first order vector perturbations.

Taking $\nabla^2$ of the equation (22) it can be rewritten using (11) as

$$\frac{1}{4a} \nabla^2 \nabla^2 B^{(v2)}_\alpha = (\dot{\varphi}^{(1)}_{,\alpha} \varphi^{(1)}_{,\beta} + \varphi^{(1)}_{,\alpha} \nabla^2 \dot{\varphi}^{(1)}_{,\beta} - \nabla^2 \varphi^{(1)}_{,\alpha} \varphi^{(1)}_{,\beta} - \varphi^{(1)}_{,\beta} \varphi^{(1)}_{,\alpha}) \quad (23)$$

$$- 4\pi G (\nabla^2 \dot{\varphi}^{(1)}_{,\alpha} + \dot{\varphi}^{(1)}_{,\beta} \varphi^{(1)}_{,\alpha} - \dot{\varphi}^{(1)}_{,\alpha} \nabla^2 \varphi^{(1)}_{,\beta} - \dot{\varphi}^{(1)}_{,\beta} \varphi^{(1)}_{,\alpha}).$$

The right hand side of this expression is in terms of first order scalar couplings which are in general non-zero. Furthermore, by direct substitution of Eq. (23) into Eq. (20), we find that for (20)-(21) and (23) to be made compatible we must have $C_\alpha(x) = 0$.

This demonstrates that, in general, scalar-scalar mode couplings can produce vector perturbations at second order, even though they are absent at the first order. It also demonstrates that these perturbations are completely determined by the behaviour of the first order scalar perturbations, since the $C(x)/a^2$ part of the solution (21) must be absent for a scalar field.

It is also important to recall that there are a number of results suggesting that vorticity is zero in scalar field settings (see e.g. [22, 23]). In our case, this is easy to demonstrate in the more familiar energy frame using the four velocity associated with the scalar field

$$u_\alpha = \frac{1}{N} \phi_{,\alpha},$$

where $N = |\phi_{,a} \phi^{,b}|^{1/2}$, which implies

$$\omega_{ab} := h^c_{[a} h^d_{b]} u_{d,c} = 0,$$

where $h_{ab} = g_{ab} + u_a u_b$. This is an exact statement and therefore valid at any perturbation order. So, we have a setting with non-zero second order vector perturbations and zero vorticity.

Now, to determine the behaviour of second order vector perturbations concretely, we need to consider particular examples of contracting universes. We shall do this in the next section.
3. Analytic solutions

In order to calculate the vector perturbations to second order we shall require solutions to the background and the first order equations. In general these can be obtained numerically. To proceed analytically, however, one needs to make assumptions. Concerning the background equations (3)-(4), it is well known that there exists an exact solution if the field is self-interacting through an exponential potential \[35\]. The result is a power-law solution in which the scale factor grows with cosmic time as

\[ a = a_0 \tau, \]

For these solutions \[H^2 \propto \dot{\phi}^2 \propto V(\phi)\] and hence the equation of state, \[w = \frac{p}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)},\] is a constant.

In both the standard inflationary scenario and scenarios involving a collapsing phase such as the ekpyrotic/cyclic scenario, it is reasonable to assume that the equation of state will be approximately constant for a significant period of their evolution, and hence the power-law solutions are a very powerful tool for studying the dynamics of these scenarios analytically. Moreover, with the background evolving in accordance with a power-law solution, there is a well known analytic solution to Eq. (12), which then allows the first order scalar perturbations in this case to be determined analytically \[36\].

The form of the potential which gives rise to the power-law behaviour \[a = a_0 \tau\] is

\[ V(\phi) = V_0 e^{-\sqrt{\frac{8\pi G}{r}} \frac{\phi}{a}}, \tag{24} \]

assuming that \(\phi\) is increasing. Changing to conformal time we have \[a = a_0 \frac{1}{\sqrt{r-\tau}}, \quad \mathcal{H} = \frac{1}{\sqrt{r-\tau}} \] and \[\phi' = \frac{1}{\sqrt{8\pi G}} \frac{\sqrt{r-\tau}}{1-r}.\] In the following subsections we shall only be interested in cases for which \(\tau\) is negative and increasing towards zero, with \(-\infty\) and zero representing the asymptotic past and future respectively. Using conformal time and taking Fourier transforms allows Eq. (12) to be written as

\[ \varphi^{(1)}(k, \tau)'' + 2 \left( \mathcal{H} - \frac{\theta''}{\theta'} \right) \varphi^{(1)}(k, \tau)' + k^2 \varphi^{(1)}(k, \tau) + 2 \left( \mathcal{H}' - \mathcal{H} \frac{\theta''}{\theta'} \right) \varphi^{(1)}(k, \tau) = 0, \tag{25} \]

where a prime denotes differentiation with respect to conformal time \(\tau\) and \(k = |k|\). This equation can be solved in terms of Bessel functions. A common procedure is to use the transformation \[\varphi^{(1)}(k, \tau) = \left( \frac{\phi'}{a} \right) u(k, \tau)\] to rewrite the above equation in the form

\[ u(k, \tau)'' + k^2 u(k, \tau) - \left( \frac{\theta''}{\theta} \right) u(k, \tau) = 0, \tag{26} \]

where \(\theta = \mathcal{H}/a\phi'\), and hence \[\frac{\theta''}{\theta} = r/(1-r)^2 \tau^{-2}.\] The solution to Eq. (26) is then readily given by

\[ u(k, \tau) = (-\tau)^{\frac{1}{2}} \left( a_k J_\nu(-k\tau) + b_k Y_\nu(-k\tau) \right), \tag{27} \]

where \(J_\nu\) and \(Y_\nu\) are Bessel functions of the first and second kind respectively, \(\nu = \frac{1}{2} \left( \frac{1}{1+r} \right)\) and \(a_k\) and \(b_k\) are arbitrary constants.

In all the scenarios we shall discuss, the perturbations represented by \(u(k, \tau)\) have their origin in quantum fluctuations which are normalised far inside the cosmological
horizon, but are pushed outside the horizon during the process of inflation or collapse. Inside the horizon this corresponds to $-k\tau \to \infty$, and therefore in this limit we must have $u(k, \tau) \approx \frac{i}{(2k)^{3/2}} e^{-ik\tau}$ in order to match to the Minkowski vacuum. This limit allows the free constants to be fixed and we arrive at

$$u(k, \tau) = \frac{\sqrt{\pi i}}{4k} \phi^{i\nu+\frac{1}{2}} e^{i\nu \frac{\pi}{2}} \sqrt{-\tau} H_\nu(-k\tau),$$

where $H_\nu = J_\nu(x) + iY_\nu(x)$ is a Hankel function. The limit corresponding to the modes pushed outside the horizon is given by $-k\tau \to 0$. Expanding the Hankel function in this limit using

$$J_\nu(x) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu - \frac{1}{\Gamma(\nu + 2)} \left(\frac{x}{2}\right)^{\nu+2} + O(x^{\nu+4}),$$

$$Y_\nu(x) = -\frac{1}{\pi} \left(\frac{x}{2}\right)^{-\nu} \left(\Gamma(\nu) + \frac{\Gamma(\nu - 1)}{4} x^2\right) + O(x^{4-\nu}),$$

we arrive at

$$\varphi^{(1)}(k, \tau) = C_r k^{\nu-1} (-\tau)^{\nu-\frac{3}{2} - \frac{i\pi}{2}} + D_r k^{\nu+1} (-\tau)^{\nu+\frac{3}{2} - \frac{i\pi}{2}} + O(\tau^{\nu+3-i\pi})$$

$$+ B_r k^{-\nu-1} (-\tau)^{-\nu+\frac{3}{2} - \frac{i\pi}{2}} + A_r k^{-\nu-1} (-\tau)^{-\nu+\frac{3}{2} - \frac{i\pi}{2}} + O(\tau^{3-\nu-i\pi}),$$

where $A_r = K_r \Gamma(\nu)$, $B_r = K_r \frac{1}{4} \Gamma(\nu - 1)$, $C_r = K_r \frac{i\pi}{\Gamma(\nu - 1)}$, $D_r = -K_r \frac{i\pi}{4 \Gamma(\nu + 2)}$, with $\mathcal{K}_r = \frac{\sqrt{\pi}}{i \pi} \frac{1}{a_0^{3/2}} \left(2^{\nu-\frac{5}{2}} e^{i(\nu+\frac{1}{2}) \frac{\pi}{2}} - \frac{1}{\sqrt{\pi} \sqrt{8\pi \alpha}}\right)$, are constants which are related to each other and depend on the initial data. This feature will play an important role in what follows.

With the exact solution for $\varphi^{(1)}$ at hand, we are now able to evaluate the second order vector perturbations $B_\alpha^{(2)}$ from Eq. [23], for a universe undergoing power–law evolution. A potentially important indicator to differentiate between collapsing models and expanding inflationary models could be sought in the ratio of the amplitudes of $B_\alpha^{(2)}$ between the expanding and contracting phases.

### 4. Amplitudes of vector perturbations in expanding and contracting phases

In this subsection we evaluate the amplitudes of $B_\alpha^{(2)}$ for the expanding and contracting phases. We restrict our attention to three scenarios which can be well modelled by the power-law solution introduced above. All these models can produce a nearly scale–invariant spectrum of first order scalar perturbations in a straightforward manner. The first order curvature perturbations $\zeta$ must have a spectrum of this form in order to be compatible with observations. Thus the scale–invariance cannot be used to distinguish between these scenarios. We shall instead study whether the resulting amplitudes of the second order vector perturbations can be used to distinguish between these models.

The first model we shall consider, $r \to \infty$, corresponds to the standard expanding inflationary scenario. As we shall see, this limit of $r$ gives rise to scale-invariant spectra for both $\varphi^{(1)}$ and curvature perturbations. The second model we shall consider is the ekpyrotic/cyclic model, in which a contracting phase with $r \to 0$ replaces the
inflationary epoch. This limit of $r$ also gives rise to a scale-invariant spectrum for the $\varphi^{(1)}$ perturbations, but not for the curvature perturbations $\zeta$. In the ekpyrotic/cyclic scenario part of the $\varphi^{(1)}$ perturbation is matched onto the curvature perturbation when the universe undergoes a bounce, and hence the observationally important quantity $\zeta$ has a scale-invariant spectrum after the bounce \[37, 38\]. Finally, we shall consider a dust-like contracting universe \[39, 40, 41\], which corresponds to a contracting universe with $r \rightarrow 2/3$. While it has been pointed out that, unlike the previous two cases, this solution is not a dynamical attractor \[42, 43\], it is still of considerable interest since $r = 2/3$ gives rise to a scale-invariant spectrum for the curvature perturbations. We note that the limiting values of $r$ we have discussed here are of course idealised. In a realistic setting $r$ would approach but not reach such limits. This is important to keep in mind as the solutions for $r = 0$ and $r = \infty$ are clearly ill defined.

A comparison of these three cases is given in Ref. \[43\]. We shall now consider the three cases in turn, and evaluate $B_\alpha^{(v2)}$ for each. We shall then proceed to calculate the ratios between the amplitudes of the second order perturbations $B_\alpha^{(v2)}$ in the inflationary case and each of the contracting cases respectively.

In what follows it is useful to define the constant quantities

$$\gamma_r = \frac{r}{1 - r} \quad \text{and} \quad \chi_r = \frac{1}{\sqrt{8\pi G}} \frac{\sqrt{2r}}{1 - r}.$$  

4.1. Expanding inflationary phase ($r \rightarrow \infty$):

This expanding phase is characterised by $r \rightarrow \infty$ (and $\nu = \frac{1}{2}$) \[43\]. Substituting in (30) we obtain

$$\varphi(k, \tau) = A_\infty k^{-\frac{3}{2}} + C_\infty k^{-\frac{1}{2}}(-\tau) + B_\infty k^{\frac{1}{2}}(-\tau)^2 + O(\tau^3).$$ (31)

For a perturbation to have a scale-invariant spectrum its Fourier components must be proportional to $k^{-3/2}$. Thus considering the expression above, we recover the well known result that as $\tau \rightarrow 0$, inflation produces a scale-invariant power spectrum for linear scalar perturbations.

To evaluate the amplitude of the second order vector perturbations we consider the expression (23). To proceed we shall employ the property that the Fourier transform $\hat{\varphi}(k, \tau) = \int \varphi(x, \tau) e^{-ikx} dx$ of the product $\varphi_1(x, \tau) \varphi_2(x, \tau)$ is equal to the convolution $\hat{\varphi}_1(k_1, \tau) * \hat{\varphi}_2(k_2, \tau) = \int \hat{\varphi}_1(k_2, \tau) \varphi_2(k_1 - k_2, \tau) dk_2$. So e.g. $\int (\nabla^2 \hat{\varphi}_\alpha - \varphi_\alpha \nabla^2 \varphi) e^{-ikx} dx = i \int k_2^2 (k_1 - k_2)(\varphi_{k_2} \hat{\varphi}_{k_1 - k_2} - \hat{\varphi}_{k_2} \varphi_{k_1 - k_2}) dk_2$. Then, by substituting (31) into (23) and using the Fourier transform we find

$$k^1 B^{(v2)} = 4i \left(1 - \frac{\gamma_\infty(\gamma_\infty + 1)}{4\pi G \chi_\infty^2}\right) A_\infty C_\infty$$
$$\times \int d^3k_2 F(k_1, k_2)(a_{k_2} c_{k_1 - k_2} - c_{k_2} a_{k_1 - k_2})$$
$$+ O(\tau),$$ (32)

where $F(k_1, k_2) = k_2^2 (k_1 - k_2) - [k_2 \cdot (k_1 - k_2)] k_2$, $a_k = k^{-\frac{3}{2}}$, $b_k = k^{\frac{1}{2}}$ and $c_k = k^{-\frac{1}{2}}$, with $k = |k|$.
4.2. Recollapsing phase \((r \to 0)\):

This recollapsing phase is characterised by \(r \to 0\) (and \(\nu = \frac{1}{2}\)). Substituting in (30) we obtain

\[
\varphi^{(1)}(k, \tau) = A_0 k^{-\frac{7}{2}} (-\tau)^{1} + C_0 k^{-\frac{1}{2}} + B_0 k^\frac{3}{2} (-\tau) + O(\tau^2) .
\] (33)

Thus we recover the well known result that the linear scalar perturbations grow (diverge) in a collapsing phase if the so-called decaying modes are taken into account. This in turn implies (from (23)) that the amplitudes of second order vector perturbations also grow during a collapsing phase, as the inverse square of the conformal time. We can also see that \(\varphi^{(1)}\) has a scale-invariant spectrum in the \(r \to 0\) limit.

In order to analyse this behaviour in more detail we substitute (33) into (23) and use the Fourier transform to obtain

\[
k_1^4 B^{(v2)} = 4i \left(1 - \frac{\gamma_0 (\gamma_0 - 1)}{4\pi G \chi_0^2} \right) \frac{A_0 C_0}{\tau^2} \times \int d^3 k_2 F(k_1, k_2)(c_{k_2} a_{k_1 - k_2} - a_{k_2} c_{k_1 - k_2}) + O(1) .
\] (34)

It is interesting to note that the form of Eq. (23) leads to a cancellation which results in the leading order time dependence in this case to be proportional to \(\tau^{-2}\) rather than \(\tau^{-3}\) as might be expected.

4.3. Recollapsing phase \((r = \frac{2}{3})\):

This dust-like collapsing phase is characterised by \(r = 2/3\) (and \(\nu = 5/2\)). Substituting in (31) we obtain

\[
\varphi^{(1)}(k, \tau) = A_2^\frac{1}{3} k^{-\frac{7}{2}} (-\tau)^{-5} + B_2^\frac{1}{3} k^{-\frac{1}{2}} (-\tau)^{-3} + O(\tau^{-\frac{3}{2}}) .
\] (35)

Again we find that the scalar perturbations \(\varphi^{(1)}(k, \tau)\) grow (diverge) in the collapsing approach to the singularity, but at a substantially different rate from the previous collapsing scenario. We shall see this has important consequences for the analysis of the next section. The second order vector perturbations in this case become

\[
k_1^4 B^{(v2)} = 8i \left(4 - \frac{4\gamma_2^2 - 17\gamma_2 \frac{4}{3} + 75}{4\pi G \chi_0^2} \right) \frac{A_2 B_2}{(-\tau)^9} \times \int d^3 k_2 F(k_1, k_2)(b_{k_2} a_{k_1 - k_2} - a_{k_2} b_{k_1 - k_2}) + O(\tau^{-4}) .
\] (36)

4.4. Comparison of expanding and recollapsing phases

As discussed above, a potentially important signature of a collapsing phase can be obtained by comparing the ratio between the amplitudes of the second order metric
vector perturbations of the collapsing and expanding phases. In this subsection we shall evaluate this ratio for both contracting scenarios.

Comparing the ekpyrotic collapsing ($r \to 0$) and the inflationary ($r \to \infty$) cases we find the ratio of the amplitudes of the second order vector perturbations in these cases to be

$$\frac{|B^{(v2)}_{\text{coll}}|}{|B^{(v2)}_{\text{exp}}|} = K_1 \frac{A_0 C_0}{A_\infty C_\infty} \frac{1}{\tau_{\text{coll}}^2} + O(1),$$

where $K_1$ is a constant in time, which shows that this ratio grows with the duration of the collapse phase as $1/\tau_{\text{coll}}^2$.

Comparing the dust-like collapsing ($r = 2/3$) and the inflationary ($r \to \infty$) cases we find the ratio of the amplitudes of the second order vector perturbations in these cases to be

$$\frac{|B^{(v2)}_{\text{coll}}|}{|B^{(v2)}_{\text{exp}}|} = K_3 \frac{A_2^4 B_2^4}{A_\infty C_\infty} \frac{1}{\tau_{\text{coll}}^9} + O(\tau_{\text{coll}}^{-4}),$$

where $K_3$ is a constant in time, which shows that the ratio in this case is radically different from the ekpyrotic case, growing with the duration of the collapse phase as $1/\tau_{\text{coll}}^9$.

Despite their usefulness these expressions are not, as they stand, sufficient to provide the complete information required for observational purposes. This is because we have not taken into account the constraints on the amplitude of the first order scalar perturbations. We shall proceed to implement these constraints in the next section.

4.4.1. Effects of imposing observational constraints from first order curvature perturbations: We have seen that in the scenarios which involve a collapsing phase the amplitude of the first order scalar perturbations $\varphi^{(1)}$ grows as the collapse proceeds. At the end of the collapse, however, the first order perturbations must have the correct amplitude to be compatible with observations. It is therefore important to study the consequences of demanding that at the end of the collapse the first order scalar perturbations produced are equal to those obtained from observations.

As was mentioned above, for expanding universes, and hence for inflation, the curvature perturbation on comoving hypersurfaces is conserved on super-horizon scales. This quantity is therefore used for comparison between theory and observations. In cosmologies with a collapsing phase we require the value of the curvature perturbation at the wavenumber corresponding to the largest scale on the CMB after the bounce to be equal to the required observational value, which is in turn equal to the value produced by a successful inflationary model. We note that since this study is only concerned with ratios we do not need to give the required observational value explicitly, and since the three scenarios considered here all produce the same spectral dependence we do not need to specify the wavenumber explicitly.

For the scaling scenarios considered here we have

$$\frac{a'}{a} = \left( \frac{r}{1-r} \right) \frac{1}{\tau}, \quad \frac{a''}{a'} = \left( \frac{r}{1-r} - 1 \right) \frac{1}{\tau}. $$
Thus in the expanding phase $r \to \infty$ we find, using Eq. (14), that
$$\zeta_{\text{exp}} = \frac{2}{3} \frac{1}{1 + w_{\text{exp}}} \left( C_\infty k^{-\frac{3}{2}} - B_\infty k^{\frac{3}{2}} (-\tau)^2 + O(\tau^3) \right),$$
while in the collapsing case $r \to 0$ we obtain
$$\zeta_{\text{coll1}} = \frac{2}{3} \frac{1}{1 + w_{\text{coll1}}} \left( \frac{1 - r}{r} \right) \left( C_0 k^{-\frac{1}{2}} - 2 B_0 k^{\frac{7}{2}} (-\tau) + O(\tau^2) \right).$$
Finally in the $r \to 2/3$ collapse case we find
$$\zeta_{\text{coll2}} = \frac{2}{3} \frac{1}{1 + w_{\text{coll2}}} \left( B_2 k^{-\frac{3}{2}} (-\tau)^{-3} + O(\tau^{-\frac{3}{2}}) \right).$$
Considering the expression (40) for $\zeta_{\text{coll2}}$, we recover the result that the collapsing scenario with a dust like equation of state produces scale-invariant curvature perturbations.

In the ekpyrotic scenario $\varphi^{(1)}$ is responsible for the curvature perturbation after the bounce. The argument used is that $\varphi^{(1)}$ and $\zeta$ mix at the bounce and part of $\varphi^{(1)}$ at the end of the collapse is matched onto the curvature perturbations after the bounce. We therefore equate, at the lowest order, the perturbations
$$\zeta_{\text{exp}} = \psi \varphi_{\text{coll1}},$$
where $\psi$ represents the proportionality factor between the two phases in the ekpyrotic scenario [37, 9]. This results in the constraint
$$C_\infty = \frac{3}{2} \frac{(1 + w_{\text{exp}})}{A_0 \psi} \frac{1}{\tau_{\text{coll1}}},$$
which can be used to obtain the ratio between the amplitudes of the second order vector perturbations in the collapsing and expanding phases from (37)
$$\frac{|B^{(v2)\text{coll1}}|}{|B^{(v2)\text{exp}}|} = K_1 \left( \frac{4/3}{1 + w_{\text{exp}} \psi} \right)^2 + O(\tau_{\text{coll1}}^2).$$
This is a constant at the lowest order (independent of the length of the collapsing phase) thus indicating that the important factor determining the ratio is the proportionality factor $\psi$ in the ekpyrotic scenario, which is determined by the physics of the bounce.

In the case of $r \to 2/3$ collapsing scenario, however, the curvature perturbations survive through the bounce and are thereafter conserved. Hence at the lowest order we can equate
$$\zeta_{\text{exp}} = \zeta_{\text{coll2}},$$
which implies
$$C_\infty = B_2 \frac{1}{3} \left( \frac{1 + w_{\text{exp}}}{1 + w_{\text{coll2}}} \right) \frac{1}{(-\tau)^3},$$
and from (38)
$$\frac{|B^{(v2)\text{coll2}}|}{|B^{(v2)\text{exp}}|} = 12 K_3 \left( \frac{1 + w_{\text{coll2}}}{1 + w_{\text{exp}}} \right)^2 \frac{1}{\tau_{\text{coll2}}^3} + O(\tau_{\text{coll2}}^2).$$
Therefore, at the lowest order, the ratio of the amplitudes of the second order vector perturbations is proportional to $\tau_{\text{coll2}}^{-3}$, which increases as the collapse time tends to 0. Hence for a long collapsing phase we expect this ratio to become very large.
5. Conclusions

A great deal of effort has recently gone into constructing alternative models to the standard inflationary scenario motivated by developments in string/M-theory. Many of these models involve a contracting phase. As a step towards distinguishing these models from the standard inflationary scenario we have studied the generation and evolution of vector perturbations in collapsing phases, in the context of scalar field cosmologies.

There are a number of reasons why vector perturbations might provide useful signatures in models with contracting phases. Such perturbations are highly suppressed by inflation and are extremely difficult to generate at the early epochs after inflation. However, they would grow if produced during collapsing phases.

Noting that first order vector perturbations cannot be supported by regimes purely sourced by a scalar field, we have considered second order vector perturbations. We have found that such perturbations can be generated by mode-mode couplings of the first order scalar perturbations and have derived their explicit dependence on first order perturbations. In principle our expressions allow the spectral dependence of second order vector perturbations to be determined, though in this work we have focused on their temporal behaviours in various early universe scenarios. Considering exponential potentials, which allow power-law solutions, we have studied the ratio of the amplitudes of second order vector perturbations in contracting and expanding phases. We have found that ignoring the details of how the hot big bang is recovered, the relative magnitudes of the second order vector perturbations in the two phases depend on the scaling solutions chosen. In particular we have considered two collapsing models, the first motivated by the ekpyrotic/cyclic scenario and the second by the dust-like collapsing scenario given in Ref. [39, 40].

For the first case, we have found the ratio to be independent of the length of collapse, at the lowest order, while depending on the details of the matching of the perturbations at the bounce. This is counter to the expectation that vector perturbations should be more prominent after a longer collapsing phase. This result is a consequence of the fact that the vector perturbations in this case grow as the square of the rate at which the scalar perturbations $\varphi^{(1)}$ grow, which in turn is a consequence of the cancellation mentioned in section 4.2. In this case, $\varphi^{(1)}$ is responsible for the curvature perturbations after the bounce. Hence, once the observational constraint, namely that the curvature perturbations produced by this collapsing phase must have the same amplitude as the curvature perturbations produced by inflation, is imposed, the ratio of vector perturbations becomes fixed, independently of the collapse time.

For the second case, we find that the ratio increases with the length of the collapse, being proportional to $\tau_{coll}^{-3}$. This result is due to the fact that the second order vector perturbations grow more rapidly during the collapse than the square of the first order curvature perturbations. This implies that the magnitude of vector perturbation is no longer limited to being of the order of the square of the first order perturbations, but could grow to be larger. Hence, even when the constraint on the first order curvature
perturbations is imposed, there is still a dependence on the collapse time. This implies that the amount of vector perturbations present at the end of the collapsing phase could, in principal, be much larger than that present at the end of the inflationary epoch, since $\tau \to 0$ as the collapse proceeds.

It is important to note that the amplitude of the vector perturbations, $B_\alpha^{(12)}$, calculated here is only valid if the power-law solutions of section 3 are applicable. These solutions are clearly only approximate and will not be valid through the entire inflationary or collapsing phases considered here. In particular, these behaviours must break down as the universe exits these phases and enters the hot big bang phase of evolution. In the inflationary scenario it is usual to assume that this occurs when the potential evolves such that it no longer gives rise to accelerated expansion, after which reheating occurs. In the collapsing scenarios the power–law evolution is expected to break down in the vicinity of the bounce, which is believed to lead to a radiation dominated phase. In principal, therefore, we would like to know how vector perturbations are affected by these transitions. This would, however, almost certainly require a numerical investigation, beyond the scope of the present study.

In conclusion, a complete understanding of second order vector perturbations from a collapsing phase and their possible observational consequences would require a more detailed understanding of how such perturbations propagate between the scalar field phase and the hot big bang phase. However, by using power-law solutions we can compare vector perturbations produced in collapsing scenarios with those obtained in the inflationary scenario at a point immediately before the transition into the hot big bang phase. Our study of the two collapsing scenarios indicates that the observable differences between the collapsing models and the inflationary scenario could be large, particularly for the dust–like collapse, if we assume that the transition we have just discussed does not significantly alter the ratios we have calculated.

Acknowledgements

We would like to thank Roy Maartens, Karim Malik, Kouji Nakamura and David Wands for helpful discussions. FCM is supported by CMAT - Univ. Minho and FCT (Portugal) through grant SFRH/BPD/12137/2003. DJM is supported by the Centre for Theoretical Cosmology, Univ. Cambridge. FCM and RT thank TIFR and IUCAA, India, for kind hospitality where part of this work was done. FCM and DJM thank the Astronomy Unit, Queen Mary, Univ. London, for hospitality while most of this work was done.

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Non-linear vector perturbations in a contracting universe