1/f fluctuations in spinning-particle motions around Schwarzschild black hole

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We study the properties of chaos in spinning-particle motions in Schwarzschild spacetime. We characterize the chaos in the motions using the power spectrum. We discover that the power spectrum shows not only white noise but also 1/f-type fluctuation, depending on the value of the spin and the angular momentum of the test particle. We obtain the phase diagram for the properties of the chaos. Furthermore, we suggest that the origin of the 1/f fluctuations is the “stagnant motions,” itinerating among regular orbits (tori).

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I. INTRODUCTION

Nature is filled with phenomena that exhibit chaotic behavior. In the chaotic systems we cannot predict the state in the future exactly [1, 2]. Such chaotic behaviors have also been found in some relativistic systems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. For example, in Schwarzschild spacetime, the motions of a spinning test particle can be chaotic [13]. If the test particle does not have spin, the motion of the test particle is regular. Researchers have found that, as the magnitude of the spin increases, the motions switch from regular to chaotic. These motions, however, have been classified merely as regular or chaotic according to the Lyapunov exponents, and the details of the properties of the chaotic motions have not been clarified.

In this paper, we look for statistical laws in the chaotic motions in the above system.
Once we find chaotic behaviors, we should characterize the chaos to extract the specific properties of the system. We can hardly learn anything about the chaos if we judge it only from the positiveness of the Lyapunov exponents. Not a few people believe that chaotic system is simply complex and completely unpredictable. Of course, we cannot predict the time evolution of the state of the spinning test particle exactly, when its system is chaotic. However, even in such cases, we can frequently find some statistical laws which are proper to the system. One measure of the chaos is the power spectrum of the time series. In some cases, the pattern of the power spectrum obeys power laws, so-called $1/f$ fluctuations, which can be distinguished from the white-noise type. That the power spectrum obeys some power laws means that the time evolution of the system is time-correlated. In other words, we can classify the chaos by the pattern of its power spectrum.

In this paper, we will characterize the properties of the chaos in the spinning-particle motions in Schwarzschild spacetime, using the pattern of the power spectrum of the time series. The motions of a test particle that does not escape to infinity and does not fall into a black hole are considered. We will find that the pattern of the power spectrum can be classified into two types, $1/f$ and white noise. We will obtain the phase diagram for the properties of the chaos, the types of the power spectrum. We will detect the origin of the $1/f$ fluctuations of the power spectrum in this system.

This paper is organized as follows. In Sec. II we shall briefly review the basic equations, i.e., the equations of motion for a spinning test particle in Schwarzschild spacetime. In Sec. III we characterize the properties of the chaos in this system using the power spectrum. In Sec. IV we detect the origin of the $1/f$ fluctuations. The final section is devoted to summary and discussion.

II. EQUATIONS FOR A SPINNING TEST PARTICLE IN SCHWARZSCHILD SPACETIME

We consider a spinning test particle in Schwarzschild spacetime,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$

(1)

where $M$ is the mass of the black hole. The equations of motions of a spinning test particle in a relativistic spacetime have been derived by Papapetrou and then reformulated by
Dixon [23]. The set of equations is given as

\[
\frac{dx^{\mu}}{d\tau} = v^{\mu},
\]

(2)

\[
\frac{Dp^{\mu}}{d\tau} = -\frac{1}{2} R^{\mu}_{\nu\rho\sigma} v^{\nu} S^{\rho\sigma},
\]

(3)

\[
\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu}v^{\nu]},
\]

(4)

where \(\tau, v^{\mu}, p^{\mu}, \) and \(S^{\mu\nu}\) are an affine parameter of the orbit, the four-velocity of a particle, the momentum, and the spin tensor, respectively. \(p^{\mu}\) deviates from a geodesic due to the coupling of the Riemann tensor with the spin tensor. We adopt the additional condition formulated by Dixon [23],

\[
p_{\mu} S^{\mu\nu} = 0,
\]

(5)

which gives a relation between \(p^{\mu}\) and \(v^{\mu}\), and consistently determines the center of mass of the spinning particle. The mass of the particle \(\mu\) is defined by

\[
\mu^2 = -p_{\mu}p^{\mu}.
\]

(6)

To make clear the freedom of this system, we have to check the conserved quantities. Regardless of the symmetry of the background spacetime, it is easy to show that the mass \(\mu\) and the magnitude of spin \(S\) defined by

\[
S^2 \equiv \frac{1}{2} S_{\mu\nu}S^{\mu\nu}
\]

are constants of motion [24]. If a background spacetime possesses some symmetry described by a Killing vector \(\xi^{\mu}\),

\[
C_{\xi} \equiv \xi^{\mu}p_{\mu} - \frac{1}{2} \xi_{\mu\nu}S^{\mu\nu}
\]

(8)

is also conserved [23]. Because the spacetime we consider in this paper is static and spherically symmetric, there are two Killing vector fields, \(\xi^{\mu}_{(t)}\) and \(\xi^{\mu}_{(\phi)}\). From (8), we find the constants of motion related with those Killing vectors as

\[
E \equiv -C_{(t)} = -p_{t} - \frac{M}{r^2} S^{tr},
\]

(9)

\[
J_z \equiv C_{(\phi)} = p_{\phi} - r(S^{\phi\rho} - rS^{\phi\theta} \cot \theta) \sin^2 \theta.
\]

(10)

\(E\) and \(J_z\) are interpreted as the energy of the particle and the \(z\) component of the total angular momentum, respectively. Because the spacetime is spherically symmetric, the \(x\)
and $y$ components of the total angular momentum are also conserved. In addition, without loss of generality we can choose the $z$ axis in the direction of total angular momentum as

$$(J_x, J_y, J_z) = (0, 0, J), \quad (11)$$

where $J > 0$. In the following sections, we integrate the above equations of motion numerically for various values of parameters $E$, $J$, and $S$, using the Bulirsch-Stoer methods [25].

### III. $1/f$ FLUCTUATIONS OF THE POWER SPECTRUM

In this section, we characterize the chaos in the spinning particle motions using the power spectrum. Here we pay attention to the chaotic motions of a test particle that does not escape to infinity and does not fall into a black hole.

First, in Fig. 1 we plot the Poincaré maps with different sets of parameters. To draw the Poincaré map, we adopt the equatorial plane ($\theta = \pi/2$) as a Poincaré section and plot the point $(r, v^r)$ when the particle crosses the Poincaré section with $v^\theta < 0$. If the orbit is chaotic, some of the tori are broken and the Poincaré map no longer consists of a set of closed curves. As shown in Fig. 1, the orbits with either set of parameters are chaotic, and we cannot distinguish them apparently. Dots with different colors correspond to the data from the orbits with different initial conditions.

Next we plot the power spectrum of the time series of $z$ components of the particle in Fig. 2. In Figs. 2(a) and (b), we choose parameter sets with as the same values as those in Figs. 1(a) and (b), respectively. Each line color in Fig. 2 also corresponds to that of the dots in Fig. 1. From Fig. 2 we find that one is $1/f^\nu$ where $\nu \simeq 1.2$ (Fig. 2(a)), while the other is white noise (Fig. 2(b)). Therefore we can distinguish them clearly using the pattern of the power spectrum. That is, the properties of the chaos depends on the system parameters, the angular momentum, and the spin of the test particle. In addition, the pattern of the power spectrum is independent of the initial conditions. A preliminary result is summarized in the phase diagram in Fig. 3. Roughly speaking, the power spectrum changes from $1/f$-type to white noise by increasing the magnitude of spin of the test particle. It is proper that the strength of the value of the spin of the test particle corresponds to the strength of the chaos. Therefore, our results can be interpreted as to mean that the pattern of the power spectrum
changes from $1/f$ type to white noise as the strength of the chaos increases.

IV. ORIGIN OF THE $1/f$ FLUCTUATIONS IN THE POWER SPECTRUM

In this section, we investigate the origin of the $1/f$ fluctuations we found in the previous section. First, we plot the time series of the $v_r$ components in the Poincaré maps (Fig. 4). We choose parameter sets in Figs. 4(a) and (b) as with the same values as those in Figs. 1(a) and (b), respectively. In Fig. 4(a) the values of the $v_r$ components fluctuate with stagnation. It seems that the duration of the stagnations varies from time to time, and are a mix of various lengths, and there is no typical one. This feature is observed commonly when the power spectrum shows the $1/f$ type. Then we expect that this behavior is related to the $1/f$ fluctuations in the power spectrum in Fig. 2(a). In Fig. 4(b), on the other hand, the values of the $v_r$ components oscillate almost monotonously. This behavior is consistent with that the power spectrum shows white noise in Fig. 2(b).

Next, we plot the same orbit in Fig. 4(a) in the two-dimensional configuration space ($r, z$) in Fig. 5. We find that the whole orbit (Fig. 5(a)) has two significant components (Figs. 5(b) and (c)). The time when the orbit stagnates around each component corresponds to the stagnation in Fig. 4(a). We never see such stagnant motions when the power spectrum is white noise (Fig. 4(b)). These results suggest strongly that the $1/f$ fluctuations in the power spectrum are originated by the stagnation between these distinguishable components.

V. SUMMARY AND DISCUSSION

In this paper we have investigated the properties of chaos in a spinning test particle in Schwarzschild spacetime. We have calculated the power spectrum of the time series of $z$ components of the test particle’s position. We have found that the pattern of the power spectrum is $1/f$ or white noise. The important point is that the pattern depends on the spin $S$ and the angular momentum $J$ of the test particle, and not depend on the initial conditions. We have obtained the phase diagram for the character of the chaos, the type of the pattern of the power spectrum. To put it another way, we can guess the properties of the system ($S$ and $J$) by observing the dynamics of the test particle, even if the motion is chaotic.
Furthermore we have detected the origin of the $1/f$ fluctuations we have found for the motion of the spinning test particle in this system. When the power spectrum is the $1/f$ type, we have found that the orbit (Fig. 5(a)) stagnates around two significant components (Figs. 5(b) and (c)). This type of motion where the phase point in chaotic orbit stays close to some regular orbits (tori) for some long time is known as “stagnant motion,” and is often observed in Hamiltonian dynamical systems [26, 27, 28]. Stagnant motions are usually accompanied by $1/f$ fluctuations and are considered to be due to the fractal structure of the phase space [29, 30, 31]. In particular, stagnant motions are often observed for weakly chaotic, nearly integrable systems. This is consistent with the fact that we have observed the $1/f$ fluctuations for smaller values of spin $S$ when the chaos is weak and white noise for larger values of spin $S$ when the chaos is strong. Until now such $1/f$ fluctuations had never been discovered in any relativistic systems. We expect that the $1/f$ fluctuations we observed for the first time in the relativistic system are also generated by such stagnant motions, itinerating among regular orbits (tori) in Schwarzschild spacetime.

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FIG. 1: The Poincaré maps. All orbits have the total angular momentum $J = 4.0\mu M$. The magnitude of spin and the total energy are $S = 1.2\mu M$ and $E = 0.93545565\mu$ in the panel (a), and $S = 1.4\mu M$ and $E = 0.92292941\mu$ in the panel (b), respectively. The dots with different colors correspond to the data from the orbits with different initial conditions.

FIG. 2: The power spectrum of the time series of \( z \) components of the particle. Each set of the parameters and the initial conditions in the panels (a) and (b) are same as Figs. 1(a) and (b), respectively.

FIG. 3: The phase diagram for the type of the pattern of the power spectrum. The spectrum is \( 1/f \) or white noise in the bottom or middle regions, respectively. The energy surface is unbounded in the upper region.

FIG. 5: The orbit in Fig. 4(a) in the two-dimensional configuration space \((r, z)\). (a) The orbit for the whole period \( 0 < t < 50000 \). (b) The orbit for the period \( 4000 < t < 11000 \). (c) The orbit for the period \( 24000 < t < 28000 \).

FIG. 4: The time series of the \( v_r(t) \) components of the Poincare map. Parameter sets in the panels (a) and (b) are chosen as with the same values as those in Figs. 1(a) and (b), respectively.
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