Constructing the Leptonic Unitarity Triangle

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Abstract

Following analogy of the ‘db’ triangle in the quark mixing case, we have constructed the ‘ν₂,ν₃’ leptonic unitarity triangle using the MNS matrix formulated in the tribimaximal scenario by Bjorken et al. In particular, for the $U_{e3}$ range 0.05 – 0.15, we find Dirac-like CP violating phase $\delta$ to be around 40°, indicating a 2.5 $\sigma$ CP violation effect in the leptonic sector.

In the last few years, apart from establishing the hypothesis of neutrino oscillations, impressive advances have been made in understanding the phenomenology of neutrino oscillations through solar neutrino experiments [1], atmospheric neutrino experiments [2], reactor based experiments [3] and accelerator based experiments [4] enabling the determination of the basic form of the MNS leptonic mixing matrix [5]. At present, one of the key issues in the context of neutrino oscillation phenomenology is to investigate the existence of CP violation in the leptonic sector.

Taking clue from the construction of the unitarity triangle in the quark sector [6], several attempts [7, 8, 9] have been made to construct the corresponding unitarity triangle in the leptonic sector. Farzan and Smirnov [8] have discussed, in detail, the desirability of investigating the construction of leptonic unitarity triangle for finding possible clues to the existence of CP violation in the leptonic sector. In particular, considering the ‘e-μ’ triangle, corresponding to the first two rows of leptonic mixing matrix, they have examined in detail the implications of different values of CP violating phase $\delta$ on the possible accuracy required in the measurement of various oscillation probabilities. Very recently, Bjorken et al. [9], by considering tribimaximal scenario, have not only constructed a MNS matrix but have also proposed a unitarity triangle, referred to as ‘ν₂,ν₃’, which could be leptonic analogue of the much talked about ‘db’ triangle in the quark sector.

In view of the importance of the tribimaximal mixing scenario [10, 11], it would be interesting to investigate, in detail, the structure of the leptonic unitarity triangle suggested by this scenario. In particular, it would be very much desirable, as a complimentary approach to the scenario investigated by Farzan and Smirnov [8], to find the possible values of CP violating phase $\delta$ suggested by tribimaximal scenario of Bjorken et al. [9].
To this end, taking clues from ‘$db$’ unitarity triangle in the quark sector, the purpose of the present paper is to explore the possibility of the construction of the leptonic unitarity triangle as well as the existence of CP violation in the leptonic sector. In particular, by considering different values of $U_{e3}$, having implications for various theoretical models, in the matrix constructed by Bjorken et al. [9] as well as by considering reasonable deviations from the tribimaximal scenario, we have explored in detail the probability of finding a non-zero value of the Jarlskog’s rephasing invariant parameter $J_l$ and the related Dirac-like CP violating phase $\delta$.

For ready reference as well as to facilitate discussion of results, we begin with the neutrino mixing phenomenon, often expressed in terms of a $3 \times 3$ neutrino mixing MNS matrix [5] given by

$$U_{\text{MNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (2)

The Majorana phases $\alpha_1$ and $\alpha_2$ do not play any role in neutrino oscillations and henceforth would be dropped from the discussion.

The constraints due to unitarity of the MNS matrix can be defined as

$$\sum_{\alpha=1,2,3} V_{i\alpha} V^*_{j\alpha} = \delta_{ij},$$ \hspace{1cm} (3)

$$\sum_{i=e,\mu,\tau} V_{i\alpha} V^*_{i\beta} = \delta_{\alpha\beta},$$ \hspace{1cm} (4)

where Greek indices run over the mass eigenstates (1, 2, 3) and Latin ones run over the flavor eigenstates ($e, \mu, \tau$). Unitarity implies nine relations, three in terms of normalization conditions, given by equation (3), and the six non-diagonal relations, given by equation (4), also expressed through the six unitarity triangles in the complex plane.

For getting viable clues to the construction of the leptonic unitarity triangle, we first consider the case of quarks wherein the CKM matrix [12] is fairly well established as well as the CP violating phase $\delta$ has also been measured recently [13]-[16]. To begin with, we consider the quark mixing matrix given by PDG 2006 [13] and attempt to reconstruct the CP violating phase $\delta$ using the Jarlskog’s rephasing invariant parameter $J$, equal to twice
the area of any of the unitarity triangle, through the relation

$$J = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin \delta.$$  (5)

In this context, we consider the usual ‘$db$’ triangle, expressed as

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 .$$  (6)

For this triangle, assuming Gaussian probability density distribution for the CKM matrix elements and by using Monte Carlo simulations for calculating the area of the unitarity triangle, the Jarlskog’s rephasing invariant parameter $J$ comes out to be

$$J = (3.03 \pm 0.373) \times 10^{-5} ,$$  (7)

which on using equation (5), yields

$$\delta = 55.42^\circ \pm 10.03^\circ ,$$  (8)

the corresponding distribution of $\delta$ is shown in figure 1. Interestingly, we find that the above mentioned $J$ value has an excellent overlap with that found by PDG group through their recent global analysis [13]. Also, this value of $\delta$ is fully compatible with the experimentally determined $\delta$ given by PDG 2006 as well as found by some of the most recent analyses [13]-[16]. It may be mentioned that in the figure we have considered only those points for which $\delta \neq 0$. The above values of $\delta$ has been found by fitting a Gaussian to the respective figure.

The above discussion immediately provides a clue for extracting the probability of non zero CP violating phase $\delta$ in the leptonic sector, even when leptonic mixing matrix is approximately known. In this context, it may be mentioned that unlike the quark mixing matrix which is strongly hierarchical in nature, the leptonic mixing matrix is very different, with one of the mixing angle almost maximal, the other being quite large and the third much smaller compared to these. As the two mixing angles are rather large, therefore the nine unitarity conditions allow a fairly reasonable determination of the mixing matrix, even though in the case of $\theta_{13}$ only upper limit is known. To this end, we consider the leptonic mixing matrix recently constructed by Bjorken et al. [9], given by

$$U \simeq \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right).$$  (9)

Out of the six triangles defined by equations (3) and (4), the ‘$\nu_2, \nu_3$’ triangle, considered by Bjorken et al. [9], which is the leptonic analogue of the ‘$db$’ triangle of the quark sector is expressed as follows

$$U_{e2}U_{e3}^* + U_{\mu 2}U_{\mu 3}^* + U_{\tau 2}U_{\tau 3}^* = 0 .$$  (10)

The above triangle can immediately be constructed in the scenario considered by
Bjorken et al. [9] in case one has definite clues about $U_{e3}$. In this regard, it is very well recognized that the value of $U_{e3}$ would have deep implications for the neutrino oscillation phenomenology [17]-[21]. In particular, very recently Albright et al. [25] have carried out a very detailed and exhaustive analysis wherein they have studied the implications of the values of $U_{e3}$ on various leptonic and grand unified models of neutrino masses and mixings. Keeping this in mind, we have chosen a few representative values which cover most of these attempts.

Broadly speaking, theoretical implications result in $U_{e3}$ taking values around 0.05, 0.10 and 0.15 which are being used for the construction of the MNS matrix and then the unitarity triangle along with the CP violating phase $\delta$. To have realistic estimates of the leptonic unitarity triangle, following Farzan and Smirnov [8], we have also attached modest errors to the mixing elements considered by Bjorken et al. [9]. In this context, we associate 5% errors with the elements $U_{e1}$, $U_{e2}$, $U_{\mu 2}$ and $U_{\tau 2}$ of the matrix given in equation (9) and for $U_{e3}$ we have taken values around 0.05, 0.10 and 0.15 and considered 5% and 10% variations to these. In all we have constructed 6 matrices, however here we present those obtained by attaching 10% errors to the above mentioned $U_{e3}$ values. The matrices corresponding to $U_{e3}$ values $0.05 \pm 0.005$, $0.10 \pm 0.01$ and $0.15 \pm 0.015$ are respectively as follows

$$U = \begin{pmatrix} 0.8165 \pm 0.0040 & 0.5774 \pm 0.0289 & 0.05 \pm 0.005 \\ 0.4516 \pm 0.0022 & 0.5774 \pm 0.0289 & 0.6821 \pm 0.0034 \\ 0.3649 \pm 0.0018 & 0.5774 \pm 0.0289 & 0.7321 \pm 0.0037 \end{pmatrix}, \quad (11)$$

$$U = \begin{pmatrix} 0.8165 \pm 0.0040 & 0.5774 \pm 0.0289 & 0.1 \pm 0.01 \\ 0.4948 \pm 0.0049 & 0.5774 \pm 0.0289 & 0.6571 \pm 0.0066 \\ 0.3216 \pm 0.0032 & 0.5774 \pm 0.0289 & 0.7571 \pm 0.0076 \end{pmatrix}, \quad (12)$$

$$U = \begin{pmatrix} 0.8165 \pm 0.0040 & 0.5774 \pm 0.0289 & 0.15 \pm 0.015 \\ 0.5382 \pm 0.0081 & 0.5774 \pm 0.0289 & 0.6321 \pm 0.0095 \\ 0.2783 \pm 0.0042 & 0.5774 \pm 0.0289 & 0.7821 \pm 0.0117 \end{pmatrix}. \quad (13)$$

It may be of interest to mention, unlike the CKM matrix, even though $U_{e3}$ is much smaller as compared to the other two angles yet we find that changes in $U_{e3}$ affects the $U_{\mu 1}$, $U_{\mu 3}$, $U_{\tau 1}$ and $U_{\tau 3}$ elements of the MNS matrix in a noticeable manner.

In the present analysis, following the quark mixing analogy, we attempt to calculate the most probable range of $\delta$ keeping in mind the present uncertainties regarding the mixing angle $U_{e3}$. To this end, considering the elements of the above matrices appearing in the '\nu_2, \nu_3' triangle, expressed in equation (10), to be Gaussian and following the same procedure as in the quark case, we obtain the corresponding respective values of $J_l$ as

$$J_l = 0.0094 \pm 0.0028, \quad (14)$$

$$J_l = 0.0169 \pm 0.0059, \quad (15)$$

$$J_l = 0.0225 \pm 0.0086. \quad (16)$$
It may be noted that similar calculations can also be carried out using the matrices obtained by giving 5% variations to $U_{e3}$, however, the corresponding values of $J_l$ are not much different from the ones given in the above equations. Further, we would like to add that the value of $J_l$ found here from the distribution of the area of unitarity triangle is fully compatible with the value found by Bjorken et al. (equation 15 of [9]) for different values of $U_{e3}$. Using these values of $J_l$ and by considering various elements of equation (5) to be Gaussian, one can find the corresponding distributions of $\delta$. Using these distributions, shown in figure 2, the $\delta$ values corresponding to $U_{e3}$ values $0.05 \pm 0.005, 0.10 \pm 0.01$ and $0.15 \pm 0.015$ are respectively as follows

$$\delta = 46.56^\circ \pm 14.79^\circ, \quad (17)$$
$$\delta = 42.46^\circ \pm 15.54^\circ, \quad (18)$$
$$\delta = 38.58^\circ \pm 15.34^\circ. \quad (19)$$

Since the input values have been taken by including reasonable errors to various matrix elements, therefore the calculated value of $\delta$ can be considered a fair estimate of likely CP violation in the leptonic sector.

The above calculated values of $\delta$, indicating more than $2\sigma$ deviation from $0^\circ$, in the tribimaximal scenario for different values of $U_{e3}$ leads to several interesting points. The above non zero values of $\delta$ are in agreement with the suggestions of Marciano and Parsa [26] as well as of Giunti and Tanimoto [27] that large value of $J_l$ in comparison to the quark sector implies the possibility of non zero CP violation in the leptonic sector. This also suggests that within the tribimaximal scenario, it looks that any reasonable value of $U_{e3}$, predicted by most of the models considered by Albright et al. [25], would lead to the existence of CP violation in the leptonic sector. Interestingly, the CP violating phase comes out to be around $40^\circ$ which is not much sensitive to variation in the value of $U_{e3}$ in the range $0.05 - 0.15$, as also emphasized by Marciano and Parsa [26]. Further, this suggests that while investigating the tribimaximal scenario in the future experiments, it would be desirable to include the implications of CP violating phase $\delta$ around $40^\circ$ in the detector sensitivities.

To summarize, following analogy of the ‘$db$’ triangle in the quark sector, we have explored the possibility of the construction of the ‘$\nu_2, \nu_3$’ leptonic unitarity triangle in the tribimaximal scenario by Bjorken et al. [9]. In particular, using the MNS matrix constructed in this scenario and considering values of $U_{e3}$ having implications on different theoretical models, we have constructed the distribution for the Jarlskog rephasing invariant parameter in the leptonic sector $J_l$ by using the area of the unitarity triangle. The distribution of $J_l$ is further used to find the Dirac-like CP violating phase $\delta$ in the leptonic sector. Interestingly, for the range $U_{e3} = 0.05 - 0.15$, the phase $\delta$ comes out to be around $40^\circ$, indicating a $2.5\sigma$ CP violation effect in the leptonic sector.

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References


Figure 1: Probability density distribution of $\delta$ in the case of quarks

Figure 2: Probability density distribution of $\delta$ in the case of neutrinos for (a) $U_{e3} = 0.05 \pm 0.005$ (b) $U_{e3} = 0.1 \pm 0.01$ (c) $U_{e3} = 0.15 \pm 0.015$