Geometric Mean Neutrino Mass Relation

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Abstract:

Present experimental data from neutrino oscillations have provided much information about the neutrino mixing angles. Since neutrino oscillations only determine the mass squared differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$, the absolute values for neutrino masses $m_i$ can not be determined using data just from oscillations. In this work we study implications on neutrino masses from a geometric mean mass relation $m_2 = \sqrt{m_1 m_3}$ which enables one to determined the absolute masses of the neutrinos. We find that the central values of the three neutrino masses and their $2\sigma$ errors to be $m_1 = (1.58 \pm 0.18)$meV, $m_2 = (9.04 \pm 0.42)$meV, and $m_3 = (51.8 \pm 3.5)$meV. Implications for cosmological observation, beta decay and neutrinoless double beta decays are discussed.

There are abundant data\textsuperscript{[1,2]} from solar, atmospheric, laboratory and long baseline neutrino experiments on neutrino mass and mixing. Neutrino oscillations provide direct evidence of non-zero neutrino masses and mixing between different species of neutrinos. The mixing can be well described by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix\textsuperscript{[3]} in the weak interaction with three neutrino oscillations. The PMNS can be parameterized as

$$V = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13}
\end{pmatrix}P, \quad (1)$$

where $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$. For Dirac neutrinos $P$ is a unit matrix, and for Majorana neutrinos $P$ is a diagonal phase matrix with two independent phases and can be written as $P = diag(1, \exp[i\phi_2], \exp[i\phi_3])$. 
Neutrino oscillations also depend on the mass squared differences \( \Delta m^2_{ij} = m_i^2 - m_j^2 \) of neutrino masses \( m_i \). The present experimental information on the mixing angles and the mass squared differences \( \Delta m^2_{ij} \) can be summarized as the following:\[^1,^2\]

\[
\sin^2 \theta_{12} = 0.314(1^{+0.18}_{-0.15}), \quad \sin^2 \theta_{23} = 0.45(1^{+0.35}_{-0.20}), \quad \sin^2 \theta_{13} = (0.8^{+2.3}_{-0.8}) \times 10^{-2},
\]

\[
\Delta m^2_{21} = 7.92(1 \pm 0.09) \times 10^{-5}\text{eV}^2, \quad |\Delta m^2_{32}| = 2.6(1^{+0.14}_{-0.15}) \times 10^{-3}\text{eV}^2.
\]

The errors are at 2\(\sigma\) level. No CP violating experiments in neutrino oscillation have been performed, the phase \( \delta_{13} \) is not known. Neutrino oscillations do not provide any information about the Majorana phases.

Although there are stringent constraints on neutrino masses from laboratory and cosmological data, and precise measurements of mass squared differences from neutrino oscillations, the absolute masses are not known. There are mechanisms proposed to understand the smallness of the neutrino masses, such as see-saw[^4] and radiative loop generation of masses[^5], a definitive mechanism to determine the absolute values of neutrino masses is still lacking. It is desirable to find some additional information, experimental or theoretical, to determine the masses. Attempts using various ansätze have been made previously[^6]. Here we find another interesting relation which can lead to the determination of neutrino masses. This is the geometric mean mass relation \( m_2 = \sqrt{m_1 m_3} \). We have chosen to work in the basis where the values of neutrino masses are all positively defined.

Geometric mean mass relation has been considered for quarks previously, in particular was used to predict the top quark mass[^7]. To have some ideas whether this is a reasonable attempt to pursue, in Fig.1 we summarize the values for \( \log_{10}(m_i) \) for quarks and charged leptons. In the figure we have used the central values for the quark masses at \( \mu = m_Z \) with[^8]:

\[
m_u = (2.22 \pm 0.24^{+0.14}_{-0.17})\text{ MeV}, \quad m_d = (4.42 \pm 0.29^{+0.29}_{-0.34})\text{ MeV}, \quad m_s = (84.7 \pm 7.2^{+5.5}_{-6.6})\text{ MeV},
\]

\[
m_c = (0.661 \pm 0.012^{+0.042}_{-0.047})\text{ GeV}, \quad m_b = (2.996 \pm 0.036^{+0.069}_{-0.074})\text{ GeV} \quad \text{and} \quad m_t = (180 \pm 13 \pm 0.02)\text{ GeV}.
\]

Using \( \log(m_i/eV) \) as vertical axis, the geometric mean mass relation is represented by a straight line for equal horizontal interval of an arbitrary unit since \( (\log(m_2/eV) - \log(m_1/eV))/(\log(m_3/eV) - \log(m_2/eV)) = \log(m_2/m_1)/\log(m_3/m_2) =1 \). The various \( \log \) plots are shown in Figure 1. Numerically we have \( \log_{10}(m_c/m_u)/\log_{10}(m_t/m_c) =1.016(1 \pm 0.166), \log_{10}(m_s/m_d)/\log_{10}(m_b/m_s) =0.828(1 \pm 0.327), \) and \( \log_{10}(m_{\mu}/m_e)/\log_{10}(m_{\tau}/m_{\mu}) =1.889 \). We see that the geometric mean mass relation holds well for the up quarks, u, c and

...
For the down quarks, d, s and b, the relation holds at one $\sigma$ level. Unfortunately the best measured charged lepton masses obviously do not satisfy the geometric mean mass relation. The electron mass seems to be anomalously small for some unknown reason. Alternatively, the tau mass is anomalously low and/or the muon mass is anomalously high. One is tempted to speculate that there is a mechanism that drives the electron mass to zero.

![FIG. 1: Summary of central values for $\log_{10}(m_i/eV)$ with equal intervals of horizontal axis for each point. From left to right, the lines are for up-quarks, down quarks and charged leptons.](image)

With the geometric mass relation $m_2 = \sqrt{m_1 m_3}$, we have

$$m_1^2 = \frac{\Delta m_{21}^2 \Delta m_{32}^2}{\Delta m_{32}^2 - \Delta m_{21}^2}, \quad m_2^2 = \frac{\Delta m_{21}^2 \Delta m_{32}^2}{\Delta m_{32}^2 - \Delta m_{21}^2}, \quad m_3^2 = \frac{\Delta m_{32}^2 \Delta m_{32}^2}{\Delta m_{32}^2 - \Delta m_{21}^2}. \quad (3)$$

Experimental data from neutrino oscillation on mass squared differences then determine the central values and $2\sigma$ errors for the neutrino masses to be

$$m_1 = (1.58 \pm 0.18)\text{meV},$$
$$m_2 = (9.04 \pm 0.42)\text{meV},$$
$$m_3 = (51.8 \pm 3.5)\text{meV}. \quad (4)$$

The masses obtained above must be checked against known experimental constraints. One of the most stringent constraints comes from cosmology consideration. In contrast to oscillation experiments, the contribution of the neutrinos to the energy density of the universe, $\Omega_\nu h^2 \approx m_{\text{sum}}/(93.5 \text{eV})$ depends on the values of $m_{\text{sum}} = m_1 + m_2 + m_3$ of course. The power spectrum of density perturbation also depends on $m_{\text{sum}}$. The present bound obtained from combining available data from CMB, large scale structure power spectrum, baryonic acoustic oscillation and small scale primordial spectrum from Lyman-alpha forest...
clouds, is very stringent with $m_{\text{sum}} < 170$ meV. Using the geometric mean mass relation, we obtain

$$m_{\text{sum}} = (62.4 \pm 3.5)\text{meV}. \quad (5)$$

It is interesting to note that this value is just 3 times smaller than the current bound from cosmology and may be probed in the future. In the near future $m_{\text{sum}}$ can be probed down to 120 meV by Planck experiment which is still about two times above the predicted value. However when combined with other data, the predicted range can be probed. For example sky survey with an order of magnitude larger survey volume would allow the sensitivity to reach 30 meV\[9\]. The mass ranges predicted with the geometric mean mass relation may be tested in the future.

There are other experimental constraints on neutrino masses. Of particular interests to neutrino masses are constraints on effective masses $\langle m_\beta \rangle$ and $m_{\beta\beta}$ from tritium $\beta$ decay and neutrinoless double $\beta$ decay, respectively. $m_\beta$ has not been measured and the present 2$\sigma$ level upper bound, combing the Mainz and Triotsk experiments, is $m_\beta < 1.8$ eV\[2\]. Planned experiments, KATRIN and MARE, can reach a sensitivity about 0.2 eV\[10\]. Currently it is still in debating whether a non-zero $m_{\beta\beta}$ has been measured in neutrinoless double beta decay of $^{76}\text{Ge}$. If the claimed observation is true it would imply a 2$\sigma$ range\[2, 11\]$0.43\text{eV} < m_{\beta\beta} < 0.81\text{eV}$. Future experiments can reach a sensitivity as low as 9 meV\[12\].

We now discuss the implications of the masses obtained from geometric mean mass relation on $\langle m_\beta \rangle$ and $m_{\beta\beta}$. These quantities are defined as

$$\langle m_\beta \rangle = (m_1^2 |V_{e1}|^2 + m_2^2 |V_{e2}|^2 + m_3^2 |V_{e3}|^2)^{1/2} = (m_{12}^2 c_{13}^2 + m_{23}^2 s_{12}^2 c_{13} + m_{33}^2 s_{13}^2)^{1/2},$$

$$m_{\beta\beta} = |m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2| = |m_1 c_{12}^2 c_{13} + m_2 s_{12}^2 c_{13} e^{2i\phi_2} + m_3 s_{13}^2 e^{2i\phi_3}|. \quad (6)$$

To have detailed information on $\langle m_\beta \rangle$ and $m_{\beta\beta}$ one needs to have more information on the mixing angles. A popular mixing matrix consistent with data is the so-called tri-bimaximal mixing\[13\] where $(V_{e1}, V_{e2}, V_{e3}) = (2/\sqrt{6}, 1/\sqrt{3}, 0)$. In this case the values for $m_i$ predicted by the geometric mass relation would give a range $(5.37 \pm 0.24)$ meV for $\langle m_\beta \rangle$. The range for $m_{\beta\beta}$ depends on the unknown Majorana phase $\phi_2$. However since the term proportional to $m_2$ again dominates, the effect of $\phi_2$ is small. The value for $m_{\beta\beta}$ is $\langle m_\beta \rangle / \sqrt{3}$ to a good approximation.
Finally we discuss how mass matrix which generating the geometric mean mass relation can be constructed. To this end we note a simple mass matrix of the form

$$M_\nu = \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & b \end{pmatrix},$$

with $b >> a$ gives,

$$m_1 \approx -\frac{a^2}{b}, \ m_2 = a, \ m_3 \approx b + \frac{a^2}{b}.$$  \hspace{1cm} (8)

The minus sign for $m_1$ can be removed by a redefinition of the phases of neutrino fields. The above satisfies the geometric mean mass related to a good approximation for neutrinos since $b$ is about 6 times larger than $a$, $a/b << 1$.

It is however a challenge to have a model which naturally give the above mass matrix. It is not difficult to have the texture zeros in the above matrix. For example, if there is a $Z_8$ discrete symmetry with the elements $\text{Exp}[i2n\pi/8]$ acts on leptons with two Higgs doublets, and the quantum numbers of the leptons and the two Higgs doublets are: $n$ for the left-handed lepton doublet $l_n$ and the right-handed charged lepton $e_{Rn}$, and “0” and “2” for $H_0, H_2$, the dimension-5 Weinberg operator $\lambda_{ij} \bar{l}_i l_j H_k H_l$ will generate a mass matrix of the form

$$M_\nu = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{pmatrix}.$$  \hspace{1cm} (9)

The above admits the desired form if $a_{22} = a_{13}$. Of course this amounts to the proposed geometric relation. We have not been able to derive $a_{22} = a_{13}$ from some symmetry principles. Further investigation is needed.

To summarize, we have studied the consequences of neutrino masses with the geometric mean relation $m_2 = \sqrt{m_1 m_3}$. With this condition the neutrino masses can be determined from measured mass-squared differences from oscillation experiments. We find that the neutrino masses are $m_1 = (1.58 \pm 0.18)\text{meV}$, $m_2 = (9.04 \pm 0.42)\text{meV}$, and $m_3 = (51.8 \pm 3.5)\text{meV}$. Although these masses are small, they can be probed by experiments from CMB measurements and large scale structure survey. We have suggested a mass matrix which produces the geometric mean mass relation, but we have not been able to derive it from some symmetry principles. It is interesting to see if a complete model with
the geometric mean mass relation for neutrino masses can be constructed.

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