A choreographic solution to the general relativistic three-body problem

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Abstract

We revisit the three-body problem in the framework of general relativity. The Newtonian N-body problem admits choreographic solutions, where a solution is called choreographic if every massive particles move periodically in a single closed orbit. One is a stable figure-eight orbit of a three-body system, which was found first by Moore (1993) and re-discovered with its existence proof by Chenciner and Montgomery (2000). In general relativity, however, the periastron shift prohibits a binary system from orbiting in a closed curve. Therefore, it is unclear whether general relativistic effects admit a choreographic solution such as the figure-eight. We carefully examine general relativistic corrections to initial conditions so that an orbit to a three-body system can be closed in a figure-eight. This solution is still choreographic without causing periastron shift. This illustration suggests that the general relativistic N-body problem also may admit a certain class of choreographic solutions.

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Introduction.— The three-body problem in the Newton gravity is one of classical problems in the astronomy and physics (e.g., [1]). In 1765, Euler found a collinear solution, and Lagrange found an equilateral triangle solution in 1772. It is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. In fact, Poincaré proved that we cannot analytically obtain all the solutions, and the number of new solutions is increasing [2]. Therefore, the three-body problem remains unsettled even for the Newtonian gravity. The Newtonian N-body problem admits choreographic solutions, which attract increasing interests. Here, a solution is called choreographic in the celestial mechanics if every massive particles move periodically in a single closed orbit. In fact, a choreographic figure-eight solution to the three-body problem was found first by Moore [3] and re-discovered with its existence proof by Chenciner and Montgomery [4].

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations. Especially, it has passed “classical” tests, such as the deflection of light, the perihelion shift of Mercury and the Shapiro time delay, and also a systematic test using the remarkable binary pulsar “PSR 1913+16” [5]. It is worthwhile to examine the three-body (in general N-body) problem in general relativity. N-body dynamics in the general relativistic gravity play important roles in astrophysics. For instance, the formation of massive black holes in star clusters is tackled mostly by Newtonian N-body simulations (e.g., [6]). However, it is difficult to work out in general relativity compared with the Newtonian gravity, because the Einstein equation is much more complicated [7] (even for a two-body system [8, 9, 10, 11]). In addition, future space astrometric missions such as SIM and GAIA [12, 13] require a general relativistic modeling of the solar system within the accuracy of a micro arc-second [14]. Furthermore, a binary plus the third body were discussed also for perturbations of gravitational waves induced by the third body [15, 16, 17, 18]. In this paper, we do not intend to solve the N-body problem in general relativity under a general situation. Instead, we shall focus on a choreographic solution. Any choreographic solutions have not been found to the general relativistic N-body problem so far.

In a two-body system, the post-Newtonian corrections cause the periastron shift so that the binary system cannot remain in the same orbit [7]. As a result, it is unclear whether general relativistic perturbations admit a choreographic solution as the figure-eight. One may thus ask, “what happens for the figure-eight in the Einstein’s gravity?”. A specific
question may arise such as, “does the figure-eight cause periastron shift?”, or “does the figure-eight make a transition to an open orbit in the general relativistic gravity?”. The purpose of this paper is to answer these questions by carefully examining general relativistic effects to initial conditions for being a choreographic solution.

This paper is organized as follows. First, we briefly summarize the choreographic figure-eight solution in the Newton gravity. Next, we analytically examine initial conditions and numerically solve the Einstein-Infeld-Hoffman equation of motion in order to obtain a choreographic solution in general relativity. Throughout this paper, we take the units of $G = c = 1$. A Newtonian choreographic solution.— As mentioned above, it is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. The simplest periodic solutions for this problem were discovered by Euler (1765) and by Lagrange (1772). The Euler’s solution is a collinear solution, in which the masses are collinear at every instant with the same ratios of their distances. The Lagrange’s one is an equilateral triangle solution in which each mass moves in an ellipse in such a way that the triangle formed by the three bodies revolves. Built out of Keplerian ellipses, they are the only explicit solutions. In these solutions, each mass moves on an ellipse (a circle for the equal mass case). A choreographic solution for which three bodies move periodically in a single figure-eight orbit was found first by Moore by numerical computations [3]. The existence of such a figure-eight orbit was proven by Chenciner and Montgomery [4]. This solution is stable in the Newtonian gravity [19, 20]. The figure-eight seems unique up to scaling and rotation according to all numerical

FIG. 1: A schematic figure for a binary orbit in general relativity. The orbit is not closed any more, because of the periastron shift.
investigations, and at the end its unicity has been recently proven \[21\]. Furthermore, it is shown numerically that fourth, sixth or eighth order polynomial cannot express the figure-eight solution \[20\]. Nevertheless, no analytic expressions in closed forms for the figure-eight trajectory have been found up to now. Therefore, in this paper, we numerically prepare the figure-eight orbit.

For simplicity, we assume a three-body system with each mass equal to \(m\). Without loss of the generality, the orbital plane is taken as the \(x-y\) plane. The position of each mass \((m_A)\) is denoted by \((x_A, y_A)\) for \(A = 1, 2, 3\). Figure 2 shows the figure-eight orbit, where we take the initial condition as \(\ell \equiv (x_1, y_1) = (-x_2, -y_2) = (97.00, -24.31), (x_3, y_3) = (0, 0)\) and \(V_{\text{Newton}} \equiv (\dot{x}_3, \dot{y}_3) = (-2\dot{x}_1, -2\dot{y}_1) = (-2\dot{x}_2, -2\dot{y}_2) = (-0.09324, -0.08647)\), where a dot denotes the time derivative \[20\]. When one mass arrives at the knot (center) of the figure-eight, \(\ell \equiv |\ell|\) is a half of the separation between the remaining two masses. It is convenient to use \(\ell\) instead of a distance between the knot and the apoapsis, because the inertial moment is expressed simply as \(2m\ell^2\). The orbital period is estimated as \(T_{\text{Newton}} = 6.326m^{-1/2}\ell^{3/2} \approx 10^4(M_\odot/m)^{1/2}(\ell/R_\odot)^{3/2}\) sec., where \(M_\odot\) and \(R_\odot\) are the solar mass and radius, respectively. Obviously this system has no Killing vector as seen in Fig. 2. Here, we should note that \(\ell\) is taken as 100, while it is the unity in the previous works. This is because we will treat the post-Newtonian correction in terms of the ratio between the mass and the separation such as \(\ell\). In our case, the ratio \(m/\ell\) is 0.01, that is, the post-Newtonian correction becomes about one percent.

*Post-Newtonian figure-eight.*—In the previous part, the motion of massive bodies follows the Newtonian equation of motion. In order to include the dominant part of general relativistic effects, we take account of the terms at the first-post Newtonian (1PN) order. Namely, the motion of the massive bodies obeys the Einstein-Infeld-Hoffman’s equation of motion. It is
expressed as
\[
\frac{d^2 \mathbf{x}_K}{dt^2} = \sum_{A \neq K} r_{AK} \frac{m_A}{r_A^3} \left[ 1 - 4 \sum_{B \neq K} \frac{m_B}{r_{BK}} - \sum_{C \neq A} \frac{m_C}{r_{CA}} \left( 1 - \frac{\mathbf{r}_{AK} \cdot \mathbf{r}_{CA}}{2r_{CA}^2} \right) \right.
\]
\[+ v_{2K}^2 + 2v_{2A}^2 - 4v_A \cdot v_K - \frac{3}{2} \left( \frac{\mathbf{v}_A \cdot \mathbf{r}_{AK}}{r_{AK}} \right)^2 \]
\[\left. - \sum_{A \neq K} \left( \mathbf{v}_A - \mathbf{v}_K \right) \frac{m_A r_{AK} \cdot (3\mathbf{v}_A - 4\mathbf{v}_K)}{r_{AK}^3} \right] + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} r_{CA} \frac{m_A m_C}{r_{AK} r_{CA}^3}, \tag{1}
\]

where we define
\[\mathbf{r}_{CA} \equiv \mathbf{r}_C - \mathbf{r}_A. \tag{2}\]

Figure 2 shows an orbit of a body starting at the Newtonian initial condition described above. In Fig. 2 a figure-eight orbit does not seem to survive at the 1PN order. However, this is not the case. We should note that the initial condition at the 1PN order does not necessarily coincide with that for the Newtonian gravity. We will thus carefully examine the initial condition by taking account of 1PN corrections. For this purpose, we assume that both the total linear momentum and the total angular momentum vanish (\(\mathbf{P} = 0\) and \(\mathbf{L} = 0\)).

The initial velocity of each mass is parameterized as
\[
\mathbf{v}_1 = k \mathbf{V} + \xi \frac{m}{\ell^3} (\mathbf{V} \cdot \ell) \ell, \tag{3}
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\]
\[
\mathbf{v}_3 = \mathbf{V}, \tag{5}
\]
where \(k\) is expressed as
\[
k = \frac{1}{2} - \alpha |\mathbf{V}|^2 + \beta \frac{m}{\ell}. \tag{6}
\]
We should remember \(\mathbf{v}_3 = -2\mathbf{v}_1 = -2\mathbf{v}_2\) for the figure-eight in the Newton gravity, for which both the total linear momentum and the total angular momentum vanish. Here, we impose the condition of \(\mathbf{P} = 0\) and \(\mathbf{L} = 0\) at 1PN order. Then, we determine the 1PN
FIG. 2: Figure-eights starting at the Newtonian initial condition. The solid curve denotes a figure-eight orbit in the Newtonian gravity. The dashed curve denotes a trajectory of one mass following the EIH equation of motion under the same Newtonian initial condition.

coefficients as

\[ \alpha = \frac{-3}{16}, \]  
\[ \beta = \frac{1}{8}, \]  
\[ \xi = \frac{1}{8}. \]  

Ut to this point, \( V \) is arbitrary. Next, we will determine \( V \).

The initial velocity of the particles can be different from that for the Newton gravity. We parameterize it as

\[ V = \left(1 + \delta \frac{m}{\ell}\right) V_{\text{Newton}} + \eta \frac{m}{\ell} \frac{\ell}{\ell} \left( V_{\text{Newton}} \cdot \frac{\ell}{\ell} \right). \]  

By numerically performing trial and error iterations until achieving a period orbit, we find out

\[ \delta = -3.3, \]  
\[ \eta = -3.7. \]
where the computations are done within the accuracy of 0.01. The numerical computation gives the orbital period as

\[ T_{GR} \approx \left( 1 + \frac{6m}{\ell} \right) \times T_{Newton}. \]  

(13)

Figure 3 shows that a figure-eight orbit is still closed even after including the dominant general relativistic effects. In Fig. 3 we can see a difference between the Newtonian figure-eight orbit and the general relativistic (GR) one. The deviation is partly fiducial, because the principal axes of the GR figure-eight orbit are not along the \( x \) and \( y \) axes. That is, \( \ell \), which defines the direction of the initial position of a particle with respect to the principal axes, changes slightly at the 1PN order. The axes are inclined by 0.012 radian with respect to those of the Newtonian figure-eight orbit. Now, we choose the \( x \)-axis as the principal axis for both the Newtonian figure-eight orbit and the general relativistic one. After choosing the principal axes, Fig. 4 shows a general relativistic choreographic solution at the first post-Newtonian order. The solution recovers line symmetry with respect to the \( x \) and \( y \) axes. Figure 5 shows the velocity of the particle starting at the origin. There are no significant differences in the velocity between the Newtonian and GR figure-eight orbits.
FIG. 4: Figure-eight orbits. The two curves are the same as those in Fig. 3. The major axes of the curves are chosen as $x$-axis.

No periastron shift in the figure-eight can be explained in the following manner. The motion of each particle is anti-clockwise in the R. H. S. of the figure-eight ($x > 0$), while it is clockwise in the L. H. S. ($x < 0$). The periastron advance cannot occur, because there are equal contributions, with exactly opposite directions, from both clockwise and anti-clockwise motions.

Finally, we mention the possibility of three-body systems in a choreographic orbit such as a figure-eight. As a new outcome of binary-binary scattering, the figure-eight orbit was discussed for presenting a way of detecting such an orbit in numerical computations [22]. According to the numerical result, the probability of the formation of figure-eight orbits is a tiny fraction of one percent. The gravitational waves emitted by the figure-eight have been recently studied by assuming the motion in the Newton gravity [23]. By evaluating the radiation reaction time scale, it is shown also that figure-eight sources emitting gravitational waves may be too rare to detect.

*Conclusion.*— We obtained a general relativistic initial condition for being a figure-eight orbit. This condition provides the first choreographic solution taking account of the post-
FIG. 5: Comparison of the velocity between the Newtonian figure-eight solution and the general relativistic one. Top panel: The $x$-component of the velocity of the particle labeled by 3 starting at the origin. Bottom panel: The corresponding $y$-component. The solid and dashed curves denote the Newtonian and general relativistic cases, respectively.
Newtonian corrections. It is interesting to include higher post-Newtonian corrections, especially 2.5PN effects in order to elucidate the backreaction on the evolution of the orbit due to the gravitational waves emission at the 2.5PN order. If the system is secularly stable against the gravitational radiation, one might see probably a shrinking ($\dot{\ell} < 0$) figure-eight orbit as a consequence of a decrease in the total energy ($\dot{E} < 0$). This speculation will be confirmed or rejected in future. It may be important also to look for other relativistic choreographic solutions for a system including four or more masses. It is possible that some of Newtonian choreographic solutions are prohibited by general relativistic effects. Further investigations along these lines will allow us to probe many-body dynamics in the Einstein gravity.


