Stable Isotropic Cosmological Singularities in Quadratic Gravity

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Abstract

We show that, in quadratic lagrangian theories of gravity, isotropic cosmological singularities are stable to the presence of small scalar, vector and tensor inhomogeneities. Unlike in general relativity, a particular exact isotropic solution is shown to be the stable attractor on approach to the initial cosmological singularity. This solution is also known to act as an attractor in Bianchi universes of types I, II and IX, and the results of this paper reinforce the hypothesis that small inhomogeneous and anisotropic perturbations of this attractor form part of the general cosmological solution to the field equations of quadratic gravity. Implications for the existence of a 'gravitational entropy' are also discussed.

1 Introduction

There has been considerable interest in the stability properties of isotropic initial singularities in general relativistic cosmologies. At first, this question was bound up with the issue of whether the initial singularity itself was a robust prediction of general relativistic cosmologies in the presence of typical fluids, like dust and radiation, or whether it was merely an aberration of poor coordinate choice or high symmetry. This question was answered unambiguously by the proofs of the first singularity theorems which approached the problem by means of the geodesic equations and the causal structure of space-time instead of the Einstein equations [1]. However, there still remained the question of whether any initial singularity was likely to be isotropic. The discovery in 1967 of the high level of temperature anisotropy in the microwave background made this question one of central importance for observational cosmology [2]. It turned out that the general behaviour of cosmological singularities dominated by matter or radiation was highly anisotropic [3, 4], even chaotic [5], and any attempt to explain the existence and rotation of galaxies by primordial turbulence required the initial singularity to be highly anisotropic [6], while attempts to dissipate strong initial anisotropies in the spirit of the chaotic cosmology programme [7] had to face the problem of entropy over-production in the very early universe [8]. However, it was possible for singularities dominated by stiff matter to be close to isotropy and 'quiescent' because the energy density of this extreme form of matter, characteristic of a massless scalar field, diverges as fast as the most rapidly diverging anisotropy modes on approach to a cosmological singularity [9, 10]. More recently, the advent of inflation [11] as a potential explanation for the observed isotropy of the visible part of the universe removed the need to appeal to a globally isotropic singularity in order to explain astronomical observations, or for anisotropy to be dissipated [12]. It is sufficient to explain the existence of local isotropy on the scale of the observable universe and this is achieved by natural means in the chaotic inflationary scenario [13]. Despite the success of this approach, there remain a number of questions whose answers depend upon the stability properties of an initial isotropic cosmological singularity. A sufficiently isotropic and homogeneous horizon-sized patch is needed somewhere in order for inflation to occur, and attempts to understand the quantum nature of gravitation will need to understand whether a physical cosmological singularity generically occurs and, if so, what its most likely properties are.
Penrose [14] has proposed that initial cosmological singularities are not simply signals that the underlying gravity theory has broken down, but play a crucial role in defining the arrow of time. Initial singularities are thereby required to be highly isotropic – so possessing low gravitational ‘entropy’, perhaps identified with the Weyl curvature – but any final singularity in a collapsing closed universe would necessarily be strongly anisotropic, with high ‘gravitational entropy’. Since such isotropic singularities are special in general relativity [15], it is necessary to introduce this gravitational entropy hypothesis as a boundary condition in order to justify choosing a highly isotropic initial singularity.

Other boundary conditions can also be chosen which make the present level of expansion isotropy understandable even if the initial state is not isotropic. Barrow [16] showed that the natural requirement that all energy densities (including that in anisotropising gravitational waves) are smaller than, or of order, the Planck density at the Planck time leads to a general requirement that the microwave background anisotropy be no larger than $O(10^{-4})$ today. Typically, inflation scenarios assume that arbitrarily large energy densities do not arise [13].

Here, we want to explore the stability of isotropic singularities in higher-order generalisations of general relativity, in which the Einstein-Hilbert lagrangian is supplemented by the addition of the quadratic scalar, $R$, and Ricci, $R_{ab}R^{ab}$, curvature invariants. Since initial cosmological singularities are expected to involve infinities in one of more of the curvature invariants of the space-time, we expect that the addition of higher-order terms to the lagrangian would produce a new dominant behaviour to such singularities. We would expect the higher-order terms to control the dynamics as the singularity is approached, and to determine the stability properties of any isotropic special solution in that limit. In this paper we are going to consider the contributions of the quadratic Ricci terms, $R_{ab}R^{ab}$, to the lagrangian. The quadratic scalar, $R^2$, contributions are conformally equivalent to the presence of a self-interacting scalar field and are understood. In a subsequent paper, we will study how our results may be modified by complicated effects that arise at general order, when $(R_{ab}R^{ab})^n$ terms are added to the lagrangian.

Specifically, we will investigate the stability of an initial isotropic zero-curvature Friedmann singularity to small inhomogeneous scalar, vector, and tensor perturbations in generalisations of general relativity which contain quadratic Ricci and scalar curvature contributions to the gravitational lagrangian of general relativity. We will show that there exists a particular isotropic solution of the pure quadratic Ricci theory which is a stable attractor as $t \to 0$ for small inhomogeneous perturbations of the metric when the gravitation theory created by the addition of quadratic Ricci terms to the general relativity lagrangian. Previous studies have shown this particular isotropic solution to be a stable $t \to 0$ attractor for spatially homogeneous vacuum universes of Bianchi types $I, II, [17]$ and $IX [18]$. Our results lend further weight to the conjecture of Barrow and Hervik [17] that this isotropic vacuum solution of the pure quadratic Ricci theory characterises part of the general cosmological solution in these particular higher-order generalisations of general relativity. This behaviour is completely different to that found in general relativity, where isotropy is unstable in the $t \to 0$ limit of an initial cosmological singularity [3, 4].

2 A Special Solution

Consider a quadratic gravity theory with action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (-2\Lambda + R + AR^2 + BR_{ab}R^{ab}) + L_m \right].$$

The field equations obtained by varying this action with respect to the metric are [20, 17, 21]:

$$G^{ab}_b + AP^{(1)}_b = T^{ab}_b - \Lambda g^{ab}_b,$$

where

$$P^{(1)}_b = -\frac{1}{2} R^2 g^a_b + 2RR^a_b + 2g^a_a \Box R - 2R^a_c R^{cb}_b,$$

$$P^{(2)}_b = \frac{1}{2} R^{cd} R_{cd} g^a_b + \Box R^a_b + \frac{1}{2} g^a_a \Box R - R^a_c R^{cb}_b - 2R^a_c e^{cd} R^{cd}_b.$$
where \(G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}\) is the usual Einstein tensor, \(T^a_b\) is the energy-momentum tensor of the matter sources, and \(\Lambda\) is the cosmological constant\(^1\).

We take \(\Lambda = 0\) and consider perturbations about the spatially flat, homogeneous and isotropic FRW spacetime,

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\]

with scale factor \(a(t)\) and associated Hubble expansion rate \(H \equiv \dot{a}/a\), for which the background equations are:

\[
\frac{1}{3}\mu = H^2 + 2(B + 3A) \left(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H}\right),
\]

(3)

\[
-\frac{1}{2}(\mu + p) = \dot{H} + 2(B + 3A) \left(\ddot{H} + 3H\dot{H} + 6\dot{H}^2\right),
\]

(4)

where we have defined a density, \(\mu = -T^0_0\), and pressure \(p = \frac{1}{3}T^\gamma_\gamma\).

For radiation obeying the equation of state \(p = \frac{1}{3}\mu\), there exists a special exact solution of these equations with [19, 21]

\[a(t) = t^{\frac{1}{2}}.\]

Importantly, this is also an exact vacuum solution (\(\mu = p = 0\)) of the purely quadratic theory\(^2\) \((B + 3A \to \infty)\), i.e. it solves

\[2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} = 0.\]

(5)

Integrating this equation, we have

\[
\frac{\dot{H}^2}{H} = \frac{M^2}{a^6},
\]

with \(M\) constant. Since \(\dot{H} = aH\frac{dH}{da}\), we can take the square root and integrate to obtain

\[
\frac{2}{3}H^{\frac{3}{2}} = D \pm \frac{M}{3a^n},
\]

with \(D\) constant. Hence, we can see that \(a \to t^{\frac{1}{2}}\) as \(t \to 0\) and \(H^2 \to \text{const} + O(a^{-3})\) to leading order as \(a \to \infty\). So the purely quadratic vacuum solution mimics radiation early on, then dust plus a cosmological constant at late times.

Clifton and Barrow [23] found that for a universe filled with a fluid with equation of state \(p = \omega\mu\) in a theory where the lagrangian depends only on the Ricci-squared term like \((R_{ab}R^{ab})^n\), there is an isotropic fluid-filled FRW solution with

\[a(t) = t^{\frac{2n}{3(1 - n)}},\]

and also that there exists an isotropic vacuum solution with

\[
a = t^{1/2} : n = 1,
\]

(6)

\[
a = t^S : n \neq 1,
\]

(7)

where [23]

\[
S \equiv \frac{3(1 - 3n + 4n^2) \pm \sqrt{3(48n^4 - 40n^3 - 5n^2 + 10n - 1)}}{6(1 - n)}.
\]

\(^1\)For a different covariant formulation see ref. [22].

\(^2\)Barrow and Hervik [17] showed that this is also true for the case of Friedmann models with non-zero curvature, in which case the curved Friedmann radiation solution is an exact solution of the full equations in the presence of radiation and also of the purely quadratic vacuum equations \((B + 3A \to \infty)\).
In particular, for the case of \( n = 1 \) and \( \omega \leq 1 \), we see that \( \frac{4n}{n(n+1)} \geq \frac{2}{3} > \frac{1}{2} \), so the special vacuum solution will dominate over the fluid-filled effects in the field equations as \( t \to 0 \).

The special isotropic vacuum solution (6) is of special interest. Previous studies have shown that it is an attractor for solutions of Bianchi types I and II as \( t \to 0 \). [17]. It is also a stable attractor in Bianchi type IX 'Mixmaster' universes as \( t \to 0 \). [18]. This is in complete contrast to the situation that exists in general relativity (\( A = B = 0 \)) where the shear and curvature anisotropy terms dominate the dynamics as the singularity is approached at \( t = 0 \), [3, 4]; in the case of Bianchi type IX, that dynamical approach is chaotic [5]. As a result an isotropic initial singularity constitutes a set of zero measure in the initial data space. In quadratic gravity, the situation is completely different, the results in Bianchi types I, II, and IX suggest that the special isotropic solution may be an attractor for the time evolution of a part of the general solution of the cosmological evolution equations. In the sections to follow we will show that this special vacuum solution is also stable against the effects of small inhomogeneous perturbations (scalar, vector, and tensor) as \( t \to 0 \).

### 3 Inhomogeneous Perturbations

The general perturbation of the FRW metric may be written as

\[
\delta s^2 = -a^2(1 + 2\alpha)dy^2 - a^2\dot{B}_\alpha dx^\alpha + a^2(\delta\alpha\beta + \dot{C}_{\alpha\beta})dx^\alpha dx^\beta,
\]

where \( \eta \) is a conformal time coordinate that is related to the comoving proper time, \( t \), by \( dt = a\eta \). We can decompose the perturbation variables into their "scalar", "vector" and "tensor" parts, as follows:

\[
\begin{align*}
\delta s^2 & = \dot{B}_\alpha + 2\alpha \eta \delta \alpha + 2B_\alpha \\
\dot{\tilde{C}}_{\alpha\beta} & = 2\phi \delta \alpha \beta + 2\gamma_{\alpha\beta} + 2C_{(\alpha\beta)} + 2C_{\alpha\beta}.
\end{align*}
\]

There are four scalar perturbation variables, \( \alpha, \beta, \phi \) and \( \gamma \), two vector variables, \( B_\alpha \) and \( C_\alpha \), and one tensor perturbation, \( C_{\alpha\beta} \). The quantities \( B_\alpha \) and \( C_\alpha \) are divergence-free, i.e. \( B^\alpha_{\alpha} = 0 = C^\alpha_{\alpha} \), and \( C_{\alpha\beta} \) is transverse and tracefree, i.e. \( C^\alpha_{\beta;\alpha} = 0 = C^\alpha_{\alpha;\alpha} \). These three types of perturbation evolve independently of each other at linear order. The linearised equations governing the perturbations in the general quadratic gravity are presented in a series of papers by Noh and Hwang [24, 25, 26].

For the gravitational-wave (tensor) perturbation modes in the general quadratic gravity, the equation for the perturbation is ([24]):

\[
\delta T^3_3 = D^3_3 + 2A \left( R\dot{D}^3_3 + \dot{R}\dot{C}^3_3 \right) - B \left\{ D^3_3 + 3H\dot{D}^3_3 - \frac{\Delta}{\alpha^2} \left( D^3_3 + 4HC^3_3 \right) \right\}
- 6 \left\{ \left( \dot{H} + H^2 \right) D^3_3 + \left( \ddot{H} + H\dot{H} \right) \dot{C}^3_3 \right\}
\]

where

\[
D^3_3 \equiv \dot{C}^3_3 + 3HC^3_3 - \frac{\Delta}{\alpha^2}C^3_3.
\]

For the vortical (vector) perturbations, following Noh and Hwang [25], we set \( B_\alpha(t, x) = b(t)Y_\alpha(x) \), with \( \Delta Y_\alpha \equiv -k^2Y_\alpha \) and use the gauge-invariant variable \( \Psi Y_\alpha \equiv B_\alpha + a\dot{C}_\alpha \). In the gauge where \( C_\alpha \equiv 0 \), the equations are

\[
\frac{k^2}{2a^2} \left\{ (1 + 2AR)\Psi - B \left[ \dot{\Psi} + \dot{H}\Psi - \left( \frac{2}{3}R - \frac{k^2}{a^2} \right) \Psi \right] \right\} = (\mu + p)v_\omega
\]

and the conservation of angular momentum transparently holds:

\[
a^4(\mu + p)v_\omega = \Omega
\]

\[^3\text{Note that there is a factor of } 2 \text{ difference in the } d\eta dx^\alpha \text{ component from the metrics given by Noh and Hwang in [25] and [26]. Defining } g_{\alpha\alpha} = -a^2(\beta_{\alpha} + B_{\alpha}) \text{ is consistent with their equations and appendix quantities.}\]
with $\Omega$ constant.

Finally, for the scalar density perturbations, we introduce the combinations $\chi \equiv a(\beta + a\dot{\gamma})$ and $\kappa \equiv 3(H\alpha - \dot{\phi}) - \frac{\Delta}{a^2}$. Then we have the following set of equations ([26]):

**Energy constraint:**

$$
\delta T^0_0 = 2 \left( \frac{\Delta \phi}{a^2} + H\kappa \right) + 2A \left\{ 2R \left( \frac{\Delta \phi}{a^2} + H\kappa \right) + \dot{R} (\kappa + 3H\alpha) - 3H\delta R + \left[ 3(\dot{H} + H^2) + \frac{\Delta}{a^2} \right] \delta R \right\} 
- B \left\{ 2 \left( H\kappa + \frac{\Delta \phi}{a^2} \right) - 6H \left( \frac{\Delta \phi}{a^2} \right) + \left[ 4(\dot{H} - 6H^2) - \frac{2\Delta}{a^2} \right] \left( H\kappa + \frac{\Delta \phi}{a^2} \right) + 3H\delta R \right\} 
- \left( 3H^2 + \frac{\Delta}{a^2} \right) \delta R + 6\dot{H}\dot{\alpha} + 6 \left( -H\ddot{H} + 2\dot{H}^2 - 9H^2 \dddot{H} \right) \alpha 
- \left( 6\dot{H} + 5H^2 \right) \kappa - \frac{8H\Delta}{3a^2} \left( \kappa + \frac{\Delta \chi}{a^2} \right) \right\}.
$$

**Momentum constraint:**

$$
T^0_\alpha = \frac{2}{3a} \nabla_\alpha \left\{ -\kappa - \frac{\Delta \chi}{a^2} + A \left( -2R\kappa - 2R\frac{\Delta \chi}{a^2} - 3\dot{R}\alpha + 3\delta R - 3H\delta R \right) \right\} 
- B \left\{ \left( \kappa + \frac{\Delta \chi}{a^2} \right) - 3H \left( \kappa + \frac{\Delta \chi}{a^2} \right) + \left( 12H^2 + \frac{\Delta}{a^2} \right) \left( \kappa + \frac{\Delta \chi}{a^2} \right) - \frac{3\delta \dot{R}}{2} + \frac{3H\delta R}{2} \right\} 
- 3\dot{H}\dot{\alpha} + \frac{3}{2} \left( 2\dot{H} + 9H^2 \right) \alpha + \frac{3\Delta}{a^2} \left( 2H\phi + \left( \dot{H} - 2H^2 \right) \chi \right) \right\}.
$$

**Tracefree propagation:**

$$
\delta T^\beta_\beta - \frac{1}{3} \delta T^\gamma_\gamma \delta^\beta_\beta = \frac{1}{a^2} \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \delta^\beta_\beta \Delta \right) \left[ \dot{\chi} + H\chi - \phi - \alpha + 2A \left( R\chi + (HR + \dot{R})\chi - \delta R - \phi - \alpha \right) \right] 
- B \left\{ \left( \dot{\chi} + H\chi - \phi - \alpha \right) - 8(\dot{H} + H^2) + \frac{\Delta}{a^2} \left( \dot{\chi} + H\chi - \phi - \alpha \right) \right\} 
+ 6\dot{H}\phi - 6 \left( \dot{H} + H^2 \right) \chi + \frac{8H}{3} \left( \kappa + \frac{\Delta \chi}{a^2} \right) \right\}.
$$

**Raychaudhuri equation:**

$$
\delta T^\gamma_\gamma - \delta T^0_0 = 2\kappa + 4H\kappa + 2 \left( 3\dot{H} + \frac{\Delta}{a^2} \right) \alpha + 2A \left\{ 2R\kappa + \left( 4HR + \dot{R} \right) \kappa + 3\dot{R}\alpha \right\} 
+ \left[ 6\dot{R} + 3H\dot{R} + 2R \left( 3\dot{H} + \frac{\Delta}{a^2} \right) \right] \alpha - 3\dot{R} + 3H\delta R + \left( 6H^2 + \frac{\Delta}{a^2} \right) \delta R \right\} 
- B \left\{ -4 \left( H\kappa + \frac{\Delta \phi}{a^2} \right) - 12H \left( H\kappa + \frac{\Delta \phi}{a^2} \right) \right\} 
+ \left[ -8 \left( \dot{H} - 6H^2 \right) + \frac{4\Delta}{a^2} \right] \left( H\kappa + \frac{\Delta \phi}{a^2} \right) + 2\delta R 
+ 6\dot{H}^2 \delta R - 28\dot{H}\kappa - 6 \left( 2\dot{H} + 5H^2 \right) \dot{\alpha} - 12 \left[ \left( 2\dot{H} + 5H^2 \right) \dot{\alpha} + 3H^2 \dddot{H} \right] \alpha 
+ \frac{16H\Delta}{3a^2} \left( \kappa + \frac{\Delta \chi}{a^2} \right) \right\}.
$$
Trace equation:

$$\delta T = -\delta R - 2(3A + B) \left[ \delta \dot{R} + 3H\delta \dot{R} - \frac{\Delta}{a^2} \delta R - \dot{R}(\kappa + \dot{\alpha}) - \left( 2\dot{R} + 3H\dot{R} \right) \alpha \right]$$

and

$$\delta R = -2 \left[ \dot{\kappa} + 4H\kappa + 3H\alpha + \frac{\Delta}{a^2}(2\phi + \alpha) \right]$$

Note that no choice of gauge has yet been made for the scalar perturbations; we will ultimately want to work in the gauge $\delta R \equiv 0$, so that we are considering perturbations in flat space.

We are primarily interested in the effects of the $R_{ab}R^{ab}$ term and so henceforth we set $A = 0$.

We consider the three perturbation modes in turn in order to determine whether the isotropic and homogeneous vacuum solution (6) is stable against their effects as $t \to 0$.

4 Gravitational-wave perturbations

The metric for the gravitational-wave perturbations takes the form

$$ds^2 = -dt^2 + a^2(\delta_{\alpha\beta} + 2C_{\alpha\beta})dx^\alpha dx^\beta,$$

$$C_\alpha^\alpha = 0 = C_{\beta}^\alpha_{,\alpha} \quad (9)$$

where $C_{\alpha\beta} = C_{\alpha\beta}(x, t)$ and is gauge-invariant. For convenience, from now on we drop the indices on $C_{\alpha\beta}$. The remaining perturbation equations in vacuum are:

$$0 = D - B \left\{ \ddot{D} + 3H\dot{D} - \frac{\Delta}{a^2}(D + 4\dot{H}C) - 6 \left[ (\dot{H} + H^2)D + (\dot{H} + H^2)\dot{C} \right] \right\}$$

$$D \equiv \ddot{C} + 3H\dot{C} - \frac{\Delta C}{a^2},$$

where $D \equiv D_\beta^\alpha$ and $C \equiv C_\beta^\alpha$.

Substituting $H = 1/2t$, and taking the $B \to \infty$ limit, keeping only $\{ \} = 0$, this reduces to

$$\ddot{D} + \frac{3}{2t} \dot{D} + \frac{3}{2t^2} D - \frac{3}{t^3} \dot{C} = \frac{\Delta}{t} \left( D - \frac{2}{t^2} C \right), \quad \text{(10)}$$

$$\ddot{C} + \frac{3}{2t} \dot{C} - \frac{\Delta C}{t} = D. \quad \text{(11)}$$

Substituting for $D$ using (11), we have the single 4th-order evolution equation for the metric perturbation $C$:

$$0 = \dddot{C} + \frac{3}{t} \ddot{C} + \frac{3}{4t^2} \dot{C} - \frac{2\Delta \dot{C}}{t} - \frac{\Delta C}{t^2} + \frac{\Delta \Delta C}{t^2}. \quad \text{(12)}$$

4.1 Large scales

In the long-wavelength limit, the last three terms in (12) can be neglected and we have

$$0 = \dddot{C} + \frac{3}{t} \ddot{C} + \frac{3}{4t^2} \dot{C},$$

Note that for our special solution, eq. (6) with $a(t) = t^{1/2}$, we have $R = 0$ to background order. Consequently, the perturbed parts of the gravitational wave and vorticity equations are independent of $A$. This is also true for the scalar perturbations when we choose the uniform curvature gauge, i.e. $\delta R \equiv 0$. So our results are also valid for the general quadratic gravity.
and so the solution is a linear sum of power-laws with
\[ C \propto t^q, \]
\[ 0 = q(q - 1)(2q - 1)(2q - 3), \]
hence
\[ C = \varepsilon + \lambda t + \nu t^{\frac{1}{2}} + \zeta t^{\frac{3}{2}}, \]
where \( \varepsilon, \lambda, \nu, \zeta \) are independent arbitrary tensor functions of \( \mathbf{x} \) satisfying the same constraints as \( C \), given by (9).

Note that as \( t \to 0 \), the perturbation \( C \to \text{const} \) and the a \( = t^{\frac{1}{2}} \) solution is stable against gravitational-wave modes. The leading-order terms in the metric perturbation expansion are therefore (restoring the tensor indices):
\[ ds^2 = -dt^2 + t(\delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta} + 2\nu_{\alpha\beta}t^{\frac{1}{2}} + 2\lambda_{\alpha\beta}t + 2\zeta_{\alpha\beta}t^{\frac{3}{2}})dx^\alpha dx^\beta, \]
where the \( \varepsilon_{\alpha\beta}(x), \nu_{\alpha\beta}(x), \lambda_{\alpha\beta}(x), \zeta_{\alpha\beta}(x) \) are arbitrary traceless symmetric 3x3 tensors subject only to (9) and coordinate transformations and the constraint equations.

### 4.2 Small scales

If we look for a separable solution to (12) of the form
\[ C = T(t)X(x) \]
then
\[ \dddot{C} + \frac{3}{t} \ddot{C} + \frac{3}{4t^2} \dot{C} = \frac{2\Delta \ddot{C}}{t} + \frac{\Delta \dot{C}}{t^2} - \frac{\Delta \Delta C}{t^2}, \]
\[ \Rightarrow \dddot{T}X + \frac{3}{t} \ddot{T}X + \frac{3}{4t^2} \dot{T}X = \frac{2\ddot{T} \Delta X}{t} + \frac{\dot{T} \Delta X}{t^2} - \frac{T \Delta \Delta X}{t^2}. \]
If \( \Delta X = \theta X \), where \( \theta \) is constant so \( \Delta \Delta X = \theta \Delta X = \theta^2 X \), then
\[ \dddot{T} + \frac{3}{t} \ddot{T} + \frac{3}{4t^2} \dot{T} = \frac{2\ddot{T} \theta}{t} + \frac{\dot{T} \theta}{t^2} - \frac{\theta^2 T}{t^2}, \quad (13) \]
\[ t^2 \dddot{T} + 3t \ddot{T} + \left( \frac{3}{4} - 2t\theta \right) \dot{T} - \dot{T} \theta + \theta^2 T = 0. \quad (14) \]
If \( \theta = 0 \), then we obtain the same results as in large-scale limit. If we take the late-time approximation or the small-scale limit by keeping only the terms in \( \theta < 0 \) in (13) then we need to solve
\[ 2t\dddot{T} + \dot{T} - \theta T = 0, \]
so as \( \theta \) will be negative,
\[ T = C_1 \cos\{\sqrt{-2\theta}t\} + C_2 \sin\{\sqrt{-2\theta}t\} \]
which approaches a constant as \( t \to 0 \).
4.3 The general separable solution

If we change to conformal time, \( t = \eta^2 \), and put \( \theta = -k^2 \), then the master equation (14) is transformed to

\[
T'''' + 2k^2 T'' + k^4 T = 0.
\]

The general solution is

\[
T^* (t) = A_1 \cos[2k\sqrt{t}] + A_2 \sin[2k\sqrt{t}] + A_3 \sqrt{t} \cos[2k\sqrt{t}] + A_4 \sqrt{t} \sin[2k\sqrt{t}]
\]

\[
T = B_1 + B_2 \sqrt{t} + B_3 t + B_4 t \sqrt{t}
\]

\( k \neq 0 \),

\( k = 0 \).

We note that, for \( k \neq 0 \),

\[
T^* (t) \rightarrow A_1 + (2kA_2 + A_3) \sqrt{t} + O(t) \rightarrow \text{const}
\]
as \( t \rightarrow 0 \).

Hence the full separable solution for the metric perturbation is

\[
C_{\alpha\beta}(t, x) = T^* (t) X_{\alpha\beta}(x),
\]

\[
\Delta X_{\alpha\beta} = -k^2 X_{\alpha\beta}.
\]

5 Vortical Perturbations

The metric expansion for the vorticity perturbations takes the form

\[
ds^2 = -a^2 d\eta^2 - 2a^2 B_{\alpha} d\eta dx^\alpha + a^2 (\delta_{\alpha\beta} + 2 C_{(\alpha, \beta)}) dx^\alpha dx^\beta,
\]

with \( B^\alpha_{,\alpha} \equiv 0 \equiv C^\alpha_{,\alpha} \).

Following Noh and Hwang [25], we set \( B_{\alpha} (t, x) \equiv b(t) Y_{\alpha} (x) \), with \( \Delta Y_{\alpha} \equiv -k^2 Y_{\alpha} \), and use the gauge-invariant variable:

\[
\Psi Y_{\alpha} \equiv B_{\alpha} + a \dot{C}_{\alpha}.
\]

In the gauge defined by \( C_{\alpha} \equiv 0 \), linearising about the special vacuum solution with \( H = \frac{1}{2t} \) and \( R = 0 \) we then have

\[
\frac{k^2}{2a^2} \left( \Psi - B \left[ \frac{\dot{\Psi}}{2t} + \frac{k^2}{t} \dot{\Psi} \right] \right) = (\mu + p) v_\omega
\]

and the conservation of angular momentum gives:

\[
a^4 (\mu + p) v_\omega = \Omega
\]

with \( \Omega \) constant.

5.1 Large scales

On large scales we have

\[
\Psi - B \left[ \frac{\dot{\Psi}}{2t} \right] = \frac{2\Omega}{k^2 t}
\]

(16)

In the case where the (quadratic) \( B \) term dominates on the LHS, we have

\[
\Psi = -\frac{D}{B} t^{1/2} - \frac{4\Omega}{Bk^2} t + \Psi_0
\]

and \( \Psi \rightarrow \text{const} \) as \( t \rightarrow 0 \) and the isotropic metric is stable.
If we keep the general-relativistic $\Psi$ term on the LHS of (16), and write $\Psi = t^2 f$, then we have

$$t^2 \ddot{f} + t \dot{f} - \left( \frac{t^2}{B} + \frac{1}{16} \right) f = -\frac{2\Omega}{k^2 B} t^4.$$

(17)

This is an inhomogeneous modified Bessel equation; so, if we set

$$y = \frac{it}{\sqrt{B}}, \quad \nu = \frac{1}{4}, \quad g = -\frac{2\Omega}{Bk^2},$$

we obtain

$$y^2 f'' + y f' + (y^2 - \nu^2) f = g(x)$$

and its solution is

$$f = C_1 J_{\frac{1}{4}} \left( \frac{it}{\sqrt{B}} \right) + C_2 Y_{\frac{1}{4}} \left( \frac{it}{\sqrt{B}} \right) + \frac{\pi}{2} Y_{\frac{1}{2}} \left( \frac{it}{\sqrt{B}} \right) \int \frac{it}{\sqrt{B}} J_{\frac{1}{4}} \left( \frac{it}{Bk^2} \right) \frac{i}{\sqrt{B}} dt.$$

Hence, on large-scales the general solution in the $C_\alpha \equiv 0$ gauge is

$$B_\alpha \equiv \Psi Y_\alpha = t^2 f(t) Y_\alpha(x),$$

(18)

$$\Delta Y_\alpha \equiv -k^2 Y_\alpha.$$

(19)

Again we note that $B_\alpha \to constant$ as $t \to 0$ and the isotropic solution (6) is stable in that limit.

5.2 The pure quadratic theory

Now consider the case where the B-term dominates, but this time without taking the long-wavelength limit, i.e. we drop just the GR term on the left hand side. In that case the governing equation is

$$\ddot{\Psi} + \frac{\dot{\Psi}}{2t} + \frac{k^2 \Psi}{t} = -\frac{2\Omega}{Bk^2 t},$$

(20)

and so

$$\Psi = D_1 \cosh(2k\sqrt{t}) + D_2 \sinh(2k\sqrt{t}) - \frac{2\Omega}{Bk^4},$$

(21)

where $D_1$ and $D_2$ are constants.

Since $B_\alpha \equiv \Psi Y_\alpha(x)$ in our chosen gauge, we again see that $B_\alpha \to constant$ as $t \to 0$ and (6) is stable in that limit.

6 Scalar Perturbations

For the scalar perturbations, the metric will take the form

$$ds^2 = -a^2(1 + 2\alpha)dt^2 - 2a^2 \beta_{\alpha} d\eta dx^\alpha + a^2(\delta_{\alpha\beta}(1 + 2\phi) + 2\gamma_{\alpha\beta}) dx^\alpha dx^\beta.$$

There are possible gauge choices that can still be made and we shall consider two useful choices.
6.1 Conformal Newtonian Gauge ($\chi \equiv 0$)

The gauge choice $\chi \equiv a(\beta + a^2) = 0$ (zero shear) greatly simplifies the perturbation equations, since we just have two perturbed variables to worry about. They reduce to the following set:

Energy constraint:

$$\delta T^0_0 = -B \left\{ \frac{\ddot{\phi}}{t} + \frac{15}{t^2} \frac{\ddot{\phi}}{t} - 6 \frac{\dot{\phi}}{t} - 4 \frac{\Delta \phi}{t} - 12 \frac{\Delta \phi}{t^2} + 2 \frac{\Delta \phi}{t^3} + 2 \frac{\Delta ^2 \phi}{t^2} - 3 \frac{\Delta \phi}{t^2} + 3 \frac{\Delta ^2 \phi}{t^2} + \frac{\Delta ^2 \alpha}{t^2} + \frac{\Delta ^2 \beta}{t^2} \right\}$$

Momentum constraint:

$$T^\alpha_\alpha = -\frac{2}{t^2} \nabla_\alpha \left[ B \left\{ -2 \frac{\ddot{\phi}}{t} + \frac{3 \ddot{\phi}}{t^2} - 6 \frac{\dot{\phi}}{t^2} + \frac{\Delta \phi}{t} - 2 \frac{\Delta \phi}{t^2} + 2 \frac{\Delta \phi}{t^3} - 3 \frac{\Delta \phi}{t^2} + \frac{\Delta \phi}{t^3} - \frac{\Delta \phi}{t^4} \right\} - \frac{\dot{\phi}}{t} + \frac{\alpha}{2t} \right].$$

Trace-free propagation:

$$\delta T^\alpha_\beta - \frac{1}{3} \delta T^\gamma_\gamma \delta^\alpha_\beta = -\frac{1}{t} \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \delta^\alpha_\beta \Delta \right) \left[ B \left\{ 5 \frac{\ddot{\phi}}{t} + \frac{17 \ddot{\phi}}{2t^2} - 3 \frac{\Delta \phi}{t} - \frac{\Delta \phi}{t^2} - \frac{\Delta \phi}{t^3} + \frac{\Delta \phi}{t^4} \right\} + \phi + \alpha \right].$$

Trace equation:

$$\delta \phi = -2B \left\{ 6 \frac{\ddot{\phi}}{t} + \frac{21 \ddot{\phi}}{t^2} - \frac{6 \ddot{\phi}}{t^2} + 6 \frac{\dot{\phi}}{t^2} - 10 \frac{\Delta \phi}{t} - 10 \frac{\Delta \phi}{t^2} + 2 \frac{\Delta \phi}{t^3} + 4 \frac{\Delta ^2 \phi}{t^2} - 3 \frac{\Delta \phi}{t^2} + \frac{\Delta \phi}{t^3} + \frac{\Delta \phi}{t^4} + \frac{2 \Delta ^2 \phi}{t^2} \right\}$$

$$-6 \frac{\ddot{\phi}}{t} + \frac{12 \phi}{t^2} + \frac{4 \Delta \phi}{t} - \frac{3 \dot{\phi}}{t^2} + \frac{2 \Delta \alpha}{t^2} = 0.$$  

(25)

6.1.1 Large-scale limit of the pure quadratic theory ($B \to \infty$) in the zero-shear gauge

If we ignore the effects of matter\footnote{\(\mu \propto a^{-3(1+\omega)} \propto t^{-2}\) for radiation, whereas on the right-hand side of the energy equation typical terms are \(\sim \frac{\dot{\phi}}{t^{3}}, \frac{\ddot{\phi}}{t^{4}}\)}, i.e. \(\delta T^\mu_\nu = 0\), and take the large-scale limit then each term in curly brackets vanishes. The combination \(\frac{2}{3}(23) + \frac{4}{3}(22) - \frac{1}{3}(24)\) gives

$$\frac{2}{3} \frac{\ddot{\phi}}{t} + \frac{6 \ddot{\phi}}{t^2} + \frac{3 \ddot{\phi}}{2t^3} = 0,$$

(26)

to which the solution is

$$\phi = C_0 + C_1 t^{-\frac{1}{2}} + C_2 t^{\frac{1}{2}}.$$

Using the energy equation, we then obtain

$$\alpha = C_1 t^{-\frac{1}{2}} + C_2 t^{\frac{1}{2}} + C_3 t^{\frac{1}{2}}.$$

So, at first sight it appears that there may be an instability as \(t \to 0\); however, we note that the coefficients of the \(t^{\frac{1}{2}}\) terms in \(\alpha\) and \(\phi\) are the same. This means that they can both be locally defined away by the freedom to choose our initial time. If we set \(\tilde{t} \equiv t - t_0\), then \(a(t) = t_0^{-\frac{1}{2}} = \tilde{t}^{-\frac{1}{2}}(1 - t_0^{-1})^{-\frac{1}{2}} \sim \tilde{t}^{-\frac{1}{2}} - \frac{1}{2} t_0^{-\frac{1}{2}} \tilde{t}^{\frac{1}{2}}\). In fact, by choosing a different gauge, we see that this is just a curvature perturbation when we compare our flat FRW background with the similar open and closed models.
6.2 Uniform Curvature Gauge ($\delta R \equiv 0$)

If we use the uniform curvature gauge then $R \equiv 0$ and we are perturbing in flat space. The set of perturbations equations are now:

Energy constraint:

\[
0 = -3\frac{\dot{\phi}}{t} + 2\frac{\Delta \phi}{t^2} + \frac{3\alpha}{2t^2} - \Delta \chi - B \left\{ -3\frac{\ddot{\phi}}{t} - \frac{3\ddot{\phi}}{t^3} + 21\frac{\dot{\phi}}{t^4} + 2\frac{\Delta \phi}{t^2} - 6\frac{\dot{\phi}}{t^2} - \frac{7\Delta \phi}{t^3} \right\}.
\]

Momentum constraint:

\[
0 = 3\frac{\dot{\phi}}{2t} - \frac{3\alpha}{2t} - B \left\{ -\frac{\ddot{\phi}}{2t} + \frac{\Delta \phi}{t} - \alpha + \frac{\dot{\alpha}}{2t} + \frac{\Delta \alpha}{2t} + \chi + \frac{3\dot{\chi}}{4t^2} - 3\frac{\chi}{4t^2} - \frac{3\Delta \chi}{2t^2} - \frac{3\Delta \chi}{t^2} \right\}.
\]

Trace-free propagation:

\[
0 = -\phi - \alpha + \dot{\chi} + \frac{\chi}{2t} - B \left\{ -\frac{\ddot{\phi}}{t} + \frac{\Delta \phi}{t} - \alpha + \frac{\dot{\alpha}}{2t} + \frac{\Delta \alpha}{t} + \chi + \frac{3\dot{\chi}}{4t^2} - 3\frac{\chi}{4t^2} - \frac{3\Delta \chi}{2t^2} - \frac{3\Delta \chi}{t^2} \right\}.
\]

The trace equation is trivial, but we have a fourth equation from $\delta R = 0$:

\[
0 = -3\frac{\ddot{\phi}}{t} - \frac{6\ddot{\phi}}{2t} - 2\frac{\Delta \phi}{t} + 3\frac{\ddot{\alpha}}{2t} + \frac{\Delta \alpha}{t} - \frac{\Delta \chi}{t^2} - \frac{\Delta \chi}{t^2}.
\]

We can now eliminate $\chi$ from our equations (from the momentum constraint, we see that $\chi$ is stable whenever $\alpha$ and $\phi$ are).

Using (28), equation (30) becomes:

\[
0 = \frac{1}{B} \left[ t^3 \ddot{\phi} + 4t^2 \dot{\phi} - t^2 \alpha - \frac{3t\alpha}{2} \right] - t^3 \ddot{\phi} - \frac{11}{2} t^2 \dot{\phi} - \frac{9}{2} \frac{\phi}{t}.
\]

Equation (27) becomes:

\[
0 = -B \left\{ \frac{3\ddot{\phi}}{t} + 3\frac{\ddot{\phi}}{t^2} - \Delta \phi + \frac{\Delta \phi}{t} + \frac{5\Delta \phi}{t^2} + \frac{\Delta^2 \phi}{t^2} - \frac{3\ddot{\phi}}{t^2} - \frac{\ddot{\alpha}}{2t} + \frac{3\ddot{\alpha}}{2t} + \frac{3\ddot{\alpha}}{2t^2} + \frac{3\alpha}{2t^2} + \frac{\Delta \alpha}{2t^2} + \frac{\Delta \alpha}{2t^2} \right\}.
\]

6.2.1 Large-scale limit of the pure quadratic theory ($B \to \infty$) in the uniform curvature gauge

In the large-scale limit of (31), setting $\phi = E_\lambda t^\lambda$, $\alpha = F_\lambda t^\lambda$ and dropping the $B^{-1}$ terms, we find that

\[
\lambda(\lambda - 1) \left( \lambda - \frac{1}{2} \right) \left( (\lambda + 1)E_\lambda - \frac{F_\lambda}{2} \right) = 0,
\]

and in the large-scale limit of (32), keeping only the terms inside {}, we have

\[
(\lambda - 1) \left( E_\lambda \lambda(2\lambda - 1) - F_\lambda \left( \lambda - \frac{3}{2} \right) \right) = 0.
\]
The right-hand bracket of each of these last two equations cannot simultaneously vanish unless \( E_\lambda = F_\lambda = 0 \), so \( \lambda = 0, \frac{1}{2}, \) or 1 and \( F_0 = F_\frac{1}{2} = 0 \). There is a further restriction on \( E_1 \) and \( F_1 \) from (29), specifically \( F_1 = \frac{2}{5}E_1 \).

Hence, we have the solution

\[
\begin{align*}
\phi &= E_0 + E_\frac{1}{2}t^\frac{1}{5} + E_1 t, \\
\alpha &= \frac{8}{5}E_1 t, \\
\Delta \chi &= \Delta(t^\frac{1}{5}\beta + t\gamma) = -\frac{3}{2}E_\frac{1}{2}t^\frac{1}{5} - \frac{9}{5}E_1 t.
\end{align*}
\]

Assuming \( \Delta \chi = -\kappa^2 \chi \), we have

\[
\begin{align*}
\beta &= \frac{3}{2\kappa^2}E_\frac{1}{2} - t^\frac{1}{5} \dot{y}(t, x), \\
\gamma &= \frac{9}{5\kappa^2}E_1 t + g(t, x).
\end{align*}
\]

There is a remaining freedom in our gauge choice. In the conformal Newtonian gauge, it is usually used to set both \( \beta = 0 \) and \( \gamma = 0 \). The other parameters are unaffected by this choice. Here, we can use this freedom to set \( g(t, x) = 0 \).

Again, we see that the metric perturbations approach constant values as \( t \to 0 \) and the isotropic solution is stable against scalar perturbations in this limit.

**7 Summary**

Combining our results from the quadratic theory, to leading order in the time as \( t \to 0 \), we have found that the metric in the neighbourhood of the special solution (6) with isotropic scale factor evolution \( a = t^{1/2} \) takes the general form

\[
\begin{align*}
\text{ds}^2 &= -a^2(1 + 2\alpha)dt^2 - a^2 \hat{B}_\alpha d\eta d\xi^\alpha + a^2(\delta_{\alpha\beta} + \hat{C}_{\alpha\beta})dx^\alpha dx^\beta, \\
& \text{(38)}
\end{align*}
\]

where

\[
\begin{align*}
\alpha &= \frac{8}{5}E_1 t, \\
\hat{B}_\alpha &= \frac{3}{\kappa^2}E_\frac{1}{2} - \alpha + 2\left(D_1 - \frac{20}{Bk^4} + 2kD_0 t^\frac{1}{5} + 2k^2D_1 t\right)Y_\alpha(x), \\
\hat{C}_{\alpha\beta} &= 2\left(E_0 + E_\frac{1}{2}t^\frac{1}{5} + E_1 t\right)\delta_{\alpha\beta} + \frac{18t}{5\kappa^2}E_1,\alpha\beta + \\
&+ 2\left(A_1 + (2jA_2 + A_3) t^\frac{1}{5} + 2j(jA_1 + A_4) t\right)X_{\alpha\beta}(x).
\end{align*}
\]

where \( \Delta Y_\alpha = -\kappa^2 Y_\alpha, \Delta X_{\alpha\beta} = -j^2 X_{\alpha\beta}, \) the \( A_i \) and \( D_i \) are constants, and \( E_i = E_i(x) \).

These results are striking. The special isotropic and homogeneous solution with scale factor evolution \( a = t^{1/2} \) is found to be stable against the effects of small scalar, vector and tensor perturbations in the \( t \to 0 \) limit. This adds to the evidence gathered from earlier studies of the stability of this solution in the Bianchi type I, II and IX universes [17, 18]. It adds further weight to the conjecture that small perturbations of this isotropic singularity form part of the general cosmological solution for theories of gravity with quadratic lagrangians of the sort studied here. This behaviour is quite different to that found in cosmological solutions of general relativity, where isotropic expansion is unstable as \( t \to 0 \). It arises because the higher-order quadratic terms in the lagrangian contribute terms which dominate on approach to an initial singularity (typically as \( O(t^{-4}) \) compared to general relativistic contributions at \( O(t^{-2}) \)) contribute isotropising stresses.
which dominate over the shear stresses arising from expansion and 3-curvature anisotropy, which dominate the initial singularity in general relativistic cosmologies.

These results have a number of implications for proposals to introduce special cosmological initial conditions in order to ensure that the initial state is isotropic, with low gravitational entropy. It appears to make additional stipulations of special initial conditions unnecessary in quadratic gravity. However, it may cause problems for the gravitational entropy scenario in closed recollapsing universes because we expect the same quadratic stresses also to drive the solution towards isotropy on approach to any future 'big crunch' singularity. This would prevent it being the high gravitational entropy state that a 'Second Law' of gravitational thermodynamics would lead us to expect.

Finally, despite the special features of the cosmological models in quadratic theories of gravity, we need to understand if these unusual results regarding the stability of isotropy persist when higher-than-quadratic-order contributions to the gravitational lagrangian are included. In a subsequent paper we will present the results of this more complicated study.

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