Logarithmic correction to scaling for multi-spin strings in the $AdS_5$ black hole background

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Abstract

We find new explicit solutions describing closed strings spinning with equal angular momentum in two independent planes in the $AdS_5$ black hole spacetime. These are $2n$ folded strings in the radial direction and also winding $m$ times around an angular direction. We especially consider these solutions in the long string and high temperature limit, where it is shown that there is a logarithmic correction to the scaling between energy and spin. This is similar to the one-spin case. The strings are spinning, or actually orbiting around the black hole of the $AdS_5$ black hole spacetime, similarly to solutions previously found in black hole spacetimes.

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There has been a lot of progress concerning the conjectured duality \cite{1,2,3} between super string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SU(N) super Yang-Mills theory in Minkowski space, beyond the super gravity limit. One step forward was obtained by Berenstein, Maldacena and Nastase \cite{4}, where one considers a particular limit of $AdS_5 \times S^5$, where it reduces to a plane wave, in which string theory is exactly solvable. Another step forward was obtained by Gubser, Klebanov and Polyakov \cite{5}, where it was observed that the first subleading term in the scaling relation between energy and spin for a rigidly rotating one-spin string in $AdS_5$ is logarithmic (actually, spinning strings \cite{6}, as well as pulsating strings \cite{7}, in $AdS$ space were originally studied more than 10 years ago)

\[ E - S \sim \ln(S) \] (1)

in the long string limit, similarly to results previously found for certain Yang-Mills operators \cite{8,9,10,11,12} at the gauge theory side.

It was soon realized that most progress came from considering multi-spin strings in $AdS_5 \times S^5$. However, most papers on multi-spin strings deal with strings where most of the spins are in the $S^5$ part of the background (in general there are two angular momenta in the $AdS_5$ part and three in the $S^5$ part), since the computation of energies at the gauge theory side then reduces to a study of certain spin chains \cite{13}. Moreover, some non-renormalization results have been obtained for large angular momenta in $S^5$; see for instance \cite{14,15} (and references given therein). Many papers take therefore only the time coordinate, and the corresponding conserved energy, from $AdS_5$ and up to three angular momenta in $S^5$, while others take only a single spin in $AdS_5$.

There have only been few papers with explicit examples of multi-spin strings in the $AdS_5$ part of the background \cite{16,17,18}. In one of them, a generalization of the string leading to equation (1) was obtained \cite{18}. Namely, a two-spin string with two equal angular momenta in two independent planes of $AdS_5$, leading to the result

\[ E - 2S \sim \ln(S) \] (2)

in the long string limit.

Also backgrounds related to $AdS_5$, in one way or the other, has been considered. Of interest to this letter, especially the $AdS_5$ black hole background, which is supposed to describe 5 dimensional $AdS$ spacetime at finite temperature \cite{19}. A one-spin string orbiting around the black hole was constructed in \cite{20}, leading in the long string and high temperature limit to a
result similar to equation (1). A two-spin string was constructed in [21], but it was not extended in the radial direction and it did not lead to a relation similar to equation (2).

The purpose of the present letter is to construct a two-spin string solution in the $AdS_5$ black hole background, which is extended in both the radial and an angular direction, and which leads to a scaling relation of the form of equation (2). These strings will be orbiting around the black hole, rather than spinning around their center of mass, similarly to solutions previously found in black hole spacetimes [6, 20, 22, 23].

We have the $AdS_5$ black hole line element [19]

$$ds^2 = -(1 + H^2 r^2 - M/r^2) dt^2 + \frac{dr^2}{1 + H^2 r^2 - M/r^2} + r^2 (d\beta^2 + \sin^2 \beta d\phi^2 + \cos^2 \beta d\tilde{\phi}^2)$$

(3)

where $M$ is proportional to the mass, which for large mass is proportional to $T^4$, where $T$ is the temperature [19]. The horizon is located at

$$r^2_h = -1 + \sqrt{1 + 4MH^2}$$

(4)

We are interested in strings which are extended in the $r$ and $\beta$ directions, and which have two identical angular momenta in the $\phi$ and $\tilde{\phi}$ directions. For our ansatz

$$X^\mu = (c_0 \tau, r(\sigma), \beta(\sigma), \omega \tau, \omega \tau)$$

(5)

where $(c_0, \omega)$ are constants, the tangent vectors are

$$\dot{X}^\mu = (c_0, 0, 0, \omega, \omega), \quad X'^\mu = (0, r', \beta', 0, 0)$$

(6)

where dot and prime denote derivative with respect to $\tau$ and $\sigma$, respectively. The 5 equations of motion (we use the orthonormal gauge)

$$\ddot{X}^\mu - X'^{\nu\mu} + \Gamma^\mu_{\rho\sigma} (\dot{X}^\rho \dot{X}^\sigma - X'^{\rho} X'^{\sigma}) = 0$$

(7)

lead to

$$r'' = \frac{H^2 r^2 - M/r^3}{1 + H^2 r^2 - M/r^2} r'^2 - (1 + H^2 r^2 - M/r^2)(H^2 r + M/r^3)c_0^2$$

$$- (1 + H^2 r^2 - M/r^2) r\beta'^2 + (1 + H^2 r^2 - M/r^2) r\omega^2 = 0$$

(8)
\[ \beta' = \frac{k}{r^2} \]  \hspace{1cm} \text{(9)}

where \( k \) is an integration constant. The induced metric on the world-sheet is

\[ g_{\tau\sigma} = G_{\mu\nu} \dot{X}^\mu X^\nu = 0 \]  \hspace{1cm} \text{(10)}

\[ g_{\tau\tau} = G_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = -(1 + H^2 r^2 - M/r^2) c_0^2 + r^2 \omega^2 \]  \hspace{1cm} \text{(11)}

\[ g_{\sigma\sigma} = G_{\mu\nu} X'^\mu X'^\nu = (1 + H^2 r^2 - M/r^2)^{-1} (r^2 + r^2 \beta'^2) \]  \hspace{1cm} \text{(12)}

Then the orthonormal gauge constraints \( g_{\tau\sigma} = 0 \) and \( g_{\tau\tau} + g_{\sigma\sigma} = 0 \), lead to

\[ r'^2 = (1 + H^2 r^2 - M/r^2) \left( (1 + H^2 r^2 - M/r^2) c_0^2 - r^2 \omega^2 - \frac{k^2}{r^2} \right) \]

\[ = \frac{1 + H^2 r^2 - M/r^2}{r^2} \left( \omega^2 - H^2 c_0^2 \right) (r_+^2 - r^2) (r^2 - r_-^2) \]  \hspace{1cm} \text{(13)}

where

\[ r_\pm^2 = \frac{1}{2(\omega^2 - H^2 c_0^2)} \left( c_0^4 + 4(H^2 c_0^2 - \omega^2)(M c_0^2 + k^2) \right) \]  \hspace{1cm} \text{(14)}

Notice that (13) is the integral of (8), so the complete dynamics is given by (9) and (13). We are interested in solutions which are oscillating (in terms of the spatial world-sheet coordinate \( \sigma \)) between positive \( r_- \) and \( r_+ \), thus we demand

\[ \omega^2 > H^2 c_0^2 \]  \hspace{1cm} \text{(15)}

and

\[ c_0^4 + 4(H^2 c_0^2 - \omega^2)(M c_0^2 + k^2) > 0 \]  \hspace{1cm} \text{(16)}

which are equivalent to

\[ 0 < \frac{\omega^2}{H^2 c_0^2} - 1 < \frac{c_0^2}{4H^2(M c_0^2 + k^2)} \]  \hspace{1cm} \text{(17)}

and notice that the string is always orbiting outside the horizon. Imposing the boundary conditions \( r(0) = r_+ \) and \( \beta(0) = 0 \), we have the solutions

\[ \sigma = -\frac{1}{\sqrt{\omega^2 - H^2 c_0^2}} \int_{r_+}^{r} \frac{xdx}{\sqrt{(1 + H^2 x^2 - M/x^2)(r_+^2 - x^2)(x^2 - r_-^2)}} \]  \hspace{1cm} \text{(18)}
\[ \beta(\sigma) = k \int_0^\sigma \frac{dx}{r^2(x)} \]  

(19)

These solutions hold for non-constant \( r(\sigma) \). The solution for constant \( r \) is given in [21]. In this letter we consider strings which are \( 2n \) folded in the \( r \) direction and winding \( m \) times around the \( \beta \) direction, thus the string is closed and consists of \( 2n \) segments. We then have the periodicity conditions \( r(\sigma + 2\pi/n) = r(\sigma) \) and \( \beta(\sigma + 2\pi) = 2\pi m + \beta(\sigma) \). The first condition becomes

\[ \pi = \frac{n}{\sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} \frac{xdx}{x \sqrt{1 + H^2 x^2 - M/x^2} (r_+^2 - x^2)(r_+^2 - r_-^2)} \]  

(20)

while the second leads to

\[ 2\pi m = \frac{2kn}{\sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} \frac{dx}{x \sqrt{1 + H^2 x^2 - M/x^2} (r_+^2 - x^2)(r_+^2 - r_-^2)} \]  

(21)

For fixed \( M \) and \( H \), we have 3 free parameters namely \( c_0, \omega \) and \( k \), but 2 of them are determined by the two previous equations. Then we have also the 2 integers \( n \) and \( m \). Notice that \( n \) and \( m \) may be constrained since \( k, c_0 \) and \( \omega \) must be real. For instance, for \( M = 0 \) we find that \( n/m > 2 \) [18].

The conserved energy is given by

\[ E = \frac{c_0}{2\pi \alpha'} \int_0^{2\pi} (1 + H^2 r^2(\sigma) - M/r^2(\sigma))d\sigma \]  

(22)

Similarly, the conserved spins are

\[ S_1 = \frac{\omega}{2\pi \alpha'} \int_0^{2\pi} r^2(\sigma) \cos^2 \beta(\sigma)d\sigma, \quad S_2 = \frac{\omega}{2\pi \alpha'} \int_0^{2\pi} r^2(\sigma) \sin^2 \beta(\sigma)d\sigma \]  

(23)

In our case, \( S_1 = S_2 \equiv S = (S_1 + S_2)/2 \). Then we get

\[ E = \frac{nc_0}{\pi \alpha' \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} xdx \sqrt{\frac{1 + H^2 x^2 - M/x^2}{(r_+^2 - x^2)(x^2 - r_-^2)}} \]  

(24)

\[ S = \frac{n\omega}{2\pi \alpha' \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} \frac{x^3dx}{\sqrt{(1 + H^2 x^2 - M/x^2)(r_+^2 - x^2)(x^2 - r_-^2)}} \]  

(25)
Now introduce

\[ r_0^2 = \frac{1 + \sqrt{1 + 4MH^2}}{2H^2} \]  

Then equations (20), (21), (24) and (25) lead to

\[ \pi = \frac{n}{H \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} \frac{x^2 \, dx}{\sqrt{(x^2 + r_0^2)(x^2 - r_+^2)(r_+^2 - x^2)(x^2 - r_-^2)}} \]  

\[ 2\pi m = \frac{2kn}{H \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} \frac{dx}{\sqrt{(x^2 + r_0^2)(x^2 - r_+^2)(r_+^2 - x^2)(x^2 - r_-^2)}} \]  

\[ E = \frac{nHc_0}{\pi \alpha' \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} x^2 \frac{(x^2 - y_0^2)(x^2 - y_h^2)}{(r_+^2 - x^2)(x^2 - r_-^2)} \]  

\[ S = \frac{n\omega}{2\pi H \alpha' \sqrt{\omega^2 - H^2 c_0^2}} \int_{r_-}^{r_+} x^4 \frac{dx}{\sqrt{(x^2 + r_0^2)(x^2 - r_+^2)(r_+^2 - x^2)(x^2 - r_-^2)}} \]  

In the special case \( M = 0 \), there is no black hole and we have pure \( AdS_5 \). Then the integrals (27)-(30) reduce to the results obtained in [18], as they should. Now introduce the dimensionless constants

\[ y_0 = H^2r_0^2, \quad y_h = H^2r_h^2, \quad y_- = H^2r_-^2, \quad y_+ = H^2r_+^2 \]  

as well as the new dimensionless integration variable

\[ y = H^2x^2 \]  

Performing changes of integration variable, similarly to the Appendix of [20], the integrals (27)-(30) lead to

\[ \pi = \frac{n}{2\sqrt{y_+ - y_-} \sqrt{\omega^2 - H^2 c_0^2}} \int_0^1 \frac{\sqrt{t + \frac{y_-}{y_+ - y_-}} \, dt}{\sqrt{(t + \frac{y_- + y_h}{y_+ - y_-})(t + \frac{y_- - y_h}{y_+ - y_-})(1 - t) t}} \]  

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The integrals (33)-(36) are of hyperelliptic type, and cannot be directly evaluated. However, we are only interested in the long string limit. In the long string limit we find from equation (14)

\[ \frac{\omega}{H c_0} = 1 + 2\eta, \quad \eta << 1 \tag{37} \]

Then

\[ \sqrt{\omega^2 - H^2 c_0^2} \approx 2H c_0 \sqrt{\eta} \tag{38} \]

\[ y_+ \approx \frac{1}{4\eta} \tag{39} \]

\[ y_- \approx H^2 (M + k^2/c_0^2) \tag{40} \]

We furthermore take the high temperature limit. Long strings and high temperature means that \( y_+ >> y_- >> 1 \). Then also \( y_- >> (y_h, y_0) \), and we can approximate

\[ (t + \frac{y_- + y_0}{y_+ - y_-})^{-1/2} \approx \left( t + \frac{y_-}{y_+ - y_-} \right)^{-1/2} \left( 1 + \sum_{j=1} \frac{(2j-1)!!}{(2j)!!} \left( \frac{y_-}{y_+ - y_-} \right)^j \right) \tag{41} \]
\[(t + \frac{y_0 - y}{y_1 - y})^{-1/2} \approx \left( t + \frac{y_0 - y}{y_1 - y} \right)^{-1/2} \left( 1 + \sum_{j=1}^{(2j-1)!!} \frac{(2j-1)!! \left( \frac{y_0 - y}{y_1 - y} \right)^j}{t + \frac{y_0 - y}{y_1 - y}} \right) \] (42)

and similarly for the powers \(1/2\) appearing in (35). This can be inserted into (33)-(36), and then the integrals reduce to elliptic integrals, which can be evaluated to all orders in our approximation. Approximating also the elliptic integrals [24], we get

\[\pi = \frac{n}{2Hc_0} \left( \ln \left( \frac{1}{4\eta H^2 (M + k^2/c_0^2)} \right) + O(\eta, 1/MH^2) \right) \] (43)

\[2\pi m = \frac{Hkn}{c_0} \left( \frac{2}{H^2 (M + k^2/c_0^2)} + O(\eta, 1/M^2H^4) \right) \] (44)

\[E = \frac{n}{4\pi\alpha' H^2} \left( 1 + \eta \ln \left( \frac{1}{4\eta H^2 (M + k^2/c_0^2)} \right) + O(\eta) \right) \] (45)

\[S = \frac{n}{8\pi\alpha' H^2\eta} \left( 1 - \eta \ln \left( \frac{1}{4\eta H^2 (M + k^2/c_0^2)} \right) + O(\eta) \right) \] (46)

where \(O(\eta, 1/MH^2)\) means \(O(\eta)\) or \(O(1/MH^2)\). From (43) and (44), we get

\[Hc_0 \approx \omega \text{ and } H^2k \approx \frac{n}{2\pi} \ln \left( \frac{1}{4\eta \left( \frac{n^2}{2\pi^2 m^2} \pm \frac{n}{\pi m} \sqrt{\frac{n^2}{4\pi^2 m^2} - H^2M} \right)} \right) \] (47)

\[H^2k \approx \frac{n}{2\pi} \left( \frac{n}{2\pi m} \pm \sqrt{\frac{n^2}{4\pi^2 m^2} - H^2M} \right) \ln \left( \frac{1}{4\eta \left( \frac{n^2}{2\pi^2 m^2} \pm \frac{n}{\pi m} \sqrt{\frac{n^2}{4\pi^2 m^2} - H^2M} \right)} \right) \] (48)

Thus we get the condition that \(n/m\) should be relatively large

\[\frac{n}{m} \geq 2\pi H\sqrt{M} \] (49)
From (45) and (46) we get

\[ \frac{E}{H} - 2S \approx \frac{n}{2\pi \alpha' H^2} \ln \left( \frac{1}{4\eta H^2 (M + k^2 / c_0^2)} \right) \]  

(50)

i.e.

\[ \frac{E}{H} - 2S \approx \frac{n}{2\pi \alpha' H^2} \ln \left( \frac{S\alpha'}{M + k^2 / c_0^2} \right) \]  

(51)

which is equivalent to

\[ \frac{E}{H} - 2S \approx \frac{n}{2\pi H^2 \alpha'} \ln \left( \frac{n^2}{2\pi^2 m^2} \pm \frac{n}{\pi m} \sqrt{\frac{n^2}{4\pi^2 m^2} - H^2 M} \right) \]  

(52)

which can also be expressed in terms of the t’Hooft coupling \( \lambda = (H^2 \alpha')^{-2} \).

Notice that we get two solutions, as in the case where the string is not extended in the radial direction [21]. In our case we get from equations (39) and (40)

\[ r_+^2 = \frac{1}{4H^2 \eta} \]  

(53)

whereas

\[ r_-^2 = \frac{1}{H^2} \left( \frac{n^2}{2\pi^2 m^2} \pm \frac{n}{\pi m} \sqrt{\frac{n^2}{4\pi^2 m^2} - H^2 M} \right) \]  

(54)

In conclusion, we have shown that it is possible to obtain a logarithmic correction to the scaling between energy and spin for a two-spin string orbiting around the black hole in the AdS\(_5\) black hole background. In order to obtain this result, we must take the long string and the high temperature limit, similarly to the one-spin case. The two-spin string consists of \(2n\) segments and is winding \(m\) times around an angular direction. Furthermore, reality of the solution demands that \(n/m\) is larger than a given quantity of the order of \(H \sqrt{M}\). It is easy to generalize these solutions to the AdS\(_5\) black hole times S\(_5\), by introducing up to three angular momenta (R-charges) in S\(_5\).

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References


