Three dimensional conformal sigma models

Takeshi Higashi, Kiyoshi Higashijima, and Etsuko Itou

Department of Physics, Graduate School of Science, Osaka University,
Toyonaka, Osaka 560-0043, Japan

Abstract

We construct novel conformal sigma models in three dimensions. Nonlinear sigma models in three dimensions are nonrenormalizable in perturbation theory. We use Wilsonian renormalization group equation method to find the fixed points. Existence of fixed points is extremely important in this approach to show the renormalizability. Conformal sigma models are defined as the fixed point theories of the Wilsonian renormalization group equation. The Wilsonian renormalization group equation with anomalous dimension coincides with the modified Ricci flow equation. The conformal sigma models are characterized by one parameter which corresponds to the anomalous dimension of the scalar fields. Any Einstein-Kähler manifold corresponds to a conformal field theory when the anomalous dimension is $\gamma = -1/2$. Furthermore, we investigate the properties of target spaces in detail for two dimensional case, and find the target space of the fixed point theory becomes compact or noncompact depending on the value of the anomalous dimension.
There are many perturbatively nonrenormalizable theories; gravity, gauge theories in higher dimensions and so on. One of such theories is a three dimensional non-linear sigma model. The model is perturbatively nonrenormalizable, but interestingly it is “renormalizable” in some nonperturbative methods. We have investigated the model using the Wilsonian renormalization group method [1, 2]. The Wilsonian renormalization group (WRG) equation describes the infinitesimal transformation of the Wilsonian effective action when we change the ultraviolet (UV) cutoff scale $\Lambda$ to $\Lambda(\delta t) = e^{-\delta t}\Lambda$ [3, 4, 5]. The equation consists of two parts; the contribution of the higher frequency modes and the terms arising from rescaling field variables to normalize the kinetic term. In general, the Wilsonian effective action has an infinite number of the local interaction terms, therefore the WRG equation consists of an infinite set of the differential equations for these coupling constants. To solve the equation, we usually use the derivative expansion. The lowest order of the approximation is the local potential approximation retaining only the potential terms. The next nontrivial approximation is to include the second derivative terms, generally written in the form of sigma model Lagrangian.

In this paper, we impose $N = 2$ supersymmetry to the theory to forbid the appearance of the local potential terms.

In the WRG approach, the renormalizability is translated to the nontrivial continuum limit to a possible ultra-violet (UV) fixed point [6, 7]. If there is a UV fixed points, we can fine tune the coupling constant when we take the continuum limit $\Lambda \to \infty$. Therefore, it is important to investigate the existence of the fixed point for WRG flow.

The WRG equation for sigma model Lagrangian gives a flow equation for the metric function of the target space. In two dimensional sigma models case, we found fixed point theories in paper [6]. One of these fixed point theories has the Witten’s Euclidean black hole solution as a target space [8]. In this paper, we will construct the three dimensional conformal sigma models using the nonperturbative renormalization group equation. The flow equation has additional terms in contrast to the two dimensional cases, and corresponds to the modified Ricci flow equation [9, 10]. The flow equation has one free parameter, arising from the field rescaling effects, and we show that the parameter is the conformal dimension of the scalar field.

This paper is organized as follows. In section 2 and 3, we will shortly review of the three dimensional nonlinear sigma model, and WRG equation for it, respectively. In section 4, we discuss the fixed point theory for a special value of the anomalous dimension. We will investigate more general cases in section 5. In this section, we will confine ourselves to two dimensional target spaces for simplicity. Finally, we will study the conical singularity of target spaces in section 6. In the appendix, we will discuss the fixed point theories in other coordinate system.

\[^4\text{Part of this work has been reported at several conferences.}\]
2 Nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry in three dimensions

Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in three dimensions are defined by the so-called Kähler potential $K(\phi, \bar{\phi})$, which is a function of the chiral and anti-chiral superfields, $\phi^i$ and $\bar{\phi}^j$. A chiral superfield $\phi^i(x, \theta)$ consists of a complex scalar field $\varphi^i(x)$ and a complex fermion $\psi^i(x)$

$$\phi^i(x, \theta) = \varphi^i(x) + \theta \psi^i(x) + \theta^2 F^i(x),$$

where $F^i(x)$ is an auxiliary field. The bosonic fields $\varphi^i(x)$ play the role of the coordinates of the target manifold $\mathcal{M}$. The metric, characterizing the target manifold $\mathcal{M}$, is obtained by the second derivative of this Kähler potential

$$g_{ij} = \frac{\partial^2 K(\varphi, \bar{\varphi})}{\partial \varphi^i \partial \bar{\varphi}^j} \equiv K_{ij}.$$

The manifold defined by a Kähler potential is called the Kähler manifold. This metric is an arbitrary function of the scalar fields. The Lagrangian of nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry reads

$$L = g_{ij} \partial_\mu \varphi^i \partial^\mu \bar{\varphi}^j + i g_{ij} \bar{\psi}^j (D_\mu \psi)^i + \frac{1}{4} R_{ijkl} \psi^i \psi^j \psi^k \psi^l,$$

where the covariant derivative for the fermion fields is given by

$$(D_\mu \psi)^i = \partial_\mu \psi^i + \partial_\mu \varphi^j \Gamma^i_{jk} \psi^k.$$

The connection and the Riemann curvature are also written in terms of the Kähler potential $K$ as follows

$$\Gamma^k_{ij} = g^{kl} g_{jl,\bar{\alpha}} = g^{kl} K_{,ijl},$$

$$R_{ijkl} = g_{im} R_{jkl}^{\bar{\alpha}} = K_{,ijkl} - g_{m\bar{\alpha}} K_{,mjl} K_{,\bar{\alpha}ik}.$$

The Ricci curvature is defined by

$$R_{ij} = -g^{kl} R_{ijkl}.$$

The first term of the Lagrangian (2.1) shows infinite number of the derivative interactions. All these interaction terms are perturbatively nonrenormalizable, since the scalar fields have the canonical dimension $d_\varphi = 1/2$ in three dimensions.

3 Renormalization group equation

Renormalization group (RG) equation for the metric of the target manifold $\mathcal{M}$ in three dimensional sigma models has been derived in [1, 2]

$$-\frac{d}{dt} g_{ij} = \frac{1}{2 \pi^2} R_{ij} + \gamma \left(2 g_{ij} + \varphi^k g_{ij,k} + \varphi^{*k} g_{ij,k} \right) + \frac{1}{2} \left( \varphi^k g_{ij,k} + \varphi^{*k} g_{ij,k} \right),$$

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where \( t \) parametrizes the change of the cutoff \( \Lambda \to e^{-t} \Lambda \), and \( \gamma \) denotes the anomalous dimension of the field \( \phi \), introduced to normalize the field at the origin

\[
g_{ij} \big|_{\varphi=0} = \delta_{ij}. \tag{3.1}
\]

The above RG equation, derived by using the so-called Kähler normal coordinate\[12\], can be written in a covariant form

\[
- \frac{d}{dt} g_{ij} = \frac{1}{2\pi^2} R_{ij} - g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i \tag{3.2}
\]

if we define a vector field

\[
\xi^i = \left( \frac{1}{2} + \gamma \right) \varphi^i \tag{3.3}
\]

in the Kähler normal coordinate. In other coordinate system, we have to choose a vector field corresponding to the scale transformation of the target manifold. The covariant derivative for vector field is defined by

\[
\nabla_k \xi^i = \partial_k \xi^i + \Gamma_{ij}^k \xi^j, \quad \nabla_i \xi_k = \partial_i \xi_k - \Gamma^{jk}_i \xi_j
\]

The RG equation (3.2), called the modified Ricci flow in mathematical literature\[10, 13\], describes the deformation of the target manifold of the effective theory.

It should be emphasized that although the RG equation obtained in the perturbation theory has the similar form with the RG equation obtained in the Wilson’s renormalization method, it is valid only in the vicinity of the free field theory, whereas the Wilsonian RG equation can be used to study even nontrivial conformal field theories located far away from the free field theory.

The fixed point, invariant under the change of the mass scale, is obtained by solving an equation

\[
\frac{1}{2\pi^2} R_{ij} - g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i = 0. \tag{3.4}
\]

The metric \( g_{ij} \) satisfying this equation defines a conformal field theory, and such solution is called the Kähler-Ricci soliton\[14\].

From now, we put the parameter \( c \) as

\[
c \equiv \frac{1}{2} + \gamma, \tag{3.5}
\]

which corresponds to the conformal dimension of the scalar fields at fixed point.

### 4 Fixed point theory for \( \gamma = -\frac{1}{2} \)

When the anomalous dimension of the fields takes a specific value \(-\frac{1}{2}\), the fixed point of the renormalization group equation has an extremely simple form

\[
\frac{1}{2\pi^2} R_{ij} - g_{ij} = 0. \tag{4.1}
\]
By comparing with the equation for the Einstein-Kähler manifolds\(^5\)

\[ R_{ij} - h\lambda^2 g_{ij} = 0, \]  

(4.2)

with a positive cosmological constant \( h\lambda^2 > 0 \), we find the coupling constant \( \lambda \) (inverse radius of the Einstein-Kähler manifold) of the fixed point theory is given by

\[ \lambda^2 = \frac{2\pi^2}{h}. \]  

(4.3)

We found that any Einstein-Kähler manifold corresponds to the conformally invariant field theory, when the radius, the inverse coupling constant, takes a specific value \( 4\lambda \).

A special class of the Kähler-Einstein manifolds is provided by the hermitian symmetric space (HSS)\(^6\) of the form \( G/H \). The compact HSS is completely classified and listed in the following table, where \( h \) denotes the dual coxeter number of the group \( G \).

<table>
<thead>
<tr>
<th>( G/H )</th>
<th>( D = \text{dim}_C(G/H) )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SU(N)/SU(N - 1) \times U(1) )</td>
<td>( N - 1 )</td>
<td>( N )</td>
</tr>
<tr>
<td>( U(N)/U(N - M) \times U(M) )</td>
<td>( M(N - M) )</td>
<td>( N )</td>
</tr>
<tr>
<td>( SO(N)/SO(N - 2) \times U(1) )</td>
<td>( N - 2 )</td>
<td>( N - 2 )</td>
</tr>
<tr>
<td>( Sp(N)/U(N) )</td>
<td>( \frac{1}{2}N(N + 1) )</td>
<td>( N + 1 )</td>
</tr>
<tr>
<td>( SO(2N)/U(N) )</td>
<td>( \frac{1}{2}N(N - 1) )</td>
<td>( N - 1 )</td>
</tr>
<tr>
<td>( E_6/SO(10) \times U(1) )</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>( E_7/E_6 \times U(1) )</td>
<td>27</td>
<td>18</td>
</tr>
</tbody>
</table>

The metric of HSS is explicitly constructed by using the gauge theory technique in Ref.\(^7\), therefore it is possible to write down the Lagrangian of conformal field theories explicitly.

### 5 Two-dimensional manifold

Although it is difficult to solve eq.\(^3\) explicitly for \( \gamma \neq -\frac{1}{2} \), it can be solved for two-dimensional target space \( \mathcal{M} \) by using a graphical method. In this section, we use real variables to describe the target manifold \( \mathcal{M} \), and choose a special gauge where the line element of \( \mathcal{M} \) takes the following form\(^7\):

\[ ds^2 = dr^2 + e^2(r)d\phi^2. \]  

(5.1)

\(^5\) A parameter \( \lambda \) has been introduced to satisfy the renormalization condition \(^3\) at the origin.

\(^6\) For \( S^2 \), \( \lambda^2 \) is related to the radius \( a^2 \) of the sphere by \( \lambda^2 = 1/2a^2 \) and \( h = 2 \).

\(^7\) We will discuss other gauges in the appendix.
Since our target spaces are complex manifolds, we have assumed rotational symmetry in the $\phi$ direction corresponding to the $U(1)$ symmetry, and normalize the range of $\phi$ to $0 \leq \phi < 2\pi$. Then $e(r)$ denotes the radius of a circle for a fixed value of $r$.

Now, components of the connection are given by

$$
\Gamma^r_{\phi \phi} = -ee', \quad \Gamma^\phi_{r \phi} = e', \quad \Gamma^\phi_{\phi \phi} = 0
$$

(5.2)

$$
\Gamma^r_{rr} = \Gamma^r_{r \phi} = \Gamma^\phi_{\phi r} = 0
$$

where prime denote derivatives with respect to $r$. In this coordinate system, the Ricci tensor takes the following form

$$
R_{rr} = R^\phi_{r \phi r} = -e''/e, \quad R_{\phi \phi} = R^r_{\phi r \phi} = -ee'',
$$

(5.3)

Corresponding to the renormalization condition (3.1), we impose a boundary condition for $e(r)$

$$
\lim_{r \to 0} e(r)/r = 1.
$$

The fixed point of the RG equation written in terms of real coordinates corresponds to the solution of

$$
a^2 R_{ij} - g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i = 0,
$$

(5.4)

where

$$
a^2 = \frac{1}{2\pi^2}.
$$

Now, we have to find the vector field $\xi^i = (\xi^r, \xi^\phi)$. The vector field $\xi^i$, representing an infinitesimal scale transformation of the target space, has to be proportional to $(cr, 0)$ at least around the origin $r = 0$, where the renormalization condition (3.1) was imposed. Since we assume the rotational symmetry ($U(1)$), it is natural to assume $\xi^\phi = 0$. Then the vector field in this coordinate system is fixed by the consistency of the coupled differential equation (5.4). The equation for $r - r$ and $\phi - \phi$ components are

$$
-a^2 e''/e - 1 + 2\xi'_r = 0, \quad (5.5)
$$

$$
-a^2 ee'' - e^2 + 2e' e \xi_r = 0. \quad (5.6)
$$

The equation for $r - \phi$ component is satisfied trivially. We find

$$
\frac{\xi'_r}{\xi_r} = \frac{e'}{e}. \quad (5.7)
$$

by requiring the compatibility of two equations. Since $\xi_r$ has to be proportional to $cr$ around the origin, we choose the vector field $\xi_i$ in this coordinate system as follows

$$
\xi^r = ce(r), \quad \xi^\phi = 0, \quad (c = \frac{1}{2} + \gamma), \quad (5.8)
$$
taking into account the boundary condition \((5.3)\). Finally, we obtain the RG equation in this gauge

\[-a^2 e'' - e + 2ce e' = 0.\]  

(5.9)

When \(c = 0\), namely for \(\gamma = -1/2\), the solution of this equation is easily obtained

\[e(r) = a \sin \frac{r}{a},\]

which defines the line element of the round \(S^2\) with radius \(a\)

\[ds^2 = dr^2 + a^2 \sin^2 \frac{r}{a} d\phi^2,\]

in conformity with the result of the previous section.

On the other hand, when \(c \neq 0\), it is convenient to rewrite the second order differential equation to a set of the first order differential equations

\[e' = p\]

(5.10)

\[p' = -\frac{1}{a^2} e(1 - 2cp)\]

with the boundary condition

\[e(0) = 0, \quad p(0) = 1\]  

(5.11)

Furthermore, if we introduce a new variable

\[Q(r) = \frac{1}{2c} \log |1 - 2cp(r)| \quad \text{and} \quad P(r) = e(r),\]  

(5.12)

the first order differential equations \((5.10)\) can be rewritten in the form of the Hamilton’s equation of motion

\[\frac{dQ}{dr} = \frac{P}{a^2} = \frac{\partial H}{\partial P}\]

\[\frac{dP}{dr} = \frac{1 - e^{2cQ}}{2c} = -\frac{\partial H}{\partial Q}\]  

(5.13)

where the "Hamiltonian" is given by

\[H(Q, P) = \frac{1}{2a^2} P^2 + V(Q)\]  

(5.14)

where \(V(Q)\) stands for the "potential energy"

\[V(Q) = -\frac{1}{2c} Q + \frac{1}{(2c)^2} e^{2cQ}.\]  

(5.15)

In eqs. \((5.13)\) and \((5.15)\), we have assumed \(1 - 2cp(r) > 0\). When \(1 - 2cp(r) < 0\), we have to introduce an extra minus sign in front of \(e^{2cQ}\) in these equations.
Once we fix the value of the parameter $c$, the sign of $1 - 2cp(r)$ at $r = 0$ is determined by the initial condition (5.11). It is easily seen that $1 - 2cp(r)$ does not change sign at any "time" $r$. To see this, let us assume $c > 0$(similar argument holds for $c < 0$). Suppose $1 - 2cp(r)$ changes sign at some "time" $r$, then $Q$ has to go to $-\infty$ at that "time" $r$ by the definition of $Q$ (5.12). However, as we will discuss below, $Q$ has the lower bound, therefore the sign of $1 - 2cp(r)$ does not change at any "time" $r$.

Since this "Hamiltonian" does not depend on $r$ explicitly, the energy is independent of "time" $r$. We draw the potential $V(Q)$ when $1 - 2cp(r) > 0$ in Fig. 1. The horizontal line represents a constant value of energy. Since the kinetic energy is positive definite, solutions exist only in the bounded region $Q_{\text{min}} \leq Q \leq Q_{\text{max}}$ where $Q_{\text{min}}$ and $Q_{\text{max}}$ are determined by $V(Q) = E$. It is also true for $1 - 2cp(r) < 0$ that $Q$ has a lower bound even when the sign of $e^{2cQ}$ has been changed in the potential (5.15). According to the initial condition (5.11), $p(r)$ starts from $p(0) = 1$. The fact that $1 - 2cp(r) < 0$ does not change sign implies that $p(r) < 1$ if $2c < 1$, and $p(r) > 1$ if $2c > 1$.

The boundary condition (5.11) gives the initial condition

$$Q(0) = \frac{1}{2c} \log |1 - 2c|, \quad P(0) = 0,$$

which determines the value of the energy

$$H(Q, P) = E = \frac{1}{(2c)^2} \left( |1 - 2c| - \log |1 - 2c| \right).$$

Then, the conservation of "energy" is expressed as follows

$$\frac{2}{(2c)^2} \left( -2c(1 - p(r)) + \log |1 - 2cp(r)| - \log |1 - 2c| \right) = \frac{e^{2(r)}}{a^2},$$

which determines the closed contour shown in Fig. 2 for $2c < 1.$
Figure 2: Flow of the first order differential equations (5.10) for $0 < 2c < 1$ in the "phase space" $(e(r),p(r))$. The solid line represents the solution specified by the boundary condition.

The vector field of the flow (5.10) is shown in Fig[2]. When $0 \leq 2c < 1$, this equation defines a compact manifold since the trajectory starting from the initial point (5.11) comes back to $e = 0$ at a finite $r = r_A$ implying the circumference of the circle at that $r$ vanishes. We may call it the "deformed shpere", whose metric function can be approximated by

$$e(r) = a \sin \left( \frac{r}{a} \right) + \frac{ca}{3} \left( 2 \sin \left( \frac{r}{a} \right) - \sin \left( \frac{2r}{a} \right) \right) + O(c^2)$$

for small $c$. Similarly, we have compact target space for $c < 0$.

On the other hand, Fig[3] shows the radius $e(r)$ becomes larger and larger when $r$ goes to infinity, then the solution corresponds to a noncompact manifold for $2c \geq 1$. To see the asymptotic behavior of $e(r)$ for large $r$, we will be able to neglect the second term in (5.9)

$$\frac{de}{dr} = 1 + \frac{c}{a^2} e^2$$

which can be integrated to obtain $e(r)$

$$e(r) = \frac{a}{\sqrt{c}} \tan \left( \frac{\sqrt{c}}{a} r \right).$$

Since $e(r)$ defines the radius of the circle when the geodesic distance from the origin $r$ is fixed, this asymptotic behavior implies that the manifold for $2c > 1$ has an increasingly large radius for large $r$. 

8
where use has been made of the covariant constancy of the metric in two dimensions, the left hand side corresponds to the Euler number when the manifold has a topological quantity at the fixed point.

Figure 3: Flow of the first order differential equations (5.10) for $2c \geq 1$ in the phase space. The solid line represents the solution specified by the boundary condition.

6 The Euler number and the volume of the manifold

Let us discuss the change of the volume of the target manifold along the flow of the RG equation. The change of the volume of the manifold along the renormalization group flow (3.2) is given by

$$\frac{d}{dt} \int \sqrt{\det g} d\varphi d\varphi^* = \frac{1}{2} \int \sqrt{\det g} \text{Tr} \left( g^{ij} \frac{dg_{ij}}{dt} \right) d\varphi d\varphi^*$$

$$= -\frac{1}{4\pi^2} \int \sqrt{\det g} R d\varphi d\varphi^* + \frac{\dim C \cdot \mathcal{M}}{2} \int \sqrt{\det g} d\varphi d\varphi^*, \quad (6.1)$$

where use has been made of the covariant constancy of the metric $\nabla g = 0$ and the scalar curvature $R$ is defined by

$$R = g^{ij} R_{ij}.$$

At the fixed point, the left hand side of the equation (6.1) vanishes, and we can obtain the following relation:

$$\frac{1}{4\pi^2} \int \sqrt{\det g} R d\varphi d\varphi^* = \frac{\dim C \cdot \mathcal{M}}{2} \int \sqrt{\det g} d\varphi d\varphi^*. \quad (6.2)$$

In two dimensions, the left hand side corresponds to the Euler number when the manifold has neither a boundary nor a singularity.

$$\chi(\mathcal{M}) = \frac{1}{4\pi} \int \sqrt{\det g} R d\varphi d\varphi^* = 2(1 - g),$$

where the genus is 0, 1 for $S^2$ and $T^2$. Thus, the volume of the target manifold is defined by the topological quantity at the fixed point.

$$V(\mathcal{M}) = \int \sqrt{\det g} d\varphi d\varphi^* = \frac{2}{\pi} \chi(\mathcal{M}).$$
Actually, when the manifold is a round sphere $S^2$ with radius $a$,

$$V(S^2) = 4\pi a^2 = \frac{2}{\pi}$$

implies the radius at the fixed point is

$$a^2 = \frac{1}{2\pi^2}. \quad (6.3)$$

The radius coincides with the footnote at page 4.

How about $c \neq 0$ cases? Unfortunately, in such cases, the target space has a conical singularity. Then the relationship between the Euler number and the volume is no longer valid. We will show that in detail.

In previous section, the Fig. 2 shows the target space with nonzero $c$ describes a deformation from the $S^2$. Then the Euler number should not change. To show the invariance of the Euler number under this deformation, we briefly review of the Gauss-Bonnet theorem.

Let us consider the metric of the target space with a conical singularity. The flat space has the following metric:

$$ds^2 = dr^2 + r^2 d\phi^2. \quad (6.4)$$

On the other hand, the cone (Fig. 4) has the additional condition

$$0 \leq \phi \leq \Phi, \quad (6.5)$$

because of the deficit angle. We rescale $\phi' = \frac{2\pi}{\Phi} \phi$, then the effect of the deficit angle appears in the metric as follow:

$$ds^2 = dr^2 + \left( r^2 \left( \frac{\Phi}{2\pi} \right)^2 \right) d\phi'^2. \quad (6.6)$$

Figure 4: The deficit angle $(2\pi - \Phi)$ of the cone.
In the Eq.(5.1), we put \( ds^2 = dr^2 + e^2(r)d\phi^2 \), and we use the initial condition \( e'(r = 0) = 1 \). If the manifold is smooth at a point \( (r = r_A) \) in Fig[2] the first differential of the function \( e(r) \) should equal to \(-1\). In other words, if the target space has a conical singularity at \( r = r_A \), we can obtain the following equation,

\[
\frac{\partial e(r)}{\partial r}|_{r=r_A} = -\frac{\Phi}{2\pi} \neq -1. \tag{6.7}
\]

In general, if the target space has several conical singularities at the point \( r = r_i, i = 1, \cdots, n \), the Gauge-Bonnet theorem is given as follow:

\[
2\pi \chi(M) = \frac{1}{2} \int_M dV \sqrt{g} R + \sum_{i=1}^{n} (2\pi - \Phi_i). \tag{6.8}
\]

In this coordinate system, the Ricci scalar and the determinant of the metric are given by \( R = -2e''(r)/e(r) \) and \( \sqrt{g} = e(r) \) respectively. Then the first term of the eq.(6.8) is

\[
\frac{1}{2} \int_M dV \sqrt{g} R = -2\pi [e'(r)|_{r=r_A} - 1]. \tag{6.9}
\]

The deficit angle at \( r = r_A \) is given by

\[
2\pi - \Phi = 2\pi - \left( -2\pi e'(r)|_{r=r_A} \right). \tag{6.10}
\]

Then, we see the Euler number remains unchanged by the \( c \)-deformation.

\[
\chi(M) = 2. \tag{6.11}
\]

On the other hand, the volume of the target space depends on the value of \( e'(r) \) at the point \( A \). From Eq.(6.2), the volume is given by

\[
V(M) = \frac{2}{\pi} \left[ 1 - e'(r)|_{r=r_A} \right], \tag{6.12}
\]

when \( 0 \leq c < 1/2 \). In this region, \( e'(r)|_{r=r_A} \) is negative and the absolute value monotonously increases with \( c \). For small \( c \), the lowest order approximation (5.19) gives \( e'(a\pi) = -(1 + \frac{4c}{\pi}) \), which implies \( \Phi = 2\pi(1 + \frac{4c}{\pi}) \) for small \( c \). At \( c = 1/2 \), the target space become flat space, then the volume goes to infinity.

### 7 Summary

Three dimensional non-linear sigma models are perturbatively nonrenormalizable. However, we derive the non-perturbative renormalization group equation for these models in Ref.[1, 2]. In this paper, we rewrite the renormalization group equation to the covariant form using the Kähler normal
coordinates. The covariant renormalization group equation has the same form with the modified Ricci flow. We investigated the fixed point theories for the renormalization group equation.

The equation has one free parameter ($c$). The parameter corresponds to the conformal dimension of the scalar fields, and is the sum of the canonical dimension ($1/2$) and the anomalous dimension. We find that the target spaces of the fixed point theories must satisfy the Einstein-Kähler condition with a specific value of the radius $c = 0$. We will discuss the stability of these fixed point target spaces in Ref. [16].

For $0 < c < 1/2$, the target space of the fixed point theory becomes the "deformed sphere", we check that numerically in the case of one complex dimension. On the other hand, for $c \geq 1/2$, we find that the target space is non-compact. At the critical value $c = 1/2$, the target space reduces to a flat space corresponding to a free field theory.

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A Ricci Flow Equation in Other Gauges

In this appendix, we discuss the Ricci flow equation in two dimensions using other coordinate system, parametrized by

$$ds^2 = A(x)(dx^1)^2 + B(x)(dx^2)^2,$$

which implies

$$g_{11} = A(x), \quad g_{22} = B(x), \quad g^{11} = \frac{1}{A}, \quad g^{22} = \frac{1}{B}. \quad (A.2)$$

By using the definition of connections

$$\Gamma^k_{ij} = \frac{1}{2} g^{mk} \{ \partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij} \}, \quad (A.3)$$

various components of connections are calculated.

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8Anomalous dimensions for some theories may be negative. Negative anomalous dimensions are allowed in non-linear sigma models[17].
\[ \Gamma^1_{11} = \frac{\dot{A}}{2A}, \quad \Gamma^1_{12} = \frac{A'}{2A}, \quad \Gamma^1_{22} = -\frac{\dot{B}}{2A}, \]
\[ \Gamma^2_{11} = -\frac{A'}{2B}, \quad \Gamma^2_{12} = \frac{B'}{2B}, \quad \Gamma^2_{22} = \frac{B'}{2B}, \quad \] (A.4)

where
\[ \dot{A} \equiv \frac{\partial A}{\partial x^1}, \quad A' \equiv \frac{\partial A}{\partial x^2}. \]

Various Riemann curvatures are given by
\[ R^1_{212} = -\partial_1 \left( \frac{\dot{B}}{2A} \right) - \frac{1}{2} \left( \log A \right)'' - \frac{A''^2}{4A^2} + \frac{\dot{B} B}{4AB} - \frac{\dot{A} \dot{B}}{4A^2} + \frac{A' B'}{4AB}, \]
\[ R^2_{121} = -\frac{1}{2} \partial_2 \log B - \partial_2 \frac{A'}{2B} + \frac{A''}{4AB} - \frac{\dot{B}^2}{4B^2} + \frac{\dot{A} \dot{B}}{4AB} - \frac{A' B'}{4B^2}, \]

where we used the definition
\[ R_{ij}^{\ell} = \partial_j \Gamma^\ell_{ik} - \partial_k \Gamma^\ell_{ij} + \Gamma^m_{ik} \Gamma^\ell_{mj} - \Gamma^m_{ij} \Gamma^\ell_{mk}. \] (A.6)

Ricci and scalar curvatures are obtained by using the Riemann curvature

**Ricci tensor**
\[ R_{11} = R^2_{121}, \quad R_{22} = R^1_{212}, \quad R_{12} = R_{21} = 0 \] (A.7)

**Scalar curvature**
\[ R = g^{ij} R_{ij} = \frac{R_{11}}{A} + \frac{R_{22}}{B} \] (A.8)

### A.1 Robertson-Walker type gauge

If the system has rotational symmetry, we can safely assume the following form of the metric
\[ x^1 = r, \quad x^2 = \phi, \quad A(x) = f^2(r), \quad B(x) = r^2, \]
then the line element is written as follows
\[ ds^2 = f^2(r)dr^2 + r^2 d\phi^2, \] (A.9)

where \( r \) denotes the radius of the circle when \( r \) is kept fixed.

\[ g_{rr} = f^2, \quad g_{\phi\phi} = r^2, \quad g'' = \frac{1}{f^2}, \quad g^{\phi\phi} = \frac{1}{r^2}. \]
Connections and curvatures in this gauge are given by
\[ \Gamma^r_{rr} = \frac{f'}{f}, \quad \Gamma^r_{r\phi} = \frac{1}{r}, \quad \Gamma^r_{\phi\phi} = -\frac{r}{f^2}, \quad \text{others} = 0 \quad \text{(A.10)} \]
\[ R_{rr} = R_{r\phi r} = \frac{f'}{rf}, \quad R_{\phi\phi} = R_{\phi r\phi} = \frac{rf'}{f^3} \quad \text{(A.11)} \]
where prime denotes the derivative with respect to \( r \).

The vector field of the rescaling is given in this coordinate system
\[ \xi^r = \frac{cr}{f(r)}, \quad \xi^\phi = 0, \]
where
\[ c = \frac{1}{2} + \gamma. \]

Since the covariant components are
\[ \xi_r = crf, \quad \xi_\phi = 0, \]
the covariant derivatives of the vector field
\[ \nabla_r \xi_r = \frac{\partial}{\partial r}(crf) - \Gamma^r_{rr}(crf) = cf \]
\[ \nabla_\phi \xi_\phi = -\Gamma^r_{\phi \phi}(crf) = c \frac{r^2}{f} \]
lead to an identical differential equation for the \( rr \)- and \( \phi\phi \)-components of the RG-equation
\[ a^2 \frac{f'}{rf} - f^2 + 2cf = 0, \quad \text{where} \quad a^2 = \frac{1}{2\pi^2}. \quad \text{(A.12)} \]

By a change of the variable \( x = r^2 \), it can be rewritten as
\[ 2a^2 \frac{df}{dx} - (f - 2c)f^2 = 0, \quad \text{(A.13)} \]
which can be integrated to give
\[ \frac{2}{(2c)^2} \left( \frac{2c}{f} + \log \left| \frac{f - 2c}{f} \right| \right) = \frac{x}{a^2} + \text{const.} \]

The boundary condition \( f(0) = 1 \) fixes the integration constant.
\[ \frac{2}{(2c)^2} \left( \frac{2c}{f} - 2c + \log \left| \frac{f - 2c}{f} \right| - \log |1 - 2c| \right) = \frac{x}{a^2}. \quad \text{(A.14)} \]

When \( c = 0 \ (\gamma = -\frac{1}{2} \text{ case}) \), the solution of this implicit equation is very simple
\[ f(r) = \frac{1}{\sqrt{1 - \frac{r^2}{a^2}}}. \]
This coordinate covers the half of the round sphere $S^2$. To cover the whole sphere, we introduce the new coordinate $\theta$ with the range $0 \leq \theta \leq \pi$

$$ds = a^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad r = a \sin \theta$$

For non-vanishing value of $c$, namely when $\gamma \neq -\frac{1}{2}$, the left hand-side of the implicit equation (A.14) is plotted in the Fig. 5. The right-hand side is represented by a horizontal line. For fixed value of $x$, $f(x)$ is given by the intersection of the graph and a horizontal line $y = \frac{x}{a^2}$. The boundary condition forces us to choose the branch I near $x \approx 0$ (the south-pole). $f(x)$ is an increasing function for small $x$. There is a maximum value of $x$ where $f(x)$ diverges to infinity. As in the case of the sphere for $c = 0$, the geodesic distance to this point is finite, and we can extend further by using the solution in the negative values of $f$ in the branch II. The north pole corresponds to a point where the graph of the branch II intersects with the horizontal axis ($x = 0$).

The approximate solution in the branch I of (A.14) around the origin $x = 0$ is given by the Taylor series

$$f(x) = 1 + \frac{1}{2} \cdot \frac{1 - 2c}{a^2} x + \frac{1}{8} \cdot \frac{3 - 10c + 8c^2}{a^4} x^2 + \cdots.$$ (A.15)

The maximum value of $x$ is given by

$$x_{\text{max}} = a^2 \left( \frac{-1}{c} - \frac{1}{2c^2} \log |1 - 2c| \right)$$ (A.16)

When $x$ approaches to $x_{\text{max}}$, $f(x)$ diverges as

$$f(x) \approx \frac{a^2}{\sqrt{x_{\text{max}} - x}}.$$ (A.17)
The line element near $r \approx r_{\text{max}} = \sqrt{x_{\text{max}}}$ is
\[ ds^2 = \frac{a^2}{r_{\text{max}}^2 - r^2} (dr)^2 + r^2 (d\phi)^2, \] (A.18)
which can be rewritten as
\[ ds^2 = a^2 (d\theta)^2 + r_{\text{max}}^2 \sin^2 \theta (d\phi)^2 \] (A.19)
if we define an angle $\theta$ by $r = r_{\text{max}} \sin \theta$. The point $r = r_{\text{max}}$ corresponding to $\theta = \pi/2$ is not a singularity but has a finite distance from the origin ($r = 0$), so that we can extend further to the region $\pi/2 \leq \theta$ where $r$ begins to decrease again.

### A.2 Relation of various gauges

Let us discuss first the relation between the Robertson-Walker type gauge (A.9) and the synchronous gauge (5.1) discussed in section 5. In this subsection, we denote the radius in the synchronous gauge as $R$. If we define $R$ in the Robertson-Walker gauge by
\[ R(r) = \int_0^r f(r)dr, \] (A.20)
the line element takes the following form
\[ ds^2 = f^2(r)(dr)^2 + r^2 (d\phi)^2 = dR^2 + e^2(R) (d\phi)^2 \]
where \( e(R) \) defined by

\[
e(R) = r(R)
\]

(A.21)
is the inverse function of (A.20). From (A.21) and (A.20), we have a relation

\[
p(R) \equiv e'(R) = \frac{dr}{dR} = \frac{1}{f(r)}.
\]

(A.22)

If we substitute eqs. (A.21) and (A.22) into (5.18), we recognize that (5.18) coincides with (A.14) exactly.

Let us discuss the relation between the Robertson-Walker type gauge and the conformal gauge when there is a rotational symmetry. If \( R \) and \( \rho(R) \) satisfy

\[
f(r) \frac{dr}{dR} = \rho(R), \quad r = R\rho(R),
\]

(A.23)

the line element (A.9) can be written in the form of the conformal gauge (A.27)

\[
ds^2 = \rho^2(R)(dR^2 + R^2d\phi^2).
\]

(A.24)

From (A.23), we obtain

\[
R(r) = re^{\int_0^r \frac{f(t)}{r(t)} dt}, \quad \rho(R) = \frac{r(R)}{R},
\]

(A.25)

(A.26)

where \( r(R) \) denotes the inverse of (A.25).

In conformal gauge

\[
A(x) = B(x) = \rho^2(x),
\]

the line element takes a very simple form

\[
ds^2 = \rho^2(x)(dx_1^2 + dx_2^2).
\]

(A.27)

Connections, curvatures in this gauge are given by

\[
\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \frac{\dot{\rho}}{\rho},
\]

(A.28)

\[
\Gamma^1_{12} = -\Gamma^2_{11} = \Gamma^2_{22} = \frac{\rho'}{\rho},
\]

(A.29)

\[
R^1_{212} = R^2_{121} = -\Delta \log \rho
\]

(A.30)

\[
R_{11} = R_{22} = -\Delta \log \rho
\]

(A.31)

\[
R = R_{11} + R_{22} = -\frac{2}{\rho^2} \Delta \log \rho
\]

(A.32)
References


