Vortex flow generated by a magnetic stirrer

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We investigate the flow generated by a magnetic stirrer in cylindrical containers by optical observations, PIV measurements and particle and dye tracking methods. The tangential flow is that of an ideal vortex outside of a core, but inside downwelling occur with a strong jet in the very middle. In the core region dye patterns remain visible over minutes indicating a pure stirring and mixing property in this region. The results of quantitative measurements can be described by simple formulas in the investigated region of the stirring bar’s rotation frequency. The tangential flow turns out to be dynamically similar to that of big atmospheric vortices like dust devils and tornadoes.

I. INTRODUCTION

Magnetic stirrers are common equipments in different types of laboratories. The main component of this equipment is a magnet rotating with adjustable frequency around a fixed vertical axis below a flat horizontal surface. The rotation of this magnet brings a magnetic stirrer bar in rotation on the bottom of a container put on the flat surface. If the container is filled up by a fluid, the bar generates strong fluid motion which is believed to generate efficient stirring and mixing. A striking pattern of such flows is a big vortex around the stirrer bar’s rotation axis and the corresponding depletion, the funnel, on the surface, which indicates that the flow is strongly structured.

Our aim is the experimental investigation of the fluid motion in cylindrical containers generated by magnetic stirrers. In the next section simple theoretical models of isolated vortices are reviewed. Then (sections 3,4) we describe the experimental set-up and the used forms of data acquisition. In section 5 we present the results of optical observations, of PIV measurements and of monitoring tracer particles and dyes. Section 6 is devoted to deriving simple relations for the vortex parameters based on the measured data. The concluding section points out the similarities and the differences of our results compared with other whirling systems: bathtub vortices, dust devils and tornadoes.

II. THEORETICAL BACKGROUND

Here we review the most important elementary models of steady isolated vortices in three-dimensional fluids of infinite extent\textsuperscript{1,2,3}. Although our system is obviously more complicated than these models, they are useful reference points in interpreting the data. The model flows shall be expressed in cylindrical (radial, tangential and axial) velocity components \((v_r, v_t, v_z)\).

\textit{Rankine vortex.} In this model the vorticity is uniformly distributed in a cylinder of radius \(c\) with a central line (the \(z\) axis) as its axis. The tangential component is continuous in \(r\) but a break appears at the radius \(c\). The two other components remain zero:

\[ v_r = 0, \quad v_t = \frac{Cr}{c^2}, \quad \text{for} \quad r \leq c, \quad v_t = \frac{C}{r}, \quad \text{for} \quad r > c, \quad v_z = 0. \]  

(1)

Within the radius \(c\) a rigid body rotation takes place, while outside a typical \(1/r\)-dependence appears with \(C\) proportional to the circulation of the flow. The limit \(c \to 0\) corresponds to an ideal vortex line.

\textit{Burgers vortex.} In a real fluid, the viscosity smoothes out the break in the tangential component of the Rankine vortex. In order to maintain a steady rotation, an inflow and an axial flow should be present. In the Burgers vortex the strength of the axial flow increases linearly with the height \(z\), measured from a certain level:

\[ v_r = -\frac{2\nu}{c^2} r, \quad v_t = \frac{C}{r} \left(1 - e^{-r^2/c^2}\right), \quad v_z = \frac{4\nu}{c^2} z. \]  

(2)

Here \(\nu\) is the kinematic viscosity of the fluid, and \(c\) remains an effective radius within which the tangential flow is approximately a rigid body rotation. Note that the tangential velocity component depends on the viscosity via a possible \(\nu\)-dependence of the radius \(c\) only.

III. EXPERIMENTAL SETUP

The experiments were carried out in glass cylinders of different radii \(R\) at different initial water heights \(H\), as summarized in Table I.

<table>
<thead>
<tr>
<th>(R) (cm)</th>
<th>(H) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.8</td>
<td>12.0</td>
</tr>
<tr>
<td>6.5</td>
<td>16.8</td>
</tr>
<tr>
<td>10.5</td>
<td>24.8</td>
</tr>
<tr>
<td>22.4</td>
<td>27.1</td>
</tr>
</tbody>
</table>

TABLE I: The radii \(R\) of the different cylinders used and the heights \(H\) to which water was filled up in each.
Two other relevant geometrical parameters are provided by the length $2a$ and the width $d$ of the magnetic stirrer bars (see Fig. 1). The different parameters of the bars used are summarized in Table II. The system’s basic geometrical parameters are thus $R, H, a$ and $d$.

When the rotation of the magnetic stirrer is switched on, the water column starts moving and after some time (which is on the order of minutes in our case) a steady state sets in. It is worth emphasizing, however, that due to the lack of a fixed axis of the stirrer bar this steady state is subjected to large fluctuations. This is the physical reason behind the relative errors of our measurements being on the order of 10 percents. The most striking optically observable object is the funnel developing on the water surface which is of height $H' > H$ around the perimeter. The characteristic sizes of the funnel, as defined in Fig. 1, can be easily measured (see Section IV).

IV. DATA ACQUISITION

The rotation frequency $\Omega$ of the stirring bar in the container filled up with water was determined by means of a stroboscope whose frequency $f$ is adjustable in a broad range. At certain frequencies $f_n$, the bar appears to be at rest (see Fig. 2a). This happens if the bar rotates an integer multiple of a half rotation between two flashes, i.e., if

$$\frac{\Omega}{f_n} = n\pi. \quad (3)$$

The largest value of the $f_n$'s uniquely determines the bar's rotation frequency as $\Omega = f_1\pi$. In order to reach a higher accuracy, we also determined the frequency $f_{1/2}$ when the bar rotates a quarter of rotation between two flashes. This case is designated by the appearance of a steady cross traced out by the bar on the bottom of the container (see Fig. 2b). The bar's rotation frequency was determined as the average of the frequencies belonging to $f_1$ and $f_{1/2}$. The uncertainty in $f$ is 0.1Hz. In the range $(20, 120) \text{ s}^{-1}$ of $\Omega$ investigated, this corresponds to a relative error of about 1 percent, which is negligible compared to the other errors.

The funnel parameters were determined via direct optical observations. The water height $H'$ in the steady state was measured by means of a ruler. The height of the funnel’s deepest point can be determined in a similar way or by using a horizontal laser sheet. From these two quantities, the funnel depth is simple $\Delta h = H' - h$ (cf. Fig. 1).

The halfwidth of the funnel was measured on the back side of the cylinder. The length obtained this way was corrected by taking into account the optical effect caused by the cylindrical lens formed by the water column to obtain the value $b$. Both the funnel depth and width are subjected to an error of about 10 percents due to the fluctuations in the steady state.

The particle image velocimetry (PIV) method can be used to determine the velocity field in a plane defined by the laser sheet at disposal. From the position of fine tracer particles on two subsequent images taken with a time difference of about $10^{-2}s$, the displacement and velocity of the particles can be determined. We used a commercial PIV equipment (ILA GmbH, Germany) and determined the flow field in the largest container ($R = 22.4\text{ cm}$) at an intermediate water height ($H = 16.8\text{ cm}$) with stirrer bar ii (cf. Table II) in different horizontal layers. The presence of the funnel and the strong downdraft make the PIV data unreliable in the middle of the container, in a region of radius of about 8cm.

Particle tracking enables us to study the central region of the flow. We used plastic beads (low density polyethylene) of diameter $\sim 1\text{ mm}$, which has a density of $\sim 0.92 \text{ g/cm}^3$. Despite of being lighter than water, they sink
below the funnel and often reach a kind of steady state (for more detail see Subsection V C). The approximate strength of the downwelling in the middle was estimated as the rising velocity of the balls in a water column at rest. From several measurements in a separate narrow glass cylinder we found this rising velocity to be 7-8cm/s, for all the balls used.

Spreading of dye can provide a qualitative picture about the flow. A particularly important region is that around the axis of the vortex, where this coloring technique reveals fine details (see Subsection V D).

V. RESULTS

A. Frequency-dependence of the funnel

The results of the optical observations of 16 cases (different containers, water heights and stirring bars) each measured at several frequencies \( \Omega \) are summarized in Figs. 3. While there is a clear frequency dependence in the funnel depth \( \Delta h \), the halfwidth \( b \) appears to be independent of \( \Omega \) (see inset), in first approximation at least.

To extract the form of the frequency-dependence, the funnel depth is plotted on log-log scale in Fig 3. The straight lines clearly indicate a power-law dependence. The exponent is read off to be 2:

\[
\Delta h \sim \Omega^2.
\]

One sees that the coefficient (not written out) depends much stronger on the stirring bar’s parameters \((a, d)\) than on the container’s geometry \((R, H)\).

B. Velocity fields

First we present, in Fig. 4 the result of a typical PIV measurement in a horizontal plane. The arrows mark velocity vectors. Arrows colored in light grey / violet ? are produced by algorithmic interpolation and cannot be considered therefore to be quantitatively reliable. The flow is not fully axially symmetric, small secondary vortices appear around the edges of the picture.

In order to understand the mean flow, we divided each PIV image into narrow concentric rings and evaluated the average tangential and radial velocity in each band. These values were further averaged over several images taken at different times in the same flow and at the same height. This way the effect of secondary vortices visible in Fig. 4 became averaged out. The procedure leads to a discrete representation of the functions \( v_t(r) \) and \( v_r(r) \). Both components appeared to be proportional to the frequency of the stirring bar, therefore we present in Figs. 5 the components already divided by \( \Omega \). Since the results in the innermost region are not reliable, the components are displayed for distances \( r > 8 \text{cm} \) only.

The agreement between the measured tangential data and the fitted hyperbolas of the form of \( A/r \) is satisfactory (cf. Fig. 4). Furthermore, the coefficient \( A \) is practically independent of the height. This indicates that for distances \( r > 8 \text{cm} \) the tangential flow is that of an ideal vortex. The vortex strength \( C \) turns out to be proportional to the frequency, i.e.,

\[
C = A\Omega
\]

with a coefficient \( A \) extracted from our data to be \( A = 0.9 \pm 0.2 \text{ cm}^2 \).
In contrast to the tangential component, the radial component depends strongly on the height (Fig. 5). In the upper layers there is an inflow \((v_r < 0)\) for \(r > 8\) cm at least, which decays towards zero as the height decreases. The level \(z = 4\) cm is dominated by outflow, but around \(r = 8\) cm a weak inflow survives.

The planar PIV algorithm does not provide any information on the vertical velocities. This component can, however, be determined from the continuity equation which takes the form

\[
\frac{\partial v_z}{\partial z} = -\frac{1}{r} \frac{\partial (rv_r)}{\partial r} \quad (6)
\]

in axisymmetric flows. We numerically integrated the left hand side to get an approximant for \(v_z(r)\) from the radial component. This component turns out to be proportional to \(\Omega\), too. The qualitative feature is that there is upwelling in the outermost \(4 - 5\) cm and a slow downwelling in the intermediate region. The strong upwelling around the boundary of the container suggests that there must be a strong downwelling in the very center, which, however, cannot be resolved by means of the PIV method.

### C. Particle tracking

The strength of downwelling in the very center of the vortex can be estimated by means of monitoring plastic particles, lighter than water. They float on the surface but become eventually trapped by the funnel, on the surface of which they slide down and become advected downwards toward the bulk of the fluid (Fig. 6). Along the axis of the vortex the particles reach a kind of steady state (subjected, of course, to considerable fluctuations). This indicates that there is a strong downward jet in which the drag acting on the particle approximately compensates the buoyancy force. As mentioned in Section IV the asymptotic rising velocity of the particles is \(7 - 8\) cm/s in a water of rest. Therefore we conclude that the strength of the downward jet is also \(7 - 8\) cm/s.

The steady state is stable. A necessary condition for this is that the velocity of downwelling decreases when moving downwards along the axis. This occurs unavoidable in our case since the velocity should approach zero close to the lower bottom of the container. Experience shows that the steady state is shifted downwards when the frequency \(\Omega\) is increased, since this makes the jet somewhat stronger.

Stability is maintained in the horizontal direction as well. This is due to the fact that in a rotating system an ‘anticentrifugal’ force acts on the ball since it is lighter then the surrounding fluid. Whenever the particle deviates from the axis, this force directs it backwards. This is accompanied by an immediate rising of the particle which indicates that the width of the strong downward jet is of the same order as the diameter of the particles. We conclude that the width is a few mm.

Due to the presence of all these effects and the permanent fluctuations of the flow, the overall motion of a particle in the steady state is rather complex. After leaving the central jet, it starts rising but the ‘anticentrifugal’ force pushes it back towards the center, at a larger height. Then it is captured by the jet and starts moving downwards again, and will leave the jet at a different height than earlier. We conclude therefore that the particle motion is chaotic (Fig. 6b). By tracking a single particle over a long period of time, a chaotic attractor is traced out in the configurational space of the flow (Fig. 6c).
D. Spreading of dye

When injecting dye into the water outside of the central region one observes a fast spreading. A drastically different behavior is found in the middle of the funnel. Very quickly, a cylindrical dye curtain develops around the vortex axis which remains observable over about a minute (Fig. 7a). This can be seen at any parameter \((R, H, a, d, \Omega)\) investigated. The radius of the dye cylinder is on the order of the magnitude of the halfwidth \(b\), it is about 1cm.

The development of this long-lived dye curtain can be explained by the fact that dye advected by the fast downward jet of a few mm becomes detrained from the center due to local turbulence. Then the dye slows down and accumulates along a surface in which the downward velocity takes its local minimum. We can therefore say that both inside and outside the cylinder surface the downward flow is stronger than within the cylinder surface. The stability of the cylinder is explained by the same effects as the stability of the particles’ steady state. When injecting the dye somewhat off the center of the funnel one often observes more then one neighboring curtains (Fig. 7b). This indicates that there might be more local minima of the downward velocity in a region around the halfwidth of the funnel.

E. Qualitative picture

Based on the observations and the measurement of the velocity components, we obtain the following qualitative picture of the time averaged flow (Fig. 8). In a given vertical slice two flow cells are formed by the upwelling at the outer walls and the downwelling in the middle. In the three-dimensional space this corresponds to a torus flow whose central line lies in a horizontal plane, below the half of the water height. The flow along the central line has no radial and vertical component, a pure rotation takes place. The most striking feature is the strong downward jet in a very narrow filament in middle of the container. It is surrounded by a region of weaker downwelling. Its local minimum is maintained along a cylinder surface whose radius is approximately the halfwidth of the funnel.

VI. QUANTITATIVE RESULTS

A. Funnel depth

To obtain a simple expression for the funnel depth, we apply dimensional arguments first, and a comparison with the measured data leads then to a particular form. As dimensionless measures of the viscous and gravitational effects in the rotating flow, a Reynolds and a Froude number is introduced as

\[
Re = \frac{\Omega a^2}{\nu}, \quad \text{and} \quad Fr = \frac{\Omega d}{(gR)^{1/2}},
\]

respectively. Small values of them indicate strong viscous and gravitational effects. With our typical data \((\Omega = 50s^{-1}, R = 10cm, a = 2.5cm, d = 1cm, \nu = 10^{-2}\text{cm}^2\text{s}^{-1})\) we obtain \(Re = 3 \cdot 10^4\) and \(Fr = 0.5\). This indicates that gravity is essential, but viscosity is not so important for the overall flow. It is, however, obviously important on small scales, like e.g. in the center of the vortex.

The ratio \(\Delta h/d\) must be a function of the dimensionless parameters, therefore we can write

\[
\frac{\Delta h}{d} = f \left( Re, Fr, \frac{a}{d}, \frac{H}{R} \right)
\]

with \(f\) as an unknown function at this point. There might other dimensionless numbers also be present. One candidate would be a measure of the surface tension. This effect we estimated to have an influence of about 10 percent. This is on the order of the typical error in our measurements, therefore, we do not include the corresponding dimensionless number into \(f\).
The fact that several measured values belong to a given values. The continuous line is a fitted hyperbola of the form of symbols is similar as in Fig. 3.

The ratio $h/R > 1$ and for sufficiently large radii ($R > 0.4H$) it scales as $R^{-1/2}$. These observations imply that no $a/d$ and $a/R$-dependence remains in $\Phi$:

$$\frac{\Delta h}{d} = \frac{\Omega^2 a^2 d}{\nu(gR)^{1/2}} \Phi \left( \frac{H}{R} \right).$$

This assumption is supported not only by (4), but by other observations as well: a careful investigation of the data shows that the funnel depth is proportional to $H/R$ is due to the use of different stirrer bars. The use of symbols is similar as in Fig. 3.

The halfwidth $b$ was found in Subsection $\langle 6 \rangle$ to be independent of $\Omega$. The ratio $b/d$ can therefore be written as

$$\frac{b}{d} = \frac{\nu}{\Omega^2 a^2 d R (gR)^{1/2}} \Psi \left( \frac{a}{d} \right).$$

Note that this expression does not contain the water height at all.

### C. Interpretation in terms of a Burgers vortex

The fact that the rotating motion in the magnetic stirrer flow is accompanied with an inflow and a downwelling resembles one on the Burgers model treated in Section $\langle 2 \rangle$. Equation $\langle 2 \rangle$ defines the plane $z = 0$ as a plane without vertical velocity. Since far away from the center there is no downwelling on the free surface at height $H'$, the $z = 0$ level should be chosen as the topmost level of the rotated water. The vortex model obtained this way corresponds to the bulk of the investigated flow, far away from the external walls and the stirrer bar. It does not describe either the upwelling near the walls or the outdraft around the stirring bar.

The steady state of the axisymmetric mean flow implies that the pressure gradient compensates the centrifugal force and gravity in the radial and vertical direction:

$$\frac{\partial p}{\partial r} = \varrho \frac{v^2}{r}, \quad \frac{\partial p}{\partial z} = -\varrho g,$$

where $\varrho$ is the fluid density. On the fluid’s free surface at $z = \eta(r) < 0$ then $\partial \eta/\partial r = -(\partial p/\partial r)/(\partial p/\partial z)$. Consequently,

$$\frac{d \eta}{dr} = \frac{v_l}{rg}.$$  \hspace{1cm} (18)

By inserting here the tangential velocity component from (2), the funnel depth can be obtained by integration:

$$\Delta h = \int_0^{R'} \frac{C^2}{r^3g} \left( 1 - e^{-r^2/c^2} \right)^2 dr.$$  \hspace{1cm} (19)

Here $R' >> c$ is the radius within which the Burgers model is valid. Due to the exponential cut-off within
TABLE III: Parameters and dimensionless numbers for the tangential velocity components of the flows compared.

<table>
<thead>
<tr>
<th>flow</th>
<th>$U$ (m/s)</th>
<th>$H$ (m)</th>
<th>$c$ (m)</th>
<th>$Re'$</th>
<th>$Fr'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>stirrer</td>
<td>0.5</td>
<td>0.2</td>
<td>$10^{-2}$</td>
<td>$5 \times 10^3$</td>
<td>0.4</td>
</tr>
<tr>
<td>bathtub</td>
<td>0.4</td>
<td>0.1</td>
<td>$2 \times 10^{-4}$</td>
<td>$8 \times 10^2$</td>
<td>0.4</td>
</tr>
<tr>
<td>dust devil</td>
<td>25</td>
<td>$10^3$</td>
<td>50</td>
<td>$8 \times 10^7$</td>
<td>0.25</td>
</tr>
<tr>
<td>tornado</td>
<td>70</td>
<td>$10^3$</td>
<td>200</td>
<td>$10^6$</td>
<td>0.7</td>
</tr>
</tbody>
</table>

The integrand the integral can well be approximated by taking $R' = \infty$. We thus obtain

$$
\Delta h = \ln 2 \frac{C^2}{c^2 g}.
$$

(20)

By equating this with the funnel depth expression derived above, we recover relation $C = A\Omega$ found in Subsection V.B with a specific coefficient

$$
A = a^2 \left( \frac{\beta^2 \nu}{\ln 2 (\alpha H + kR)(gR)^{1/2}} \right)^{1/2}
$$

(21)

Similarly, from $c \approx b$ and the halfwidth expression based on the measured data, we obtain

$$
c = \frac{\beta \alpha \nu}{d(gR)^{1/2}}.
$$

(22)

a relation already used in (21). Thus we are able to express both basic parameters $C$, $c$ of the Burgers model with the directly measurable parameters of the flow investigated.

VII. DISCUSSION

Here we compare the fluid dynamical properties of our model with that of other whirling systems: bathtub vortices, sand devils and tornadoes.

Detailed measurements of the velocities in these flows (see5,6, and7,8, respectively) indicate that, in spite basic differences in the other components, the tangential component outside of the vortex core decays with distance $r$ as $C/r$, independently of height. The most dominant tangential component of all the flows is thus practically identical.

To estimate the degree of dynamical similarity in this component, we use another set of the dimensionless numbers which are based on data which can clearly be identified in all these flows:

$$
Re' = \frac{UC}{\nu}, \quad Fr' = \frac{U}{(gH)^{1/2}}.
$$

(23)

$U$ represents here the maximum velocity of the tangential flow component and $c$ denotes the radius of the vortex core.

The value of $U$ can be estimated in our case to be a few dm/s, for the estimate we take 50 cm/s. The core radius and the height are $c \approx b \approx 1$ cm and $H \approx 20$ cm, respectively. The data for the other flows are taken from the literature and are summarized, along with the resulting dimensionless numbers in Table III.

The Froude numbers are on the same order of magnitude but the Reynolds numbers are rather different. This shows that the role of viscosity is much stronger in small scale flows then in the atmosphere. Viscous flow is present in the vortex core, therefore the detailed flow patterns do not match there. The global flow is, however, in all cases practically that of an ideal fluid. Therefore we conclude that outside of the vortex cores the tangential flows are dynamically similar, i.e. our experiment faithfully models all these whirling systems, an observation which can be utilized in undergraduate teaching.

Finally we mention that the analog of the dye curtain can be seen in tornadoes, typically in the vicinity of the bottom since it is the Earth surface which is the source of ‘dye’ (in the form of dust or debris). In some tornadoes this ‘dye’ curtain is clearly separated from the funnel (cf. e.g., http://www.oklahomalighting.com).

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