The Universe as a topological defect

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Abstract

Four-dimensional Einstein’s General Relativity is shown to arise from a gauge theory for the conformal group, SO(4,2). The theory is constructed from a topological dimensional reduction of the six-dimensional Euler density integrated over a manifold with a four-dimensional topological defect. The resulting action is a four-dimensional theory defined by a gauged Wess-Zumino-Witten term. There is a pure gravitational sector in which the Einstein’s field equations hold and the unique coupling constant in the action is shown to be restricted to take integer values.

1 Introduction

Besides its observational success in the solar system, in measurements of the binary pulsar, and in the early universe through primordial nucleosynthesis, Einstein’s general relativity (GR) has a beautiful mathematical formulation. One of the appealing
mathematical features is its connection with a topological invariant in two dimensions. The well known relation of the Einstein-Hilbert Lagrangian and the Euler characteristic can be summarized as follows:

\[ S_{EH} = \frac{c^3}{16\pi G} \zeta_4(M), \quad \chi_2(M) = \frac{1}{4\pi} \zeta_2(M), \quad \zeta_D(M) = \int_M R \sqrt{|g|} d^D x. \quad (1) \]

This fact, sometimes referred to as the dimensional continuation of the Euler density, has a straightforward generalization to higher dimensions, giving rise to the Lovelock series \[1, 2\]. This series in dimension \( D \) contains \( \lceil \frac{D+1}{2} \rceil \) terms, where \( \lfloor \cdots \rfloor \) denotes the integer part. The terms are the dimensionally continued Euler densities of all dimensions below \( D \), and the cosmological constant term.

Although the dimensional continuation process gives a well defined prescription to obtain the most general, ghost-free \(^1\), gravitational Lagrangian \[^3\], its Kaluza-Klein (KK) reduction to four dimensions gives standard GR with an arbitrary cosmological constant and with additional constraints that force, for instance, the four dimensional Euler density to vanish \[^4, 5\]. This is a generic feature of the dimensional reduction of theories that contain higher powers of curvature. It is commonly believed that higher curvature corrections to the Einstein-Hilbert action produce small deviations from GR, but this is actually not true: the field equations, obtained from the variation of the reduced action with respect to the four-dimensional scalars, produce constraints additional to the Einstein equations which rule many solutions of GR, including the gravitational field of a spherically symmetric source \[^6\]. This problem is analogous to the one encountered in standard KK reductions for the gauge theory sector in four dimensions, when starting from the Einstein-Hilbert action in \( D > 4 \), where the Yang-Mills density is forced to vanish in backgrounds with constant scalars. Thus, although the behavior of theories obtained by the KK reduction of Lovelock Lagrangians could be reasonable at the galactic scale or at the beginning of our Universe, at the scale of our solar system their departure from the GR behavior is not experimentally acceptable.

On the other hand, there is the largely unsolved problem of the non-renormalizability, in the power counting sense \[^7\], of the gravitational interaction. Although pure gravity has a finite one-loop \( S \) matrix \[^8\], until now all matter couplings –except supergravity \[^9\]–, destroy this one-loop behavior. At two loops, pure gravity diverges \[^10\] and at three loops also supergravity contains divergences \[^11\], although the coefficient in front of the divergence has not been calculated until now \[^12\]. One is left with an uncomfortable scenario, in which there is no field theory formulation to compute a simple graviton scattering in a consistent way.

\(^1\)For perturbations around flat space.
These facts motivate the search for new theories that include Einstein’s field equations in some way, but that also contain other dynamical sectors, such that other phenomena can be explained within these theories.

An interesting guide can be taken from the three dimensional case which, in the first order formalism can be seen as a gauge theory, where the vielbein $e$ and the spin connection $\omega$ are part of a single connection $[13]$. This Chern-Simons (CS) theory for gravity contains a larger set of field configurations than metric GR. Indeed, by a gauge transformation any of the components of a flat connection can always be set equal to zero in a open neighborhood. Thus, a generic field configuration of CS gravity does not have a metric interpretation. Projection of the gauge theory to the sector where the vielbein is invertible and the connection is torsion-free, allows one to recover the usual metric theory of gravity.

Three-dimensional CS theory is renormalizable as follows from the fact that the unique, dimensionless, coupling constant can only take integer values (in fact, it is finite at the quantum level) $[14], [15]$. Renormalization of three-dimensional gravity can be done then, by embedding the theory in a gauge theory with principal bundle structure, something not very surprising since all known physical interactions which make sense at the quantum level are explained by gauge theories. These qualities motivate the search for an embedding of four dimensional GR in a gauge theory, such that the $e$ and $\omega$ are part of a single connection.

The theoretical motivation is quite natural. Instead of considering the dimensional continuation of the two dimensional Euler density, the four dimensional Lagrangian will be given by a topologically induced dimensional reduction of the six dimensional Euler density. The dimensional reduction mechanism occurs due to the introduction of a four-dimensional topological defect in the six dimensional manifold where the Euler density is integrated. This approach was already studied in $[16], [17]$. Those authors, however, restrict the connection in the action such that the only degrees of freedom left at the defect are the components which correspond to the four-dimensional $e$ and $\omega$.

Here, instead, no restrictions are imposed in the reduction process and the non triviality of the bundle is always assumed. This gives rise to a four-dimensional theory, gauge invariant under the conformal group $SO(4,2)$. The theory is defined by the metric independent sector of the gauged Wess-Zumino-Witten (gWZW) action: the kinetic term $Tr(D_\mu g D^\mu g^{-1})$, where $D_\mu g = \partial_\mu g + [A_\mu, g]$ and $Tr$ is the bilinear invariant of the Lie group, never arise in our construction $[18]$. The resulting action resembles in many ways its three-dimensional, quantum mechanically finite sibling: in both cases $e$ and $\omega$ are part of a single $SO(m,n)$ connection, $\mathcal{A}$; both theories admit a vacuum configuration $e = \omega = 0$, in which the space-time causal structure completely
disappears; both have a quantized dimensionless coupling constant in front of the action. The discreteness of the coupling constant makes any continuous process of renormalization impossible, suggesting that the beta function must be zero.

The mechanism of dimensional reduction is discussed in Section 2. For the sake of simplicity, the discussion is done first reducing the four dimensional Euler density to two dimensions. The extension of results to reduce from six to four dimensions together with the field equations, is stated. In Sect. 3, the on shell configuration that reproduces Einstein’s gravity is discussed. The conditions under which the coupling constant takes integer values are discussed in Sect. 4, and Sect. 5 contains the discussion and conclusions.

2 Topologically induced dimensional reduction

Compactification processes always make assumptions about the geometry of the compact manifold, and at the end of the day physics depends on this geometry. Given a higher-dimensional theory, the compactification to four dimensions introduces extra information in the process, making the theory incomplete and transforming it into a model. It is natural to think that if the higher dimensional theory is topological, the dimensional reduction mechanism could be done independently of the metric, giving a more satisfactory result for the four dimensional theory compared with the standard compactification procedure.

Observing that four dimensional gravity is the dimensional continuation of the two-dimensional Euler density, the natural object to dimensionally reduce is the six-dimensional Euler density

$$\chi(M) = \frac{1}{48\pi^2} \int_{M^6} \langle \mathcal{F} \mathcal{F} \rangle = \frac{1}{48\pi^2} \frac{1}{2^3} \int_{M^6} \varepsilon_{ABCDEF} F^{AB} F^{CD} F^{EF},$$

(2)

where the indices $A, B, ...$ go from 0 to 5, $\mathcal{F} = \frac{1}{2} J_{AB} F^{AB} = dA + AA$ is the pseudo-Riemannian curvature of the six-dimensional manifold\textsuperscript{3}. Depending on the signature of the six dimensional metric, the generators $J_{AB}$ can be assumed to span any of the algebras $so(6)$, $so(5,1)$, $so(4,2)$ or $so(3,3)$. The symmetric trace $\langle ... \rangle$ is the Levi-Civita invariant tensor of these groups, $\langle J_{AB} J_{CD} J_{EF} \rangle = \varepsilon_{ABCDEF}$ and $\partial M^6 = \emptyset$. As will be shown, a dimensional reduction occurs if a four-dimensional sub-manifold

\textsuperscript{2}In this work the exterior product between forms is omitted, i.e $F \wedge F \equiv FF$. Since pullback and exterior derivatives commute, they are usually omitted in physics literature, and we follow that convention. For more conventions see the appendix.

\textsuperscript{3}We call it $\mathcal{F}$ so as not to confuse it with its four dimensional analog $\mathcal{R}$. 

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is removed from $M^6$, producing a topological defect. However, in order to be able to use the standard exterior calculus (e. g., Stokes Theorem), and pass from the six-dimensional integral to a four-dimensional one, a limiting process is needed. Here the topological defect will be created by removing a six-dimensional cylinder $M^4 \times D^2$, and then taking the limit in which the radius of the two dimensional disc $D^2$ shrinks to zero (Fig. 1), this is known as a regularization process to remove a sub-manifold of codimension two.

### 2.1 The two-dimensional case

In order to describe the process in a simpler setting, let us consider the case of a four-dimensional manifold with a two-dimensional defect, as depicted in figures 1 and 2.

We define the four-dimensional integral over $M^4 - M^2$ as the integral over $M^4 - D^2_R \times M^2$, in the limit in which the radius $R$ of the 2-disk $D^2_R$ goes to zero,

$$\int_{M^4 - M^2} \langle \mathcal{F} \rangle := \lim_{R \to 0} \int_{M^4 - D^2_R \times M^2} \langle \mathcal{F} \rangle .$$

The excision of $D^2_R \times M^2$ from $M^4$, introduces the boundary $\partial (M^4 - D^2_R \times M^2) = S^1_R \times M^2 \ (\partial M^4 = \partial M^2 = \emptyset)$. Stokes theorem can be applied introducing two charts.
that cover the entire region and overlap on a three-dimensional region, $M^3$, where a transition function, $h$, relating the connections on both sides of $M^3$, is defined. The integral of (3) becomes,

$$\lim_{R \to 0} \int_{M^4_D^2 \times M^2} \langle FF \rangle = \int_{M^3_+} CS(A_+) + \int_{M^3_-} CS(A_-)$$

$$- \lim_{R \to 0} \int_{S^1_R \times M^2} CS(A_t),$$

where

$$CS(A) := \left\langle AdA + \frac{2}{3} A^3 \right\rangle, \quad A_+ := h^{-1} A_- h + h^{-1} dh,$$

and $A_t$ interpolates between the connections defined on $M^3_-$ and $M^3_+$, that is $A_{t=0} = A_+, \ A_{t=2\pi} = A_-$. The introduction of the $S^1$ boundary is due to the regularization process needed to resolve the topological defect, $M^2$. The $t$ dependence of the interpolating connection, $A_t$, also corresponds to its coordinate dependence along the $S^1$; it should be coordinate invariant in order to ensure that the regularization process is independent of this structure. Thus, we further require that the dependence on $t$ be through arbitrary scalar functions such as

$$A_t = [1 - p(t)] A_+ + p(t) A_-,$$

where $p(0) = 0$ and $p(2\pi) = 1$.

The first two integrals on the RHS of (4) take the form of the difference of two gauge-related Chern-Simons forms and can be recognized as the same expression that appears in the calculation of the instanton number of an (anti) self-dual Yang-Mills configuration. The last term, induced by the presence of the topological defect, contains the interpolation $A_t$, integrated over the $S^1$ “angle” $t$ (the minus in front of the integral is due to the reverse orientation of that boundary). Now, one of the coordinates in the integral $\int_{S^1_R \times M^2} CS(A_t)$ can be chosen as $Rdt$, so that in the limit $R \to 0$ one finds

$$\lim_{R \to 0} \int_{S^1_R \times M^2} CS(A_t) = \lim_{R \to 0} \int_{S^1_R \times M^2} \left\langle A_t Rdt \frac{\partial}{\partial t} A_t \right\rangle$$

$$= \int_{S^1 \times M^2} \frac{dp}{dt} \left\langle A_t (A_+ - A_-) \right\rangle = \int_{M^2} \langle A_- A_+ \rangle.$$

This ensures that the regularization process is the same for any curve homeomorphic to $S^1$. If $A_1$ and $A_2$ are connections, then $\lambda_1 A_1 + \lambda_2 A_2$ is a connection only if $\lambda_1 + \lambda_2 = 1$. The form of the interpolation is then defined up to an arbitrary function of the parameter $t$, see [20].
The first equality results from the observation that if the connection is bounded in the limit, its derivative with respect to $Rt$ grows to infinity, while the other terms, $\frac{2}{3}A_\mu A_\nu A_\lambda \varepsilon^{\mu \nu \lambda} R dx^0 dx^1 dt$, as well as the remaining exterior derivatives, vanish in the limit. The third equality comes from the integration in $dt$. The integral has support on the topological defect, $M^2$, which can be recognized as the boundary of $M^3$. Then,

$$\int_{M^4 - M^2} \langle \mathcal{F} \mathcal{F} \rangle = \int_{\Sigma} CS(A) - CS(A^h) - \int_{\partial \Sigma} \langle AA^h \rangle.$$  \hfill (8)

where $\partial \Sigma = M^2$. Using the identity $CS(A^h) \equiv CS(A) - \frac{1}{3} \langle (h^{-1} dh)^3 \rangle + d \langle h^{-1} Adh \rangle$, it is direct to conclude that

$$\int_{M^4 - M^2} \langle \mathcal{F} \mathcal{F} \rangle = \int_{\Sigma} \frac{1}{3} \langle (h^{-1} dh)^3 \rangle - \int_{M^2} \langle (A - h^{-1} dh) A^h \rangle.$$  \hfill (9)

The RHS of (9) is a gWZW term, invariant under the local transformations,

$$h \rightarrow g^{-1} hg, \quad A \rightarrow g^{-1} A g + g^{-1} dg,$$  \hfill (10)

and it defines a theory on the manifold described by the topological defect, $M^2$. It has been recognized that CS theory on a Riemann surface times $S^1$ is equivalent to a WZW model [21]. The equality (9) was conjectured to exist in [22].

### 2.2 A closer look at the regularization

The regularization process introduced here has a major drawback: it is not gauge invariant under transformations that belong to a non trivial homotopy class. When the singularity at $M^2$ is resolved, an $S^1 \times M^2$ boundary is added to the manifold, but gauge transformations that are not single-valued on $S^1$ become ill-defined in the limiting process. Let us consider the gauge invariant object $\langle \mathcal{F} \mathcal{F} \rangle$. Under an infinitesimal gauge transformation, it changes by $\delta \langle \mathcal{F} \mathcal{F} \rangle = 2 \langle D \delta A \mathcal{F} \rangle = 2 d \langle \delta A \mathcal{F} \rangle$. Now, if $\delta A = D \lambda$, with $\lambda$ not single valued around $S^1$, the integral

$$\int_{M^4 - D^2 \times M^2} \delta \langle \mathcal{F} \mathcal{F} \rangle$$  \hfill (11)

has a non trivial contribution with support on $M^2$. Of course, in order for the RHS of (3) to be invariant under these infinitesimal gauge transformations, one should add an adequate boundary term that restores gauge invariance.

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5From now on, we will call $A_\lambda = A$, and $M^3 = \Sigma$. 

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A topological defect of dimension $m$ in a $d$-dimensional space is classified by the $(d - m - 1)$-th homotopy group. Since here $m = d - 2$, the topological defect is classified by the homotopy group $\pi_1(M^d - M^{d-2})$ for any $d$. This implies the existence of gauge transformations with non trivial holonomy around the defect which break the symmetry [21]. These singular gauge transformations live on the plane transverse to the topological defect.

Let us consider a gauge transformation, $g$, restricted to a $U(1)$ subgroup, such that the non-contractible loops around the defect are mapped into gauge transformations with non trivial holonomy. Here $g : S^1 \times M^{d-2} \to U(1)$ can be chosen as $\exp(\psi(t, x) J)$. The generators that commute with $J$ span the algebra of rotations in $d - 2$ dimensions, that is, the symmetry associated with the local rotations in the non coordinate tangent space of the topological defect, $M^{d-2}$. The singularity of the gauge transformation originates in the non single-valued nature of $\psi$ around the loop $S^1_R$ of Fig.2.

Consider the gauge transformations, $\delta$, with non trivial holonomy around the $S^1$. The LHS of (9) transforms as

$$
\lim_{R \to 0} \delta \int_{M^4 - D^2_R \times M^2} \langle \mathcal{F} \mathcal{F} \rangle = - \lim_{R \to 0} \int_{S^1_R \times M^2} 2 \langle \delta \mathcal{A}_t \mathcal{F}_t \rangle 
= - \lim_{R \to 0} \int_{S^1_R \times M^2} 2 \langle D_{\mathcal{A}_t} (\psi) \mathcal{F}_t \rangle 
= - \int_{S^1 \times M^2} 2 dt \frac{\partial}{\partial t} \langle \psi \mathcal{F}_t \rangle 
= - 2 \int_{M^2} \langle \psi(x) J \mathcal{F} \rangle, 
$$

(12)

where, after integrating in $t$, we have defined $\psi(2\pi, x) = \psi(x), \psi(0, x) = 0$.

This gauge transformation can be seen in the RHS of (9) as a change in the field $h = e^{\phi}$ alone. Indeed, since gauge transformations are given by the right action of the group on the fiber, it can be realized in a representative connection $\mathcal{A}^h$ in two ways,

$$
\mathcal{A}^h \to g^{-1} \mathcal{A}^h g + g^{-1} dg \quad \text{and} \quad h \to h, \quad \text{or} \quad \mathcal{A} \to \mathcal{A} \quad \text{and} \quad h \to h g.
$$

(13)

Thus, it is possible to identify $\psi(x) J = \delta \phi$, where $\delta$ singles out the component of the infinitesimal version of the transition functions $h$ along $J$, and $\delta \mathcal{A} = 0$. In view of this, one can identify the last expression in (12) as

$$
-2 \int_{M^2} \langle \delta \phi \mathcal{F} \rangle.
$$
We conclude that the action to be considered is:

\[ S(h, A) = \kappa \int_\Sigma \frac{1}{3} \left\langle \left( h^{-1} dh \right)^3 \right\rangle - \kappa \int_{M^2} \left\langle (A - h^{-1} dh) A_h \right\rangle + 2\kappa \int_{M^2} \left\langle \phi \mathcal{F} \right\rangle, \quad (14) \]

The mechanism presented here has generated well known structures in two dimensions: the first two terms are the gWZW terms that define the minimal extension necessary to render \( \left\langle (h^{-1} dh)^3 \right\rangle \) gauge invariant. (When a kinetic term for the Goldstone fields is added, most of the two dimensional physics can be translated to this non-linear sigma model language; the description of the super-string [23], the characterization of exact string backgrounds [24] and the non-abelian bosonization phenomena [25], to name a few.) The last term, \( \int_{M^2} \left\langle \phi \mathcal{F} \right\rangle \) generalizes the two-dimensional gravity of Jackiw and Teitelboim [26], can be recognized as two copies of the anomaly-free finite topological gravity of Chamseddine and Wyler [27] and is related to the non-local Polyakov action,

\[ \int R \frac{1}{\partial^2} Rd^2 x. \quad (15) \]

This prescription has unambiguously produced a well defined action with all relative coefficients fixed. The construction can also be extended to build gravitational actions in \( d - 2 \) dimensions beginning from the Euler density in \( d \) dimensions.

### 2.3 The four-dimensional case

The application of the same procedure to the six dimensional Euler density (2) yields

\[ S(h, A) = \frac{\kappa}{48\pi^3} \int_\Sigma \text{CS}(A) - \text{CS}(A^h) - \frac{\kappa}{48\pi^3} \int_{M^4} B(A, A^h) + \frac{3\kappa}{48\pi^3} \int_{M^4} \left\langle \phi \mathcal{F} \mathcal{F} \right\rangle, \quad (16) \]

\[ \text{CS}(A) := \left\langle A d A d A + \frac{3}{2} A^3 d A + \frac{3}{5} A^5 \right\rangle, \quad h = e^\phi = \exp(\frac{1}{2} J_{AB} \phi^{AB}), \quad (17) \]

\[ B(A, A^h) := \left\langle A A^h \left( \mathcal{F} + \mathcal{F}^h - \frac{1}{2} A^2 - \frac{1}{2} (A^h)^2 + \frac{1}{2} A A^h \right) \right\rangle. \quad (18) \]

Replacing the identity,\(^6\)

\(^6\)As usual, the action is defined up to a multiplicative constant.
\[ CS(A) \equiv CS(A^h) + d \left( (h^{-1} dh) \left( A^h \mathcal{F}^h - \frac{1}{2} (A^h)^3 \right) \right) - \frac{1}{10} (h^{-1} dh)^5 \]
\[ -d \frac{1}{2} \left( (h^{-1} dh)^2 \mathcal{F}^h - \frac{1}{2} (h^{-1} dh) A^h (h^{-1} dh) A^h \right) \]
\[ -d \frac{1}{2} (h^{-1} dh)^3 A^h \] (19)

back in (16) the action takes the form

\[ S(h, A) = -\kappa \int_\Sigma \frac{1}{480 \pi^3} \left\langle (h^{-1} dh) (h^{-1} dh)^2 (h^{-1} dh)^2 \right\rangle \]
\[ + \frac{\kappa}{48 \pi^3} \int_{M^4} \left\langle (dh^{-1}) A \left( dA + \frac{1}{2} A^2 \right) \right\rangle \]
\[ - \frac{\kappa}{96 \pi^3} \int_{M^4} \left\langle (dh^{-1}) A \left( (dh^{-1})^2 + A (h^{-1} dh) \right) \right\rangle \]
\[ - \frac{\kappa}{48 \pi^3} \int_{M^4} \left\langle AA^h \left( \mathcal{F} + \mathcal{F}^h - \frac{1}{2} A^2 - \frac{1}{2} (A^h)^2 + \frac{1}{2} A A^h \right) \right\rangle \]
\[ + \frac{\kappa}{16 \pi^3} \int_{M^4} \langle \phi \mathcal{F} \mathcal{F} \rangle \] (20)

It must be stressed that the right normalization of the Wess-Zumino term was obtained from the normalized Euler characteristic (2) as a by-product of the construction, without a need for adjusting the parameters in the action (20). The normalized Wess-Zumino term for a group with \( \pi_5(G) = \mathbb{Z} \) satisfies \([28]\).

\[ \int_{S^5} \frac{1}{480 \pi^3} \left\langle (h^{-1} dh)^5 \right\rangle = n \in \mathbb{Z}, \] (21)

where \( n \) is the homotopy class to which the map \( h : S^5 \to G \) belongs.

Actions of the type (20) are widely used in particle physics to describe the infrared behavior of QCD \([29, 30]\). The gauged version was introduced originally by Witten in ref \([14]\), where the motivation was to find a gauge invariant extension of the global \( G \times G \) symmetry present in the five-dimensional closed form \( \left\langle (h^{-1} dh)^5 \right\rangle \). This problem is far from trivial, since the naive gauge extension of this term obtained by replacing the exterior derivative by a covariant derivative, doesn’t work: if this is done, the
5-form is no longer closed and the equations of motion have support in the five-dimensional manifold $\Sigma$. Although far from obvious, the same gWZW structures arise in the description of QCD and GR. While in QCD the gWZW term describes the interactions of the infrared sector of the theory, here it might correspond to an ultraviolet extension of GR.

The action (20) was proposed as a gravitational action in [18] where, in order to obtain Einstein’s field equations, a field was fixed in the action. That is a rather unpleasant situation since this is a condition imposed on a theory by an $a$ posteriori expected result. The derivation presented here gives a contribution not considered before, $3 \int_{M^4} \langle \phi \mathcal{F} \mathcal{F} \rangle$, that, as will be shown, allows to obtain a background configuration where the Einstein equations hold, without breaking the gauge symmetry present in (20).

The field equations associated with the Goldstone fields $h = e^\phi$ are

$$\int_{M^4} \left\langle h^{-1} \delta h \left\{ \left( \mathcal{F}^h \right)^2 + \mathcal{F}^2 + \mathcal{F}^h \mathcal{F} - \frac{3}{4} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}] (\mathcal{F}^h + \mathcal{F}) \right. \right. $$

$$\left. \left. + \frac{1}{8} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}]^2 + \frac{1}{2} (\mathcal{A}^h - \mathcal{A}) [\mathcal{F}^h + \mathcal{F}, \mathcal{A}^h - \mathcal{A}] \right) \right\} - 3 \delta \phi \mathcal{F}^2 \right\rangle = 0,$$

while those associated with the connection $\mathcal{A}$ are

$$0 = \int_{M^4} \delta \mathcal{A} \left( \mathcal{A}^h - \mathcal{A} \right) \left( \mathcal{F}^h + 2 \mathcal{F} - \frac{1}{4} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}] \right) - 3 \mathcal{F} \delta \phi \right\rangle$$

$$(h \leftrightarrow h^{-1}).$$

If one wishes to describe a four-dimensional world with Lorentzian signature, the gauge group to be chosen can only be $SO(5, 1)$, $SO(4, 2)$ or $SO(3, 3)$. The discussion will be restricted from now to the $SO(4, 2)$ group, as it is particularly interesting, allowing for the quantization of the coefficient $\kappa$ in front of the action [31].

The minimal requirement for a theory that describes gravity is to reproduce the phenomenology of GR in some limit. That is the subject of the next section.

### 3 The Einstein dynamical sector

The topological action $^7$ (20) gives rise to first order field equations, is invariant by construction under coordinate transformations, and is also invariant under the local transformations,

$$h \rightarrow g^{-1}hg \implies \phi \rightarrow g^{-1}\phi g, \quad \mathcal{A} \rightarrow g^{-1}\mathcal{A}g + g^{-1}dg.$$

$^7$Topological in the sense that no metric is needed to construct it.
The theory contains 30 fields, the 15 independent components of \( h = \exp[\frac{1}{2} \phi AB J_{AB}] \), and the 15 fields in the connection \( A = \frac{1}{2} A^{AB} J_{AB} \). The introduction of a four-dimensional topological defect in the six-dimensional manifold splits the generators \( J_{AB} \) into those that rotate the tangent space of \( M^4 \) into itself, \( J_{ab}, J_{a5} \), and those that move the tangent space of \( M^4 \) into the 4 and 5 directions, \( J_{a4}, J_{a5} \). It is therefore natural to separate the generators into their irreducible Lorentz covariant parts \( (J_{ab}, J_{a5}), (J_{ab}, J_{a4}) \), where \( a, b = 0, \ldots, 3 \) are Lorentz indices. Correspondingly, the connection and the Goldstone fields are written as

\[
A = \frac{1}{2} \omega^{ab} J_{ab} + c^a J_{a5} + b^a J_{a4} + \Phi J_{a4}, \quad (25)
\]

\[
\phi = \frac{1}{2} J_{ab} \phi^{ab} + J_{a4} \phi^{a4} + J_{a5} \phi^{a5} + \theta J_{a4}, \quad (26)
\]

and the curvature reads

\[
F = \frac{1}{2} (R^{ab} + c^a c^b - b^a b^b) J_{ab} + [D b^a + c^a \Phi] J_{a4} + [D c^a + b^a \Phi] J_{a5} \\
+ [d \Phi - b_a c^a] J_{a4}. \quad (27)
\]

Here \( (J_{ab}, J_{a5}) \) and \( (J_{ab}, J_{a4}) \) span the \( SO(3, 2) \) and \( SO(4, 1) \) subalgebras of \( SO(4, 2) \), respectively; \( R^{ab} = d \omega^{ab} + \omega^a_c \omega^{cb} \) is the Lorentz curvature two-form and \( D c^a = dc^a + \omega^a_b c^b \).

Note that the vielbein should be identified as a vector under local Lorentz rotations. At this point there is no good reason to choose either \( c \) or \( b \) or any linear combination thereof, as the vielbein of GR. This arbitrariness can be resolved by the field equations.

The vacuum configuration of the theory should be invariant under the group that maps the tangent space of the topological defect into itself, \( SO(3, 1) \times SO(1, 1) \). This reduces the field \( h \) to be

\[
h = e^{\theta J_{a5}}. \quad (28)
\]

This simplifies the field equation enough to write them down by components. From (23) it is straightforward to obtain (see appendix)

\[
\delta \Phi : 0 = 0, \quad (29)
\]

\[
\delta c^a : \varepsilon_{abcd} c^b \left( R^{cd} + (c^c c^d - b^c b^d) \left( 1 - \frac{\sinh \theta (1 - \cosh \theta)}{3 (\sinh \theta - \theta)} \right) \right) = 0, \quad (30)
\]

\[
\delta b^a : \varepsilon_{abcd} b^b \left( R^{cd} + (c^c c^d - b^c b^d) \left( 1 - \frac{\sinh \theta (1 - \cosh \theta)}{3 (\sinh \theta - \theta)} \right) \right) = 0, \quad (31)
\]

\[
\delta \omega^{ab} : \varepsilon_{abcd} D \left( (-2 \sinh \theta + 3 \theta) (b^c b^d - c^c c^d) \right) = 0, \quad (32)
\]
The variations with respect to $h$ are also simplified \((22)\). Consider first just the field equation corresponding to $\theta$, $\delta \theta : \varepsilon_{abcd} \left( c^a c^b - b^a b^b \right) \left( R^{cd} + \left( c^c c^d - b^c b^d \right) \left( 1 - \frac{(1 - \cosh \theta)}{3} \right) \right) = 0. \quad (33)

Subtracting the difference between $c^a$ times \((30)\) and $b^a$ times \((31)\), from \((33)\) one obtains

$$
\varepsilon_{abcd} \left( c^a c^b - b^a b^b \right) \left( c^c c^d - b^c b^d \right) \left( 1 - \frac{\sinh \theta}{(\sinh \theta - \theta)} \right) = 0. \quad (34)
$$

The term $\left( 1 - \frac{\sinh \theta}{(\sinh \theta - \theta)} \right)$ vanishes when $\theta$ is $0$ or $\infty$, however in the first case the equations of motion are trivial, and in the second, some of them blow up. Equation \((34)\) has two simple Lorentz covariant solutions for generic values of $\theta$, $c^a = b^a$ or $c^a = -b^a$. These solutions can be understood decomposing the connection in the conformal basis as

$$
\mathcal{A} = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{2} \left( c^a + b^a \right) P_a + \frac{1}{2} \left( c^a - b^a \right) K_a + \Phi J_{45}
$$

with $(J_{ab}, P_a)$ the Poincaré subgroup of $SO(4,2)$ and $K_a$ the conformal boost. It follows that eq \((34)\) can be read as selecting the vielbein associated to $P$ or to $K$, equal to zero. It is enough for the subsequent discussion to take as the vielbein the one associated to $P_a, e^a = (c^a + b^a)/2$, and $c^a = b^a$ to satisfy \((34)\). In this dynamical sector the set of equations \((29-33)\) is self consistent and reduces to

$$
\varepsilon_{abcd} e^b R^{cd} = 0. \quad (36)
$$

Next, consider the other field equations that follows from \((22)\). Under the conditions already stated the equation associated to $\phi^a 4$ coincides with that associated to $\phi^a 5$, and is equal to

$$
\delta \phi^a 4 : \varepsilon_{abcd} \left( 2 \left( D c^b + c^b \Phi \right) - \frac{3}{2} c^b d \theta \right) R^{cd} = 0. \quad (37)
$$

After using \((36)\) this reduces to

$$
\varepsilon_{abcd} D e^b R^{cd} = 0, \quad (38)
$$

namely, the covariant derivative of \((36)\). In this field configuration, the equations associated to $\phi^{ab}$ are identically satisfied. Indeed, \((36)\) reduces to Einstein’s equations when the torsion vanishes

$$
T^a = de^a + \omega^a_b e^b = 0. \quad (39)
$$
In this dynamical sector of the theory, the fields $\Phi$ and $\theta$ remain completely undetermined. This indicates that the dynamical sector defined by this vacuum is degenerate and possesses fewer degrees of freedom than in a generic sector \[32, 33\]. As in the three dimensional case, when GR is regarded as a gauge theory \[14\], in order to make contact with the metric phase of the theory, it is necessary to restrict the vielbein to the case in which it is invertible, $e^a_\mu e^v_a = \delta^v_\mu$, $e^a_\mu e^b_\mu = \delta^a_b$. The introduction of a parameter with dimensions of length, $l$, is also necessary, in order to make $\bar{e}^a_\mu = le^a_\mu$ dimensionless. These two conditions allow to regard $\bar{e}^a_\mu$ as an isomorphism between the coordinate tangent space and the non-coordinate one, such that the relation $g_{\mu\nu} = \bar{e}^a_\mu \bar{e}^b_\nu \eta_{ab}$ makes sense. Using this, plus the torsionless condition (39), the usual form of the Einstein field equations for the metric, $g_{\mu\nu}$, is recovered \[8\],

$$
\varepsilon_{abcd} e^b R^{cd} = 0 \implies R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0,
$$

(40)

where $R_{\mu\nu}$ is the metric Ricci tensor.

4 Quantization of $\kappa$ and Euclidean continuation

The conditions under which the constant $\kappa$ takes integer values are well known \[19, 31\]. Consider a non-linear sigma model defined by the map

$$
h : S^4 \longrightarrow G.
$$

(41)

If $S^4$ is viewed as the boundary of some compact manifold $D^5$, one can consider the extension of the map $h$ to the interior ($\partial D^5 = S^4$),

$$
h : D^5 \longrightarrow G.
$$

(42)

However, $S^4$ can be the boundary of many different interiors. Requiring the independence of the functional

$$
S = \kappa \int_{D^5} \frac{1}{480\pi^3} \left< (h^{-1} dh)^5 \right>
$$

(43)

from the particular $D^5$ chosen as a possible interior of $S^4$, implies

$$
\kappa \int_{D^5} \frac{1}{480\pi^3} \left< (h^{-1} dh)^5 \right> = \kappa \int_{D^5} \frac{1}{480\pi^3} \left< (h^{-1} dh)^5 \right>.
$$

(44)

\[8\]A small section that explicitly show how to pass from from forms to tensors is given in the appendix.

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Since $D^5$ and $D^\bar{5}$ have the same orientation, one finds

$$0 = \kappa \int_{D^5} \frac{1}{480 \pi^3} \langle (h^{-1} dh)^5 \rangle - \kappa \int_{D^\bar{5}} \frac{1}{480 \pi^3} \langle (h^{-1} dh)^5 \rangle$$

(45)

$$= \kappa \int_{S^5} \frac{1}{480 \pi^3} \langle (h^{-1} dh)^5 \rangle$$

(46)

Now, if $\pi_5(G) = \mathbb{Z}$ from (21), one concludes that $\kappa n = 0$ for any $n$, and therefore, requiring independence of the action from the extension $D^5$ leads to the trivial result $\kappa = 0$. However quantum mechanics allows for (43) to be defined up to a integer multiple of $2 \pi \hbar$,

$$\kappa n = 2 \pi m \hbar.$$  

(47)

As this must be true for all $n$, this implies that $\kappa$ itself must be an integer multiple of $2 \pi \hbar$.

The relevance of these arguments to our case arise through the analytic continuation of our theory, defined by the map of the connection

$$A^{\bar{a}} \to \overline{A}^{\bar{a}} = \tilde{A}^{\bar{a}}, \quad A^{\bar{a}0} \to \overline{A}^{\bar{a}0} = i \tilde{A}^{\bar{a}0}$$

$$A^{\bar{a}5} \to \overline{A}^{\bar{a}5} = i \tilde{A}^{\bar{a}5}, \quad A^{05} \to \overline{A}^{05} = - \tilde{A}^{05}$$

and similarly for the Goldstone fields. The resulting action is invariant under $SO(6)$, as can be seen by the equivalent map of the generators

$$J_{\bar{a}0} \to \tilde{J}_{\bar{a}0} = i J_{\bar{a}0}, \quad J_{\bar{a}5} \to \tilde{J}_{\bar{a}5} = i J_{\bar{a}5}$$

$$J_{05} \to \tilde{J}_{05} = - J_{05}, \quad J_{\bar{a}\bar{b}} \to \tilde{J}_{\bar{a}\bar{b}} = J_{\bar{a}\bar{b}}$$

(48)

where the indexes $\bar{a}$ cover the range $\bar{a} = 1, \ldots, 4$ such that the metric in these indexes is Euclidean $\delta_{\bar{a}\bar{b}} = (+, +, +, +)$. Under these changes the invariant tensor $\langle J_{AB}J_{CD}J_{EF} \rangle$ reverses sign and so the action (20) changes as $S \to -S$.

On the other hand, the Euclidean continuation of the groups $SO(5, 1)$ and $SO(3, 3)$ instead produce

$$SO(5, 1) \to SO(6) \implies e^{\pm \kappa S} \to e^{\mp \kappa S}; \quad SO(3, 3) \to SO(6) \implies e^{\pm \kappa S} \to e^{\mp \kappa S}.$$  

We see that the group $SO(4, 2)$ has the particular property that since its Euclidean continuation changes the action by a sign and not by an imaginary factor, it allows for the existence of the freedom in the phase (47). Conversely, requiring that the coefficient in front of the action be quantized, singles out the gauge group to be $SO(4, 2)$. 

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5 Discussion and Outlook

Here a gauge theory for four dimensional gravity has been proposed. The action (20) is metric independent, and all the fields have a geometrical interpretation. Besides the usual connection $A$, the transition functions $h$ are also present. These two objects are completely defined once a principal bundle is given. Conversely, a theory in which these two objects are dynamically determined allows to dynamically characterize the principal bundle, up to homotopies.

The theory generalizes GR in two ways: first, it contains a dynamical sector in which Einstein’s equations hold, reproducing all the experimental tests that are compatible with GR. Second, the observation that GR is the dimensional continuation of the Euler density is generalized to the first non-trivial case: instead of thinking of the Einstein-Hilbert Lagrangian as the dimensional continuation of the two-dimensional Euler density, we consider the topological dimensional reduction of the six-dimensional Euler density. In this way, a theory that contains other fields besides GR is obtained, something that could have been seen as a drawback some years ago, but not in the current state of affairs, where there is a proliferation of models that try to explain the dynamics of the galaxies or the inflation process of our Universe, among others phenomena that cannot be explained using only GR and standard matter fields.

Goldstone fields represent topological information of the six dimensional theory and becomes dynamical in the four dimensional theory. Their presence could be interpreted as the deconfining phase of the higher dimensional, topological theory and, if the theory makes sense, they could be relevant to the description of our Universe. The theory naturally includes torsion. The presence of propagating torsion in a background configuration changes many of the known results in GR, including those about the generic existence of singularities in spacetime [34].

The emergence of the space-time causal structure in the theory defined by (20) arises only after a vielbein is chosen from amongst all the invertible linear combinations of the $b$ and $c$. When the gauge invariance of the theory is on shell reduced to local Lorentz transformations, the invertibility of what is chosen as a vielbein is not changed by the remanent gauge symmetry. Note that with the choice of vielbein $e^a = (b^a + c^a)/2$, the symmetry generated by $g = \exp (\Theta J_{15})$ is a conformal transformation on the metric,

$$A \rightarrow A' \Rightarrow g_{\mu\nu} \rightarrow \exp (2\Theta) g_{\mu\nu}.$$ (49)

The obtention of a gravitation theory that is metric independent; in which GR could be seen as a broken phase of a topological field theory has been a long sought
The construction presented here is a step in this direction.

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A Appendix

The following convention for $SO(4,2)$ algebra was used:

$$[J_{AB}, J_{CD}] = -J_{AC} \eta_{BD} + J_{BC} \eta_{AD} - J_{BD} \eta_{AC} + J_{AD} \eta_{BC};$$  \hspace{1cm} (50)

$$A = 0, \ldots, 5 \quad \eta_{AB} = (-, +, +, +, +, -).$$  \hspace{1cm} (51)

A simple exercise to pass from differential forms to tensors:

$$\varepsilon_{abcd} e^b R^{cd} = 0,$$  \hspace{1cm} (52)

$$\Rightarrow \varepsilon_{abcd} e^b R^{cd} dx^\mu = 0$$  \hspace{1cm} (53)

$$\Rightarrow \frac{1}{2} \varepsilon_{abcd} e^b R^{cd} dx^v dx^\lambda dx^\rho dx^\mu = 0$$  \hspace{1cm} (54)

$$\Rightarrow \delta_{\alpha\beta}^v \lambda^\mu \eta_{\alpha\beta} R^{\gamma\delta} \det(e) dx^4 = 0$$  \hspace{1cm} (55)

$$\Rightarrow R^\mu_{\alpha} - \frac{1}{2} \delta^\mu_{\alpha} R = 0;$$  \hspace{1cm} (56)

where in the second line the eq (52) is multiplied by the differential $dx^\mu$, in the third line the definition of $R^{cd} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu dx^\nu$ is used, in the fourth line the $\varepsilon_{abcd}$ in the
non coordinate tangent space is passed to the coordinate tangent space using the vielbeins, so that a determinant of them appears in that transformation and the identity \( \varepsilon_{\alpha\beta\gamma\delta}dx^\nu dx^\lambda dx^\rho dx^\mu = \delta^{\nu\lambda\rho\mu} dx^4 \) was used. Finally contracting the generalized delta \( \delta^{\nu\lambda\rho\mu}_{\alpha\beta\gamma\delta} \) with the Riemann tensor and multiplying by \( e^\alpha_a \) the Einstein’s field equation in its tensorial form appears.

Some useful formulas, given \( h = e^a_{J45} \) is possible to compute:

\[
A^h = \frac{1}{2} \omega_{ab} J^{ab} + J_{a4} (b^a \cosh \theta + c^a \sinh \theta) + J_{a5} (c^a \cosh \theta + b^a \sinh \theta)
+ (\Phi + d\theta) J_{45}
\]

\[
F^h = \frac{1}{2} (R^{ab} + c^a c^b - b^a b^b) J_{ab} + J_{a4} [(D b^a + c^a \Phi) \cosh \theta + \sinh \theta (D c^a + b^a \Phi)]
+ J_{a5} [(D c^a + b^a \Phi) \cosh \theta + \sinh \theta (D b^a + c^a \Phi)] + (d\Phi - b^a c_a) J_{45}
\]

\[
[A^h - A, A^h - A] = 2 (1 - \cosh \theta) (c^b c^b - b^a b^b) J_{ab} + 2 ((\cosh \theta - 1) c^a + \sinh \theta b^a) d\theta J_{a4}
+ 2 ((\cosh \theta - 1) b^a + \sinh \theta c^a) d\theta J_{a5} + 4 (1 - \cosh \theta) c^b b^b J_{45} \eta_{ab}
\]

References


Singularity theorems generically include as hypotheses that the connection is metric compatible and torsion free [see S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge U.P. (1973)]. Although the first hypothesis is physically motivated, the second is not, and eliminating it changes the form of the equation of geodesic deviation: take a smooth 1-parameter system of geodesics described by a smooth map from a strip \( \{(t,v) | t_0 < t < t_1, -\varepsilon < v < \varepsilon \} \) into \( M \), where each geodesic is given by setting \( v = \text{const} \), define the coordinates vectors \( X = \partial_t, V = \partial_v \) and the acceleration relative as \( a = \nabla_X \nabla_X V \). With usual definition of torsion and curvature, it follows that

\[
a = \nabla_X \nabla_V X + \nabla_X T(X,V) = R(X,V)X + \nabla_X T(X,V).
\]

The second term is normally ignored in the equation of geodesic deviation. Thus, the inclusion of torsion could change the bounds for the energy momentum tensor to cause singularities.