BRST Symmetric Gaugeon Formalism for Higgs Model

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We reinvestigate Yokoyama’s gaugeon formalism for the spontaneously broken Abelian gauge theory. Within the framework of the covariant linear gauges, we give a general gauge-fixing Lagrangian which includes the gauge field, the Goldstone mode, the multiplier \( B \)-field and Yokoyama’s gaugeon fields (as well as Faddeev-Popov ghosts). As special choices of the values of the gauge-fixing parameters, our theory includes the usual covariant gauges and \( R_\xi \)-like gauges. Although some of the gauge-fixing parameters can be shifted by the \( q \)-number gauge transformation, the \( \xi \) parameter cannot be shifted in any of the \( R_\xi \)-like gauges.

\section{Introduction}

In the standard formalism of canonically quantized gauge theories,\textsuperscript{1}–\textsuperscript{3}) we do not consider the gauge transformation on the quantum level. There is no quantum gauge freedom, since the quantum theory is defined only after the gauge fixing.

Yokoyama’s gaugeon formalism\textsuperscript{4}–\textsuperscript{9}) provides a wider framework in which we can consider the quantum gauge transformation, \( q \)-number gauge transformation, among a family of Lorentz covariant linear gauges. In this formalism, a set of extra fields, so-called gaugeon fields, is introduced as the quantum gauge freedom. This formalism was proposed for quantum electrodynamics,\textsuperscript{4,5}) spontaneously broken \( U(1) \) gauge theory,\textsuperscript{7}) spontaneously broken chiral \( U(1) \) gauge theory\textsuperscript{8}) and Yang-Mills gauge theory.\textsuperscript{6}) Owing to the quantum gauge freedom, it becomes almost trivial to check the gauge parameter independence of the physical \( S \)-matrix.\textsuperscript{9})

BRST symmetric theories of this formalism have been also developed for quantum electrodynamics,\textsuperscript{11}–\textsuperscript{13}) Yang-Mills theory,\textsuperscript{10,14}) and the spin-3/2 Rarita-Schwinger gauge field.\textsuperscript{15}) By virtue of the BRST symmetry, Yokoyama’s physical subsidiary conditions have been improved;\textsuperscript{10}–\textsuperscript{12}) the Gupta-Bleuler-type subsidiary conditions are replaced by a single Kugo-Ojima-type condition.\textsuperscript{2,3}) The BRST symmetry is also very helpful to define the gauge invariant physical Hilbert space.\textsuperscript{12,13,15})

In this paper we apply the BRST symmetric gaugeon formalism to the Higgs model.\textsuperscript{16,17}) The gaugeon formalism of this model was first studied by Yokoyama and Kubo.\textsuperscript{7}) In their theory, the unphysical Goldstone boson mode appears as a massless dipole field; corresponding standard formalism is the theory of the usual covariant gauge.\textsuperscript{2,17,18}) Thus, their theory does not include other types of gauges, such as \( R_\xi \) gauge,\textsuperscript{20,21}) where the Goldstone boson mode becomes massive. The main purpose of the present paper is to construct a gaugeon formalism for the Higgs model.
model which includes both the usual covariant gauges and $R_\xi$-like gauges. In the $R_\xi$-like gauges, we also explore the possibility that the $\xi$ parameter might be shifted by the $q$-number gauge transformation.

In §2 we briefly review the Lorentz covariant quantization of the Higgs model; the theory in the usual covariant gauge (Lorenz gauge), the gaugeon formalism corresponding to this Lorenz gauge by Yokoyama and Kubo,\textsuperscript{7} and the theory of $R_\xi$-gauges by Fujikawa, Lee and Sanda and by Yao. In §3 we consider the most general gauge-fixing Lagrangian which consists of the gauge field, the Goldstone mode, the multiplier field and the gaugeon fields. The Lagrangian has seven gauge-fixing parameters. As a special choice of the values of the parameters the theory would be equivalent, for example, to the gaugeon formalism of the Lorenz gauge by Yokoyama and Kubo. By introducing two pairs of the Faddeev-Popov (FP) ghost fields to the general Lagrangian given in §3, we present a general form of the BRST symmetric gaugeon formalism for the Higgs model in §4; the theory has the BRST symmetry, and admits the $q$-number gauge transformation under which some of the gauge-fixing parameters change their values. In §5 we study the $R_\xi$-like gauges of our theory. By choosing special values for the gauge-fixing parameters we show that our theory includes a gaugeon formalism for the Yao’s $R_\xi$ gauge, where one of two gauge-fixing parameters of Yao’s theory can be shifted by the $q$-number gauge transformation, while the other ($\xi$-parameter) cannot be shifted. We also show that, in more general case, the $\xi$-parameter in any of the $R_\xi$-like gauges cannot be shifted by the $q$-number gauge transformation. §6 is devoted to a summary and discussion. The number of the conserved BRST-like charges in our theory is also discussed.

§2. Higgs model

The Lagrangian of the Higgs model is given by

$$\mathcal{L}_{cl} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^\dagger (D^\mu \varphi) + \mu^2 \varphi^\dagger \varphi - \frac{\lambda}{2} (\varphi^\dagger \varphi)^2, \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the Abelian gauge field $A_\mu$, $\varphi$ is a complex scalar field, $D_\mu \varphi = (\partial_\mu - ieA_\mu)\varphi$, $e$ is the charge of $\varphi$, and $\mu^2$ and $\lambda$ are positive constants. The vacuum expectation value of $\varphi$ is given by $\langle 0|\varphi|0 \rangle = v/\sqrt{2} = \sqrt{\mu^2/\lambda}$, where we have adjusted the phase of $\varphi$ so that the vacuum expectation value is real.

The Lagrangian $\mathcal{L}_{cl}$ is invariant under the gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \quad (2.2)$$
$$\varphi(x) \to e^{ieA(x)} \varphi(x). \quad (2.3)$$

To quantize (2.1) we should choose a suitable gauge by adding a suitable gauge-fixing term (and a corresponding Faddeev-Popov(FP) ghost term) to $\mathcal{L}_{cl}$. 

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2.1. Lorenz gauge

The quantum Lagrangian in the usual covariant gauge (Lorenz gauge) is given by\(^{(17), (18)}\)

\[
L_N = L_{\text{cl}} + B \partial_\mu A^\mu + \frac{1}{2} \alpha B^2, \tag{2.4}
\]

where \(B\) is the multiplier \(B\)-field of Nakanishi-Lautrup\(^{(1), (19)}\) and the numerical constant \(\alpha\) is the gauge-fixing parameter. The BRST symmetric version is obtained by adding FP ghost term to (2.4).\(^2\) The global \(U(1)\) symmetry remains unbroken in this gauge, and the massless Goldstone boson arises as an unphysical mode. In particular, this massless Goldstone boson appears as a dipole field except for the Landau gauge \(\alpha = 0\).

2.2. gaugeon formalism for the Lorenz gauge

By introducing the gaugeon fields \(Y\) and \(Y^\ast\) into (2.4), we get the Lagrangian of Yokoyama and Kubo:\(^7\)

\[
L_{\text{YK}} = L_{\text{cl}} + B \partial_\mu A^\mu + \frac{\varepsilon}{2} (a B + Y_s)^2 - \partial_\mu Y_s \partial^\mu Y, \tag{2.5}
\]

where \(\varepsilon\) is a sign factor \((\varepsilon = \pm 1)\) and \(a\) is a numerical gauge-fixing parameter. The gauge-fixing parameter \(\alpha\) in (2.4) corresponds to \(\varepsilon a^2\): the propagator \(\langle A_\mu A_\nu \rangle\) followed from (2.5) is exactly the same with that followed from (2.4) if we assume \(\alpha = \varepsilon a^2\).

The Lagrangian (2.5) admits a \(q\)-number gauge transformation. Under the field redefinition

\[
\hat{A}_\mu = A_\mu + \tau \partial_\mu Y, \quad \hat{\varphi} = e^{i \varepsilon \tau Y} \varphi, \tag{2.6}
\]

\[
\hat{Y}_s = Y_s - \tau B, \quad \hat{B} = B, \quad \hat{Y} = Y, \tag{2.7}
\]

with \(\tau\) being a numerical parameter, the Lagrangian is \textit{form invariant}, that is, it satisfies

\[
L_{\text{YK}}(\phi_A, a) = L_{\text{YK}}(\hat{\phi}_A, \hat{a}), \tag{2.8}
\]

where \(\phi_A\) stands for any of the relevant fields and \(\hat{a}\) is defined by

\[
\hat{a} = a + \tau. \tag{2.9}
\]

From the form invariance (2.8), it can be immediately shown that the fields \(\hat{\phi}_A\) and \(\phi_A\) satisfy the same field equations and the same commutation relations except for the parameter \(a\) which should be replaced by \(\hat{a}\) for \(\hat{\phi}_A\).

2.3. \(R_\xi\) gauge

The quantum Lagrangian of \(R_\xi\) gauge of Fujikawa, Lee and Sanda\(^{20}\) is given by

\[
L_{\text{FLS}} = L_{\text{cl}} - \frac{\xi}{2} \left( \partial_\mu A^\mu + \frac{1}{\xi} M^2 \right)^2 \tag{2.10}
\]

\(^{1}\) In this paper, we use the symbol \(\langle \phi_A \phi_B \rangle\) to denote free propagators in momentum representation: \(\langle \phi_A \phi_B \rangle = \text{F.T.}(0\langle T \phi_A(x) \phi_B(y) \rangle 0)\), in the interaction picture.
in the Abelian case, where $\xi$ is a numerical gauge-fixing parameter, $M = ev$ denotes the acquired mass of $A_\mu$ through the spontaneous symmetry breaking, and the hermitian field $\chi$ is the Goldstone mode defined by

$$\varphi = \frac{1}{\sqrt{2}}(v + \psi + i\chi), \quad (2.11)$$

with $\psi$ being a physical Higgs mode. The global $U(1)$ symmetry is also broken through this gauge fixing so that the Goldstone mode $\chi$ acquires non-zero mass-squared $M^2/\xi$. In particular, by taking the limit of $\xi \to 0$, we can reach the unitary gauge. The BRST symmetric version of (2.10) is discussed in the textbook by Kugo.\(^3\)

Yao has discussed a similar gauge.\(^2\) His Lagrangian can be read in our notation as

$$L_{Yao} = L_{\text{cl}} + B\left(\partial_\mu A_\mu + \frac{M}{\xi} \chi \right) + \frac{1}{2\eta} B^2, \quad (2.12)$$

where $\xi$ and $\eta$ are numerical gauge-fixing parameters. If we put $\xi = \eta$, the Lagrangian (2.12) becomes identical with (2.10), after eliminating the multiplier field $B$.

Free propagators among $A_\mu$, $\chi$ and $B$ fields followed from (2.12) are given by

$$\begin{align*}
\langle A_\mu A_\nu \rangle &= \frac{1}{p^2 - M^2 - \xi^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right) + \frac{p_\mu p_\nu}{M^2} \left[ -\frac{1}{p^2 - \xi^{-1} M^2} + \frac{(\xi^{-1} - \eta^{-1}) M^2}{(p^2 - \xi^{-1} M^2)^2} \right], \\
\langle A_\mu \chi \rangle &= -i(\xi^{-1} - \eta^{-1}) \frac{M p_\mu}{(p^2 - \xi^{-1} M^2)^2}, \\
\langle A_\mu B \rangle &= \frac{ip_\mu}{p^2 - \xi^{-1} M^2}, \\
\langle \chi \chi \rangle &= \frac{p^2 - \eta^{-1} M^2}{(p^2 - \xi^{-1} M^2)^2} \left( \frac{1}{p^2 - \xi^{-1} M^2} + \frac{(\xi^{-1} - \eta^{-1}) M^2}{(p^2 - \xi^{-1} M^2)^2} \right), \\
\langle \chi B \rangle &= -\frac{M}{p^2 - \xi^{-1} M^2}, \\
\langle BB \rangle &= 0. \quad (2.13)
\end{align*}$$

which shows the characteristic features of the $R_\xi$ gauge:

1. for finite $\xi$, the unphysical Goldstone mode $\chi$ propagates with finite mass and
   the ultraviolet behavior of the propagator $\langle A_\mu A_\nu \rangle$ is $O(1/p^2)$.
2. in the limit of $\xi \to 0$, the theory becomes that of the unitary gauge: the mass of
   the Goldstone mode becomes infinitely large and does not propagate any more,
   and the second term of $\langle A_\mu A_\nu \rangle$ vanishes so that $\langle A_\mu A_\nu \rangle$ becomes the propagator
   of the Proca field (see Eq.(4.45)) whose ultraviolet behavior is $O(1)$.

§3. General gauge fixing including gaugeon fields

Now we consider the general gauge-fixing Lagrangian $L_{GF}$ which includes gaugeon fields $Y$ and $Y^*$. To this end, we use a polar decomposition of $\varphi$ field rather than (2.11). If we use the parameterization (2.11) to construct a gaugeon formalism for the $R_\xi$ gauge, it is inevitable to introduce non-polynomial terms in the Lagrangian.
To avoid this, we use the following parameterization for $\varphi$:

$$\varphi(x) = \frac{1}{\sqrt{2}} \left( v + \rho(x) \right) e^{i\pi(x)/v}, \quad (3.1)$$

where hermitian fields $\rho(x)$ and $\pi(x)$ correspond to the fields $\psi(x)$ and $\chi(x)$ of (2.11), respectively; $\pi(x)$ is the Goldstone mode. In terms of these polar variables, $L_{cl}$ is expressed by

$$L_{cl} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( 1 + \frac{e}{M} \rho \right)^2 \left( A_\mu - \frac{1}{M} \partial_\mu \pi \right)^2$$

$$+ \frac{1}{2} \left( \partial_\mu \rho \partial^\mu \rho - m^2 \rho^2 \right) - \frac{1}{2} m \sqrt{\lambda} \rho^3 - \frac{\lambda}{8} \rho^4 + \frac{1}{8} m^2 v^2, \quad (3.2)$$

where $M = ev$ is the mass of $A_\mu$ and $m = \sqrt{\lambda v^2}$ is the mass of the Higgs boson $\rho$.

The gauge transformation (2.3) is now

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x),$$

$$\pi(x) \rightarrow \pi(x) + M \Lambda(x),$$

$$\rho(x) \rightarrow \rho(x), \quad (3.3)$$

under which the Lagrangian (3.2) is invariant.

### 3.1. General gauge-fixing Lagrangian

We impose the following conditions on the gauge-fixing Lagrangian $L_{GF}$:

(a) Lorentz invariance.

(b) quadratic in the fields $A_\mu$, $\pi$, $B$, $Y$ and $Y^\ast$.

(c) The mass dimension\footnote{Note that the mass dimensions of $B$, $Y$, and $Y^\ast$ are different from those of usual fields; $B$ and $Y^\ast$ have dimension two while $Y$ has dimension zero.} of each term does not exceed four. For example, we do not include such terms as $\partial_\mu Y^\ast \partial^\mu \pi$, which has dimension five. Dimension six operators such as $\partial_\mu B \partial^\mu Y^\ast$ are also excluded from $L_{GF}$.

(d) BRST invariance (by incorporating suitable FP ghost terms). This condition excludes those terms expressed as a product of two BRST parent fields, such as $\partial_\mu Y \partial^\mu \pi$.

The most general gauge-fixing Lagrangian satisfying these conditions can be written by

$$L_{GF} = -\left( \omega_1 \partial_\mu B + \omega_3 \partial_\mu Y^\ast \right) A^\mu - \left( \omega_2 \partial_\mu B + \omega_4 \partial_\mu Y^\ast \right) \partial^\mu Y + \left( \beta_1 B + \beta_3 Y^\ast \right) M \pi$$

$$+ \left( \beta_2 B + \beta_4 Y^\ast \right) M^2 Y + \frac{1}{2} \alpha_1 B^2 + \alpha_2 B Y^\ast + \frac{1}{2} \alpha_3 Y^\ast^2, \quad (3.4)$$

where $\omega_i$, $\beta_i$ and $\alpha_i$ are numerical parameters. In order to ensure that the $L_{GF}$ properly fixes the gauge, these parameters should satisfy at least one of the following three conditions:

$$\omega_1 \omega_4 - \omega_2 \omega_3 \neq 0,$$

$$\omega_1 \beta_4 + \beta_1 \omega_4 - \omega_2 \beta_3 - \beta_2 \omega_3 \neq 0,$$

$$\beta_1 \beta_4 - \beta_2 \beta_3 \neq 0. \quad (3.5)$$
If all these three condition are not satisfied, we cannot obtain the propagators from the Lagrangian \( \mathcal{L}_{cl} + \mathcal{L}_{GF} \).

If we put \( \omega_1 = \omega_3 = 0 \) and thus the first term of (3.4) vanishes, then the gauge-fixing Lagrangian \( \mathcal{L}_{GF} \) leads to the unitary gauge, and thus the ultraviolet behavior of the propagators \( \langle A_\mu A_\nu \rangle \) becomes \( O(1) \) rather than \( O(1/p^2) \). To avoid this case, we assume that at least one of the parameters \( \omega_1 \) and \( \omega_3 \) is not equal to zero, or equivalently, \( \omega_1^2 + \omega_3^2 \neq 0 \).

Now we can consider the field redefinition of \( B \) and \( Y \):

\[
\begin{pmatrix}
B' \\
Y'_* 
\end{pmatrix} = \begin{pmatrix}
\omega_1 & \omega_3 \\
\omega_3 & -\omega_1 
\end{pmatrix} \begin{pmatrix}
B \\
Y_* 
\end{pmatrix},
\]

by which the first term of (3.4) is transformed to \(-B' \partial_\mu A^\mu\); the matrix in (3.6) is invertible because of the assumption \( \omega_1^2 + \omega_3^2 \neq 0 \). Thus, without loss of generality, we can assume \( \omega_1 = 1 \) and \( \omega_3 = 0 \) in (3.4) owing to this field redefinition.

Next, we consider the following field redefinitions:

\[
A'_\mu = A_\mu + \omega_2 \partial_\mu Y, \\
\pi' = \pi + \omega_2 MY,
\]

by which the second term of (3.4) is transformed to \(-\omega_4 \partial_\mu Y_* \partial^\mu Y\), while \( \mathcal{L}_{cl} \) remains invariant. Thus, without loss of generality, we can put \( \omega_2 = 0 \) in (3.4).

Here, we further assume that \( \omega_4 = 1 \) from the following reason. If \( \omega_4 = 0 \) (in addition to \( \omega_2 = 0 \)), the second term of (3.4) vanishes. Then, the field equations for the fields \( Y \) and \( Y_* \) become algebraic ones, and thus these fields should be eliminated from the Lagrangian. Namely, \( \mathcal{L}_{GF} \) no longer admits \( q \)-number gauge transformations. To exclude this situation, we assume \( \omega_4 \neq 0 \). With this assumption, \( Y \) can be rescaled as \( \omega_4 Y \rightarrow Y \) to absorb the parameter \( \omega_4 \). In terms of the rescaled field, the value of \( \omega_4 \) is equal to one.

Our general gauge-fixing Lagrangian is now given by

\[
\mathcal{L}_{GF} = -\partial_\mu B A^\mu - \partial_\mu Y_* \partial^\mu Y + (\beta_1 B + \beta_3 Y_*) M \pi \\
+ (\beta_2 B + \beta_4 Y_*) M^2 Y + \frac{1}{2} \alpha_1 B^2 + \alpha_2 BY_* + \frac{1}{2} \alpha_3 Y_*^2.
\]

In the following, we also use a matrix notation to express this Lagrangian as

\[
\mathcal{L}_{GF} = -\partial_\mu B^T A^\mu + B^T M \beta \Pi + \frac{1}{2} B^T \alpha B,
\]

where \( A_\mu, B, \) and \( \Pi \) denote column matrices defined by

\[
A_\mu = \begin{pmatrix} A_\mu \\ \partial_\mu Y \end{pmatrix}, \quad B = \begin{pmatrix} B \\ Y_* \end{pmatrix}, \quad \Pi = \begin{pmatrix} \pi \\ MY \end{pmatrix},
\]

\( \alpha \) and \( \beta \) are \( 2 \times 2 \) matrices defined by

\[
\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}.
\]
and $T$ represents the matrix transpose.

Note here that the Lagrangian (3.8) or (3.9) satisfies the first condition of (3.5) and that no more constraints on $\alpha_i$ and $\beta_i$ are necessary. Consequently, we may consider the case such that all the $\beta_i$ are equal to zero ($\beta = 0$); in this case, the gauge-fixing Lagrangian is identical with that of Lorenz gauge by Yokoyama and Kubo (2.5) (with $\alpha_i$ chosen appropriately). We can also take another case that $\alpha_2 = \beta_2 = \beta_3 = 0$ (both $\alpha$ and $\beta$ are diagonal); in this case, the gaugeon sector of $L_{GF}$ decouples and the rest of the Lagrangian become identical to that of Yao’s $R_\xi$ gauge (2.12).

§4. BRST symmetric theory

We introduce FP ghost term $L_{FP}$ into the general gauge-fixing Lagrangian $L_{GF}$ given by (3.8) or (3.9).

$$L_{GF+FP} = L_{GF} + L_{FP}$$

$$= - \partial_\mu B^\mu - \partial_\mu Y_\ast \partial^\mu Y + (\beta_1 B + \beta_3 Y_\ast) M\pi + (\beta_2 B + \beta_4 Y_\ast) M^2 Y$$

$$+ \frac{1}{2} \alpha_1 B^2 + \alpha_2 BY_\ast + \frac{1}{2} \alpha_3 Y_\ast^2$$

$$- i \partial_\mu c_\ast \partial^\mu c - i \partial_\mu K_\ast \partial^\mu K + i (\beta_1 c_\ast + \beta_3 K_\ast) M^2 c + i (\beta_2 c_\ast + \beta_4 K_\ast) M^2 K,$$

(4.1)

where $c$ and $c_\ast$ are usual FP ghost fields and $K$ and $K_\ast$ are the FP ghost fields for gaugeon fields. In the matrix notation, the Lagrangian can be expressed as

$$L_{GF+FP} = - \partial_\mu B^T A^\mu + B^T M \beta \Pi + \frac{1}{2} B^T \alpha B$$

$$- i \partial_\mu C^T \partial^\mu C + i C^T M^2 \beta C,$$

(4.2)

where $C$ and $C_\ast$ denote column matrices defined by

$$C = \begin{pmatrix} c \\ K \end{pmatrix}, \quad C_\ast = \begin{pmatrix} c_\ast \\ K_\ast \end{pmatrix}.$$  

(4.3)

4.1. field equations

Field equations derived from the Lagrangian $L_{cl} + L_{GF+FP}$ are, for non-FP-ghost fields,

$$\partial_\mu E^{\mu\nu} + M^2 (1 + \frac{e}{M \rho})^2 (A^\nu - \frac{1}{M} \partial^\nu \pi) - \partial^\nu B = 0,$$

(4.4)

$$\partial_\mu \left\{ M \left( 1 + \frac{e}{M \rho} \right)^2 \left( A^\mu - \frac{1}{M} \partial^\mu \pi \right) \right\} + \beta_1 MB + \beta_3 MY_\ast = 0,$$

(4.5)

$$\left( \Box + m^2 \right) \rho + \frac{3}{2} m \sqrt{\rho^2 + \frac{\lambda}{2} \rho^3} - e M \left( 1 + \frac{e}{M \rho} \right) \left( A_\mu - \frac{1}{M} \partial_\mu \pi \right)^2 = 0,$$

(4.6)

$$\partial_\mu A^\mu + \beta_1 M \pi + \alpha_2 Y_\ast + \alpha_1 B + \beta_2 M^2 Y = 0,$$

(4.7)

$$\left( \Box + \beta_4 M^2 \right) Y + \alpha_3 Y_\ast + \alpha_2 B + \beta_3 M \pi = 0,$$

(4.8)

$$\left( \Box + \beta_4 M^2 \right) Y_\ast + \beta_2 M^2 B = 0,$$

(4.9)
from which we also have
\[(\Box + \beta_1 M^2)B + \beta_3 M^2 Y_\ast = 0.\]  
(4.10)

In the matrix notation (3.10) and (3.11), the equations (4.7) and (4.8) can be written by
\[\partial_\mu A^\mu + M\beta\Pi + \alpha B = 0,\]  
(4.11)
and the equations (4.9) and (4.10) by
\[(\Box + M^2 \beta^T)B = 0.\]  
(4.12)

Field equations for FP ghost fields are given by
\[(\Box + M^2 \beta)C = 0,\]  
(4.13)
\[(\Box + M^2 \beta^T)C_\ast = 0,\]  
(4.14)
where we have used the matrix notation (4.3) and (3.11).

The Proca field $U_\mu$ can be defined by
\[U_\mu = A_\mu - \frac{1}{M} \partial_\mu \pi - \frac{1}{M^2} \partial_\mu B,\]  
(4.15)
which satisfies
\[\Box U_\mu = 0,\]  
(4.16)
\[\partial_\mu U^\mu = 0,\]  
(4.17)
where we have assumed the free field approximation, that is, we consider the case $e \to 0$ but $M \neq 0$.

In the free field approximation, (4.5) and (4.7) lead to the field equation for the $\pi$ field:
\[\Box (\alpha - \beta_1) \pi + M^3 \beta_2 Y + M(\alpha_1 - \beta_1) B + M(\alpha_2 - \beta_3) Y_\ast = 0,\]  
(4.18)
which, together with (4.8), may be expressed as
\[\Box (\alpha - E(11)\beta^T)B = 0,\]  
(4.19)
where $E(11)$ is a matrix defined by
\[E(11) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\]  
(4.20)

4.2. BRST symmetry

Our Lagrangian $\mathcal{L}_{cl} + \mathcal{L}_{GF+FP}$ is invariant under the BRST transformation,
\[\delta_B A_\mu = \partial_\mu c, \quad \delta_B \pi = M c, \quad \delta_B \rho = 0,\]
\[\delta_B c_\ast = iB, \quad \delta_B B = \delta_B c = 0, \quad \delta_B Y = K, \quad \delta_B K_\ast = iY_\ast, \quad \delta_B Y_\ast = \delta_B K = 0,\]  
(4.21)
which can be also expressed as
\[ \begin{align*}
\delta_B A_\mu &= \partial_\mu C, \\
\delta_B \Pi &= MC, \\
\delta_B \rho &= 0, \\
\delta_B B &= 0.
\end{align*} \tag{4.22} \]

This obviously satisfies the nilpotency, \( \delta_B^2 = 0 \). Because of the nilpotency, the BRST invariance of \( L_{GF+FP} \) can be easily seen if we rewrite the Lagrangian as
\[ L_{GF+FP} = -i \delta_B \left[ -\partial_\mu C^T A^\mu + C^T \left( M_\beta \Pi + \frac{1}{2} \alpha B \right) \right]. \tag{4.23} \]

The corresponding BRST current \( J_B^\mu \) is given by
\[ J_B^\mu = B^T \partial^\mu C. \tag{4.24} \]

Conservation of this current can be easily seen as
\[ \partial_\mu J_B^\mu = B^T (\overleftarrow{\square} - \overrightarrow{\square}) C = 0, \tag{4.25} \]
where we have used the field equations (4.12) and (4.13).

The corresponding BRST charge is thus given by
\[ Q_B = \int d^3 x B^T \overleftarrow{\partial^\mu} C = \int d^3 x [B \overleftarrow{\partial^\mu} c + Y_\ast \overleftarrow{\partial^\mu} K]. \tag{4.26} \]

By the help of this charge we can define the physical subspace \( V_{phys} \) as a space of states satisfying
\[ Q_B |_{phys} = 0. \tag{4.27} \]

By using this subsidiary condition, we can remove all unphysical modes of this theory.\(^2,3\)

Note that the Proca field \( U_\mu \) defined by (4.15) is BRST invariant:
\[ \delta_B U_\mu = 0. \tag{4.28} \]

4.3. \textit{q-number gauge transformation}

We define \( q \)-number gauge transformations by
\[ \begin{align*}
A_\mu &\to \hat{A}_\mu = A_\mu + \tau \partial_\mu Y, \\
\pi &\to \hat{\pi} = \pi + \tau MY, \\
Y_\ast &\to \hat{Y}_\ast = Y_\ast - \tau B, \\
B &\to \hat{B}, \\
Y &\to \hat{Y}, \\
c &\to \hat{c} = c + \tau K, \\
K_\ast &\to \hat{K}_\ast = K_\ast - \tau c_\ast, \\
c_\ast &\to \hat{c}_\ast = c_\ast, \\
K &\to \hat{K} = K.
\end{align*} \tag{4.29} \]
where $\tau$ is a numerical parameter. We also define the transformation for the physical Higgs field $\rho$, which should be invariant:

$$\rho \rightarrow \hat{\rho} = \rho.$$  

(4.30)

Note that the Proca filed $U_\mu$ is also invariant under the transformation:

$$U_\mu \rightarrow \hat{U}_\mu = U_\mu.$$  

(4.31)

If we introduce a one-parameter matrix $g(\tau)$ by

$$g(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix},$$

(4.32)

the $q$-number gauge transformed fields in (4.29) can be expressed as

$$\hat{A}_\mu = g(\tau) A_\mu,$$

$$\hat{H} = g(\tau) H,$$

(4.33)

$$\hat{B}^T = B^T g(\tau)^{-1},$$

(4.34)

$$\hat{C} = g(\tau) C, \quad \hat{C}^*_T = C^*_T g(\tau)^{-1},$$

(4.35)

where $g(\tau)^{-1} = g(-\tau)$ is the inverse matrix of $g(\tau)$.

Under the $q$-number gauge transformation $L_{\text{cl}}$ is invariant while $L_{\text{GF+FP}}$ transforms as

$$L_{\text{GF+FP}} = -\partial_\mu \hat{B}^T \hat{A}^\mu + B^T M \beta \hat{H} + \frac{1}{2} B^T \hat{\alpha} \hat{\beta}$$

$$- i \partial_\mu \hat{C}^*_T \partial^\mu \hat{\alpha} + i \hat{C}^*_T M^2 \hat{\beta},$$

(4.36)

with $\hat{\alpha}$ and $\hat{\beta}$ being

$$\hat{\alpha} = g(\tau) \alpha g(\tau)^T,$$

(4.37)

$$\hat{\beta} = g(\tau) \beta g(\tau)^{-1}.$$  

(4.38)

Thus, under the $q$-number gauge transformation, the total Lagrangian $L = L_{\text{cl}} + L_{\text{GF+FP}}$ is form invariant:

$$L(\phi_A, \alpha_i, \beta_j) = L(\hat{\phi}_A, \hat{\alpha}_i, \hat{\beta}_j),$$

(4.39)

where $\hat{\alpha}_i$ and $\hat{\beta}_j$ are components of the matrices $\hat{\alpha}$ and $\hat{\beta}$, that is,

$$\hat{\alpha}_1 = \alpha_1 + 2\alpha_2 \tau + \alpha_3 \tau^2,$$

$$\hat{\alpha}_2 = \alpha_2 + \alpha_3 \tau,$$

$$\hat{\alpha}_3 = \alpha_3,$$

$$\hat{\beta}_1 = \beta_1 + \beta_3 \tau,$$

$$\hat{\beta}_2 = \beta_2 + (\beta_4 - \beta_1) \tau - \beta_3 \tau^2,$$

$$\hat{\beta}_3 = \beta_3,$$

$$\hat{\beta}_4 = \beta_4 - \beta_3 \tau.$$  

(4.40)
We emphasize here that this $q$-number gauge transformation commutes with the BRST transformation (4.21). As a result, the BRST charge (4.26) is invariant under the $q$-number gauge transformation,

$$\hat{Q}_B = Q_B,$$

and therefore the physical subspace is also invariant:

$$\hat{\mathcal{V}}_{\text{phys}} = \mathcal{V}_{\text{phys}}. \quad (4.42)$$

4.4. free propagators

The quadratic part of our Lagrangian $\mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{GF+FP}}$ can be written as

$$L_{\text{quadratic}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} U_\mu U^\mu + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{m^2}{2} \rho^2$$

$$- \frac{1}{M} \partial_\mu B^T \partial^\mu \Pi + B^T M \beta \Pi - \frac{1}{2M^2} \partial_\mu B^T E_{(11)} \partial^\mu B + \frac{1}{2} B^T \alpha B$$

$$- i \partial_\mu C^T \partial^\mu C + i C^T M^2 \beta C,$$

where $U_\mu$ is the Proca field (4.15) and $E_{(11)}$ is the matrix defined by (4.20). This expression shows that $U_\mu$ and $\rho$ decouple from other fields $\Pi, B, C$, and $C^*$; $F_{\mu\nu}$ can be also expressed as $F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$. The propagators among $U_\mu$'s and $\rho$ is thus given by

$$\langle \rho \rho \rangle = \frac{1}{p^2 - m^2}, \quad (4.44)$$

$$\langle U_\mu U_\nu \rangle = \frac{1}{p^2 - M^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right). \quad (4.45)$$

Any other propagators which include $U_\mu$ or $\rho$ are equal to zero.

Before discussing the propagators for other fields, we first define the quantity,

$$D(p^2; \beta) = \det(-p^2 1 + M^2 \beta)$$

$$= (p^2)^2 - p^2 M^2 t + M^4 d, \quad (4.46)$$

where 1 stands for the unit matrix and

$$t = \text{tr} \beta = \beta_1 + \beta_4, \quad (4.47)$$

$$d = \det \beta = \beta_1 \beta_4 - \beta_2 \beta_3, \quad (4.48)$$

Then, we get

$$(-p^2 1 + M^2 \beta)^{-1} = \frac{1}{D(p^2; \beta)} (-p^2 1 + M^2 \tilde{\beta}), \quad (4.49)$$

where $\tilde{\beta}$ is the cofactor matrix of $\beta$,

$$\tilde{\beta} = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}, \quad (4.50)$$
which satisfies
\[ \beta \tilde{\beta} = \tilde{\beta} \beta = \det \beta \cdot 1, \]  
\[ \beta + \tilde{\beta} = \text{tr} \beta \cdot 1. \]  

Note that \( \tilde{\beta} \) transforms as \( \hat{\beta} \equiv \tilde{\beta} = g(\tau)\tilde{\beta}g(\tau)^{-1} \) under the q-number gauge transformation.

The propagators among \( \Pi, B, C \) and \( C_* \) are now given by
\[ \langle \Pi \Pi^T \rangle = \frac{1}{D(p^2;\beta)}(-p^2 1 + M^2 \beta)(p^2 E_{(11)} + M^2 \alpha)(-p^2 1 + M^2 \tilde{\beta})^T \]
\[ = \frac{1}{D(p^2;\beta)}(p^2 E_{(11)} - M^2 \alpha') + \frac{M^4}{D(p^2;\beta)^2}(p^2 \gamma - M^2 \delta), \]  
\[ \langle \Pi B^T \rangle = 0, \]
\[ \langle B B^T \rangle = \frac{-i}{D(p^2;\beta)}(-p^2 1 + M^2 \tilde{\beta}), \]
\[ \langle C C_*^T \rangle = \frac{-i}{D(p^2;\beta)}(-p^2 1 + M^2 \tilde{\beta}), \]

where \( \alpha', \gamma \) and \( \delta \) in (4.53) are symmetric matrices defined by
\[ \alpha' = \alpha + \tilde{\beta} E_{(11)} + E_{(11)} \beta^T - tE_{(11)} = \begin{pmatrix} \alpha' \alpha' \alpha' \\ \alpha' \alpha' \alpha' \end{pmatrix}, \]  
\[ \gamma = -t\alpha' - dE_{(11)} + \tilde{\beta} \alpha + \alpha \beta^T + \tilde{\beta} E_{(11)} \beta^T = \begin{pmatrix} \gamma \gamma \gamma \\ \gamma \gamma \gamma \end{pmatrix}, \]  
\[ \delta = -d\alpha' + \beta \alpha \beta^T = \begin{pmatrix} \delta \delta \delta \\ \delta \delta \delta \end{pmatrix}, \]
which lead to
\[ \begin{aligned}
\alpha'_1 &= \alpha_1 - \beta_1 + \beta_4, \\
\alpha'_2 &= \alpha_2 - \beta_3, \\
\alpha'_3 &= \alpha_3, \\
\gamma_1 &= -\alpha_1(\beta_1 - \beta_4) - 2\alpha_2\beta_2 + \beta_1^2 - d, \\
\gamma_2 &= -\alpha_1\beta_3 - \alpha_3\beta_2 + \beta_1\beta_3, \\
\gamma_3 &= -2\alpha_2\beta_3 + \alpha_3(\beta_1 - \beta_4) + \beta_3^2, \\
\delta_1 &= \alpha_1(-d + \beta_4^2) - 2\alpha_2\beta_2\beta_4 + \alpha_3\beta_2^2 + (\beta_1 - \beta_4)d, \\
\delta_2 &= -\alpha_1\beta_3 \beta_4 + 2\alpha_2\beta_2\beta_3 - \alpha_3\beta_1 \beta_2 + \beta_3 d, \\
\delta_3 &= \alpha_1\beta_3^2 - 2\alpha_2\beta_1 \beta_3 - \alpha_3(-d + \beta_1^2). 
\end{aligned} \]

It can be easily confirmed that under the q-number gauge transformation (4.57) and (4.58) the matrices \( \alpha', \gamma \) and \( \delta \) transform as \( \hat{\alpha}' = g(\tau)\alpha'g(\tau)^T \), \( \hat{\gamma} = g(\tau)\gamma g(\tau)^T \) and \( \hat{\delta} = g(\tau)\delta g(\tau)^T \).

Since \( A_\mu = U_\mu + \partial_\mu \pi/M + \partial_\mu B/M^2 \), the propagators for \( A_\mu \)'s can be evaluated from the propagators for other fields, such as,
\[ \langle A_\mu A_\nu \rangle = \langle U_\mu U_\nu \rangle + \frac{p_\mu p_\nu}{M^2} \left[ \langle \pi \pi \rangle + \frac{1}{M} (\langle \pi B \rangle + \langle B \pi \rangle) + \frac{1}{M^2} \langle B B \rangle \right], \]  
(4.63)
\[ \langle A_\mu \pi \rangle = \frac{-ip_\mu}{M} \left[ \langle \pi \pi \rangle + \frac{1}{M} \langle B \pi \rangle \right]. \quad (4.64) \]

Thus, from (4.45) and (4.53)-(4.55), we get

\[ \langle A_\mu A_\nu \rangle = \frac{1}{p^2 - M^2} \left[ -g_{\mu \nu} + \frac{P_\mu P_\nu}{M^2} \right] \]
\[ + \frac{p_\mu P_\nu}{M^2} \left[ -p^2 + (t - \alpha_1)M^2 \right] + \frac{M^2 \gamma_1 p^2 - \delta_1 M^2}{D(p^2; \beta)} \], \quad (4.65) \]
\[ \langle A_\mu \rho \rangle = 0, \quad (4.66) \]
\[ \langle A_\mu \pi \rangle = ip_\mu M \left[ \frac{\alpha_1 - \beta_1}{D(p^2; \beta)} - \frac{M^2 (\gamma_1 p^2 - \delta_1 M^2)}{D(p^2; \beta)^2} \right], \quad (4.67) \]
\[ \langle A_\mu B \rangle = ip_\mu \frac{p^2 - \beta_1 M^2}{D(p^2; \beta)}, \quad (4.68) \]
\[ \langle A_\mu Y \rangle = ip_\mu \left[ \frac{\alpha_2}{D(p^2; \beta)} - \frac{M^2 (\gamma_2 p^2 - \delta_2 M^2)}{D(p^2; \beta)^2} \right], \quad (4.69) \]
\[ \langle A_\mu Y_s \rangle = ip_\mu \frac{\beta_2 M^2}{D(p^2; \beta)}. \quad (4.70) \]

Note here the propagator (4.65) reads

\[ \langle A_\mu A_\nu \rangle = -\frac{g_{\mu \nu}}{p^2 - M^2} - (\alpha_1 - 1) \frac{P_\mu P_\nu}{D(p^2; \beta)} \]
\[ + M^2 P_\mu P_\nu \left[ \frac{1 - t + \delta_1 (p^2 - M^2)}{D(p^2; \beta) D(p^2; \beta)} + \frac{\gamma_1 p^2 - \delta_1 M^2}{D(p^2; \beta)^2} \right], \quad (4.71) \]

which shows that \( \langle A_\mu A_\nu \rangle \) has the usual ultraviolet behavior,

\[ \langle A_\mu A_\nu \rangle = O \left( \frac{1}{p^2} \right), \quad \text{as } p^\mu \to \infty \quad (4.72) \]

rather than that of the Proca filed, \( \langle U_\mu U_\nu \rangle = O(1) \).

§5. \( R_\xi \)-like gauges

Our theory includes \( R_\xi \)-like gauges. By choosing the gauge-fixing parameters \( \alpha \) and \( \beta \) appropriately, we find that some of the parameters can be considered as a \( \xi \) parameter. That is, some of the parameter has a property similar to that of the \( \xi \) parameters of \( R_\xi \) gauge\(^{20,21}\) mentioned in §2.

5.1. Yao’s gauge

We choose the parameters \( \alpha \) and \( \beta \) as

\[ \alpha = \begin{pmatrix} \eta^{-1} & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \beta = \xi^{-1} \mathbf{1} = \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad (5.1) \]
then the gauge-fixing term of the Lagrangian is now
\[ \mathcal{L}_{GF} = B \left( \partial_\mu A^\mu + \frac{1}{\xi} M \pi \right) + \frac{1}{2\eta} B^2 + \frac{1}{2} B Y_s - \partial_\mu Y_s \partial^\mu Y + \frac{1}{\xi} M^2 Y Y, \] (5.2)
up to total derivatives. The first two terms of the right hand side are the same gauge-fixing term as Yao’s (2.12). Furthermore, we can easily see that \( A_\mu, \pi (\approx \chi) \) and \( B \) have the same propagators as Yao’s (2.13).

Under the \( q \)-number gauge transformation (4.29), the parameters \( \xi \) and \( \eta \) transform as
\[ \xi^{-1} \rightarrow \hat{\xi}^{-1} = \xi^{-1} \] (5.3)
\[ \eta^{-1} \rightarrow \hat{\eta}^{-1} = \eta^{-1} + \tau. \] (5.4)

Thus the parameter \( \eta \) can be shifted freely. In particular, by using the \( q \)-number gauge transformation, we can put \( \hat{\xi} = \hat{\eta} \); the theory become equivalent to that of Fujikawa-Sanda-Lee.\(^{20} \) In contrast to \( \eta \), the parameter \( \xi \) cannot be shifted. Consequently, we cannot take the limit \( \xi \rightarrow 0 \) by the \( q \)-number gauge transformation.

In addition to the BRST charge (4.26), we can define the following BRST-like conserved charges:
\[ Q_{B(KO)} = \int c \overleftrightarrow{\partial_0} B d^3x, \] (5.5)
\[ Q_{B(Y)} = \int K \overleftrightarrow{\partial_0} Y_s d^3x, \] (5.6)
\[ Q'_{B(KO)} = \int \overleftrightarrow{K} \partial_0 B d^3x, \] (5.7)
\[ Q'_{B(Y)} = \int \overleftrightarrow{c} \partial_0 Y_s d^3x. \] (5.8)

All of these satisfy the nilpotency condition. The conservation of these charges are due to the fact that in the case of \( \beta = \xi^{-1} 1 \) the fields \( B, Y_s, c \) and \( K \) satisfy the same Klein-Gordon equation with the same mass squared \( M^2/\xi \).

Instead of the physical condition (4.27), we may consider the condition
\[ Q_{B(KO)}|_{phys} = 0, \]
\[ Q_{B(Y)}|_{phys} = 0, \] (5.9)
to define the physical subspace. The unphysical Goldstone mode is removed by the first equation while the gaugeon modes are removed by the second. Let \( \mathcal{V}^{(\eta)}_{phys} \) denote the space of states satisfying (5.9). This space is a subspace of \( \mathcal{V}^{(\eta)}_{phys} \) defined by (4.27): \( \mathcal{V}^{(\eta)}_{phys} \subset \mathcal{V}^{(\eta)}_{phys} \). The definition of the space \( \mathcal{V}^{(\eta)}_{phys} \) depends on the parameter \( \eta \). In fact, under the \( q \)-number gauge transformation, the charges \( Q_{B(KO)} \) and \( Q_{B(Y)} \) transform as
\[ Q_{B(KO)} \rightarrow \hat{Q}_{B(KO)} = Q_{B(KO)} + \tau Q'_{B(Y)}, \]
\[ Q_{B(Y)} \rightarrow \hat{Q}_{B(Y)} = Q_{B(Y)} - \tau Q'_{B(KO)}. \] (5.10)
Consequently, the subspace $\mathcal{V}_\text{phys}^{(\eta)}$ transforms into another subspace $\mathcal{V}_\text{phys}^{(\eta+\tau)}$.

Let us define a subspace $\mathcal{V}_\text{Yao}^{(\eta)}$ of the total Fock space $\mathcal{V}$ by

$$
\mathcal{V}_\text{Yao}^{(\eta)} = \ker Q_{B(Y)} = \{|\Phi\rangle \in \mathcal{V}; Q_{B(Y)}|\Phi\rangle = 0\} \subset \mathcal{V},
$$

(5.11)

which includes $\mathcal{V}_\text{phys}^{(\eta)}$ as a subspace. Since the gaugeon modes are excluded from $\mathcal{V}_\text{Yao}^{(\eta)}$, the space $\mathcal{V}_\text{Yao}^{(\eta)}$ corresponds to the total Fock space of the Yao’s theory. Under the $q$-number gauge transformation this subspace transforms as

$$
\mathcal{V}_\text{Yao}^{(\eta)} \rightarrow \mathcal{V}_\text{Yao}^{(\eta+\tau)} = \ker \hat{Q}_{B(Y)} \subset \mathcal{V}.
$$

(5.12)

Thus various Fock spaces of the Yao’s theory (including the theory of Fujikawa, Sanda and Lee) corresponding to various values of $\eta$ are embedded in the single Fock space $\mathcal{V}$ of our theory.

5.2. more complicated cases

We choose the parameters $\alpha$ and $\beta$ as

$$
\alpha = \begin{pmatrix} \eta^{-1} & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \xi^{-1} & \beta_2 \\ 0 & \xi^{-1} \end{pmatrix},
$$

(5.13)

then the determinant $D(p^2; \beta)$ and matrices $\alpha'$, $\gamma$ and $\delta$ become

$$
D(p^2; \beta) = (p^2 - \xi^{-1}M^2)^2,
$$

(5.14)

$$
\alpha' = \alpha, \quad \gamma = -\beta_2E_{(11)}, \quad \delta = -\xi^{-1}\beta_2E_{(11)}.
$$

(5.15)

Free propagators for $A_\mu$ and $\pi$ are given by

$$
\langle A_\mu A_\nu \rangle = \frac{1}{p^2 - M^2} \left[ -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right]
$$

$$
+ \frac{p_\mu p_\nu}{M^2} \left[ \frac{1}{p^2 - \xi^{-1}M^2} - \frac{(\xi^{-1} - \eta^{-1})M^2}{(p^2 - \xi^{-1}M^2)^2} - \frac{\beta_2M^4}{(p^2 - \xi^{-1}M^2)^3} \right],
$$

$$
\langle A_\mu \pi \rangle = -ip_\mu \left[ \frac{(\xi^{-1} - \eta^{-1})M}{(p^2 - \xi^{-1}M^2)^2} - \frac{\beta_2M^3}{(p^2 - \xi^{-1}M^2)^3} \right],
$$

$$
\langle \pi \pi \rangle = \frac{1}{p^2 - \xi^{-1}M^2} + \frac{(\xi^{-1} - \eta^{-1})M^2}{(p^2 - \xi^{-1}M^2)^2} - \frac{\beta_2M^4}{(p^2 - \xi^{-1}M^2)^3}.
$$

(5.16)

As easily seen, if we put $\beta_2 = 0$, the theory becomes Yao’s gauge.

Under the $q$-number gauge transformation, the parameter $\xi$, $\eta$ and $\beta_2$ transforms as

$$
\xi^{-1} \rightarrow \hat{\xi}^{-1} = \xi^{-1},
$$

$$
\eta^{-1} \rightarrow \hat{\eta}^{-1} = \eta^{-1} + \tau,
$$

$$
\beta_2 \rightarrow \hat{\beta}_2 = \beta_2.
$$

(5.17)

*) If we put $\beta = 0 (\xi \rightarrow \infty)$, the theory becomes the BRST symmetric version of the gaugeon formalism for the Lorenz gauge of Yokoyama and Kubo. The same gauge structure of the Fock spaces as (5.12) still holds for this theory.
The \( \xi \) parameter again cannot be shifted. A bit different case is

\[
\alpha = \begin{pmatrix} \alpha_1 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & \beta_4 \end{pmatrix}, \quad (\beta_1 \neq \beta_4)
\] (5.18)

In this case, the determinant \( D(p^2; \beta) \) becomes

\[
D(p^2; \beta) = (p^2 - \beta_1 M^2)(p^2 - \beta_4 M^2),
\] (5.19)

and it can be seen that \( \beta_1^{-1} \) has the same property as the \( \xi \) parameter. Under the \( q \)-number gauge transformation, parameters \( \alpha_1, \beta_1, \beta_2 \) and \( \beta_4 \) transform as

\[
\hat{\alpha}_1 = \alpha_1 + \tau, \quad \hat{\beta}_1 = \beta_1, \quad \hat{\beta}_2 = \beta_2 + (\beta_4 - \beta_1)\tau, \quad \hat{\beta}_4 = \beta_4.
\] (5.20)

Thus, the \( \xi \) parameter \( \beta_1 \) is again invariant.

It should be commented that if \( \beta_2 = 0 \) the propagators among \( A_\mu, \pi \) and \( B \) fields are the same propagators in Yao’s gauge with \( \xi = \beta_1^{-1} \) and \( \eta = \alpha_1^{-1} \). In this case, however, the \( q \)-number gauge transformation (5.20) shifts the value of \( \beta_2 \) to non-zero so that the propagators no longer maintain the form of Yao’s gauge.

5.3. \( \xi \)-parameter in general case

We have exhibited above some examples of the \( R_\xi \)-like gauge which are included in our theory as the special choices of \( \alpha \) and \( \beta \). In any of these examples, the \( \xi \)-parameter has been invariant under the \( q \)-number gauge transformation. We show here that even in more general cases the possible \( \xi \)-parameter is always invariant under the \( q \)-number gauge transformation.

First, we define the \( R_\xi \) gauge of our theory as follows: There exits some gauge-fixing parameter(s) denoted by \( \xi \) such that in the limit of \( \xi \to 0 \) the propagators among \( A_\mu \)'s and \( \pi \) become the propagators of the unitary gauge, that is,

\[
\begin{cases}
\langle A_\mu A_\nu \rangle \to \langle U_\mu U_\nu \rangle, \\
\langle A_\mu \pi \rangle \to 0, \\
\langle \pi \pi \rangle \to 0,
\end{cases}
\] (5.21)

where \( \langle U_\mu U_\nu \rangle \) is the propagator for the Proca field (4.45). Owing to the equations (4.63) and (4.64), the condition (5.21) can be read as

\[
\begin{cases}
\langle \pi \pi \rangle \to 0, \\
\langle \pi B \rangle \to 0, \\
\langle BB \rangle \to 0.
\end{cases}
\] (5.22)

We find that from (4.53)-(4.55) this condition is equivalent to

\[
D(p^2; \beta) = \det(-p^2 1 + M^2 \beta) \to \infty.
\] (5.23)
The determinant $D(p^2; \beta)$ can be factorized as
\[
D(p^2; \beta) = (p^2 - \xi_1^{-1}M^2)(p^2 - \xi_2^{-1}M^2),
\]
where $\xi_1^{-1}$ and $\xi_2^{-1}$ are two eigenvalues of the matrix $\beta$. The condition (5.23) shows that the possible $\xi$-parameter is one or both of the parameters $\xi_1$ and $\xi_2$. Since the eigenvalues of the matrix $\beta$ are invariant under the $q$-number gauge transformation (4.38), our possible $\xi$-parameters $\xi_1$ and/or $\xi_2$ cannot be shifted by the $q$-number gauge transformation.

§6. Summary and discussion

Starting from the most general gauge-fixing Lagrangian including $Y$ and $Y_*$ fields, we present a general form of BRST symmetric gaugeon formalism for the Higgs model. Our theory has seven gauge-fixing parameters $\alpha_i$ ($i = 1, 2, 3$) and $\beta_j$ ($j = 1, 2, 3, 4$), some of which can be shifted by the $q$-number gauge transformation. The $q$-number gauge transformation commutes with the BRST transformation. As a result, the BRST charge is invariant, $\hat{Q}_B = Q_B$ and thus the physical subspace $\mathcal{V}_{\text{phys}} = \ker Q_B$ is also gauge invariant.

As a special choice of the gauge-fixing parameters ($\alpha_1 = \varepsilon a^2, \alpha_2 = \varepsilon a, \alpha_3 = \varepsilon; \beta_j = 0$), our theory includes the BRST symmetric version of the gaugeon formalism for Lorenz gauge by Yokoyama and Kubo (2.11). Other choices of the parameters $\alpha$ and $\beta$ lead us to the theories of $R_\xi$-like gauges. Especially, by choosing (5.1), we get the gaugeon formalism for the Yao’s $R_\xi$ gauge (2.12), where one of the two gauge-fixing parameters, $\eta$, can be shifted by the $q$-number gauge transformation. In particular, the $q$-number gauge transformation can shift the $\eta$ to be equal to $\xi$, where the theory becomes equivalent to the $R_\xi$ gauge of Fujikawa, Lee and Sanda (2.10). In any case of these $R_\xi$-like gauges, the $\xi$-parameter is shown to be invariant under the $q$-number gauge transformation.

The invariance of the $\xi$-parameter under the $q$-number gauge transformation might be understood by the following arguments. The propagator $\langle A_\mu A_\nu \rangle$ in the $R_\xi$ gauge of Fujikawa, Lee and Sanda is given by
\[
\langle A_\mu A_\nu \rangle = \frac{1}{p^2 - M^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right) - \frac{p_\mu p_\nu}{M^2 (p^2 - \xi^{-1}M^2)}. \tag{6.1}
\]
Now assume that there might exist a $q$-number gauge transformation $\hat{A}_\mu = A_\mu + \tau \partial_\mu A$, under which the $\xi$-parameter would transform into $\hat{\xi} = \xi + \Delta \xi (\neq \xi)$. Then,
\[
\langle \hat{A}_\mu \hat{A}_\nu \rangle - \langle A_\mu A_\nu \rangle = \tau \left( \langle A_\mu \partial_\nu A \rangle + \langle \partial_\mu A A_\nu \rangle \right) + \tau^2 \langle \partial_\mu A \partial_\nu A \rangle
\]
\[
= -\frac{p_\mu p_\nu}{M^2} \left[ \frac{1}{p^2 - (\xi + \Delta \xi)^{-1}M^2} - \frac{1}{p^2 - \xi^{-1}M^2} \right]
\]
\[
= -\frac{p_\mu p_\nu}{M^2} \left[ \frac{1}{p^2 - \xi^{-1}M^2} \left( \frac{\Delta M^2}{p^2 - \xi^{-1}M^2} \right)^{-1} - 1 \right]
\]
\[
= -\frac{p_\mu p_\nu}{M^2} \left[ \frac{\Delta M^2}{(p^2 - \xi^{-1}M^2)^2} + \frac{\Delta M^4}{(p^2 - \xi^{-1}M^2)^3} + \cdots \right], \tag{6.2}
\]
where \( \Delta M^2 = (\xi + \Delta \xi)^{-1}M^2 - \xi^{-1}M^2 \). This shows that the field \( \Lambda \) should include dipole modes, tripole modes, quadrupole modes, \ldots, and any other higher-pole modes. The \( Y \) field of our theory, however, does not satisfy this condition: \( Y \) includes at most quadrupole modes (see, for example, (4.53)). Thus we may infer that the gaugeon formalism with \( \xi \)-parameter which might be shifted by the \( q \)-number gauge transformation, if exists, would require infinite series of multi-pole fields \((n\text{-pole fields with } n = 2, 3, 4, \ldots)\).

We have seen in the section 5-1 that the Fock space of Yao’s \( R_\xi \) gauge is embedded in the total Fock space of our theory (if we choose \( \alpha_3 = 0, \beta = \xi^{-1}1 \)). In the arguments, the four BRST-like charges \((5.5)-(5.8)\) (or equivalently, \( \tilde{Q}_{(KO)} \), \( \tilde{Q}_{(Y)} \), \( \tilde{Q}_{(KO)} \)) play an essential role. Similar arguments on the gauge structure of the Fock spaces are applicable to the theory in Lorenz gauge of Yokoyama and Kubo \((\beta = 0)\), since the four BRST-like charges \((5.5)-(5.8)\) also exist in this gauge.

Here we shall consider the number of the conserved BRST-like charges in general case, and in what case the number becomes four. A BRST-like current may be expressed as

\[
J_R^\mu = \mathcal{B}^T R \partial_\mu \mathcal{C},
\]

(6.3)

where \( R \) is a real and constant \( 2 \times 2 \) matrix. By using the field equations \((4.12)\) and \((4.13)\) we can evaluate the divergence of the current:

\[
\partial_\mu J_R^\mu = \mathcal{B}^T (-\Box) R \mathcal{C} + \mathcal{B}^T R \Box \mathcal{C}
= \mathcal{B}^T M^2 (\beta R - R \beta) \mathcal{C}.
\]

(6.4)

Thus, the current \( J_R^\mu \) is conserved if and only if

\[
[R, \beta] = 0.
\]

(6.5)

The number of the independent matrices \( R \) satisfying \((6.5)\) is just the number of the conserved BRST-like charges. If \( \beta \) is not proportional to the unit matrix, two types of the matrix \( R = \text{const.} \times 1 \) and \( R = \text{const.} \times \beta \) commute with \( \beta \), thus in this case the number of the conserved BRST-like charges is two. On the other hand, if \( \beta = \text{const.} \times 1 \), an arbitrary matrix \( R \) commutes with the \( \beta \), thus there exist four independent conserved BRST-like charges in this case. This is nothing but the case of Yao’s gauge \((\beta = \xi^{-1}1)\) and the Lorenz gauge of Yokoyama and Kubo \((\beta = 0)\); no other case ensures the existence of four conserved currents.

Acknowledgements

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References

10) M. Abe, “The Symmetries of the Gauge-Covariant Canonical Formalism of Non-Abelian