Non-Metric Gravity I: Field Equations

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Abstract

We describe and study a certain class of modified gravity theories. Our starting point is Plebanski formulation of gravity in terms of a tripple of 2-forms, a connection $A$ and a “Lagrange multiplier” field $\Psi$. The generalization we consider stems from presence in the action of an extra term proportional to a scalar function of $\Psi$. As in the usual Plebanski general relativity (GR) case, the equations coming from variations with respect to $\Psi$ imply that a certain metric can be introduced. However, unlike in GR, the connection $A$ no longer coincides with the self-dual part of the metric-compatible spin-connection. Field equations of the theory are shown to be relations between derivatives of the metric and components of field $\Psi$, as well as its derivatives, the later being in contrast to the GR case. The equations are of second order in derivatives. An analog of the Bianchi identity is still present in the theory, as well as its contracted version tantamount to energy conservation equation. The arising modifications to the later are possibly of experimental significance.

1 Introduction

Modified gravity theories have become popular recently, motivated mainly by the fact that Einstein’s general relativity (GR) interprets the available observational data in terms of most of the content of the universe being in the form of some dark energy and matter that have never been observed directly. Modified gravity theories are essentially of two main types. One type introduces new degrees of freedom (new fields). Some examples of these are: Brans-Dicke [1] (a generalization of GR in which the gravitational constant is replaced by a scalar field), a scalar-vector-tensor theory of Bekenstein [2] that is designed to give modified Newtonian dynamics (MOND) as its non-relativistic limit, a scalar-tensor theory [3] for dynamical light velocity, brane-world theories [4], [5] that introduce an extra spacetime dimension (an thus an infinite number of new degrees of freedom). Another interesting recent proposal [6] calls for gravity to be described by the so-called area metric instead of the metric in the usual sense. One of the major challenges for all these models is to show how Einstein’s GR arises in “usual circumstances”. The other type of modifications considered is that of pure GR, i.e. with no new degrees of freedom (DOF) added, but with modified equations of motion, typically in a way that introduces higher derivatives. The quantum gravity induces modifications (quantum corrections) are precisely of this type, which is one of the reasons for interest in this models. Some examples of these are: theories used for string cosmology, see e.g. [7], Weyl conformal gravity considered in e.g. [8], the so-called $f(R)$ theories [9]. There are other modified gravity theories considered in the literature, and the above list makes no attempt to be complete. The two types described are not necessarily mutually exclusive, as integrating the new DOF out in a theory of the first type one typically gets a theory of the second type. One of the lessons learnt from all the above developments is that it is extremely hard to modify Einstein’s general relativity in any interesting way and to remain consistent with what is observed.

In this paper we introduce another class of modified gravity theories which falls into the second of the two categories described above. That is, no new degrees of freedom are added, modification considered is that of DOF existing already in GR. In spite of this, the theories we consider also share some similarities with the proposal of [6], in that the basic fields of the theory have very little to do with the usual spacetime metric field. It is for this reason, as well as for the fact that the main derivative operator that appears in the theory has little to do with any metric-compatible one, that we decided to refer to the class of models we study as “non-metric” gravity.

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In broad terms, the nature of modification we consider is that a certain new field, referred to as the Lagrange multiplier field \( \Psi \) is introduced. Such a field is also present in Plebanski formulation [10] of GR. In that case, one of the equations of motion of the theory expresses \( \Psi \) in terms of second derivatives of the metric (or, in terms of the curvature of \( g \)). After such \( \Psi \) is substituted into the action, one gets back the usual Einstein-Hilbert functional. In modified theories we consider \( \Psi \) is no longer so simply expressable through \( g \) relates \( D \) where can be solved, at least in principle, for \( \Psi \), with the solution being, again schematically a where \( \Psi = D^2 g + \sum_{n,m=1}^{\infty} a_{nm} (\tilde{D}^2)^n (D^2 g)^m \), where \( a_{nm} \) are some coefficients. Substituted back into the action this gives the usual Einstein-Hilbert functional corrected by an infinite series of higher order terms in the curvature of \( g \).

At the very least, the class of models we consider is an interesting compact (and useful for quantum considerations, see more comments on this below) way of re-writing such infinite expansions. As is clear from [11] the coupled equations for \( g, \Psi \) are second order, which guarantees that many unpleasant problems with higher derivative theories are automatically avoided. Most optimistically, this formulation of (quantum corrected) gravity could allow one to come to terms with the behavior of gravity under renormalization, as was argued in [11]. Another point is that, as we shall demonstrate in this paper, the modification of gravity we introduce affects the energy conservation equation for matter in quite an interesting way. Thus, instead of \( \nabla^\mu T_{\mu \nu} = 0 \), where \( \nabla \) is the metric-compatible derivative operator, the energy conservation in theories we consider takes the schematic form \( D_{\mu} = 0 \), where \( D \) is an operator different from \( \nabla \), and the contraction of indices in this equation is done more non-trivially, see the main text. This modification of the energy conservation equation rests on the assumption that the matter fields couple not to the metric \( g \), but directly to the fundamental 2-form field of the theory, as we explain below. This non-metric modification implies that many relations that follow from GR and are usually taken in cosmology for granted may have to be re-thought. It would be extremely interesting to describe what the available cosmological data imply about the universe if interpreted using the theories from the class we consider. We hope to perform such an analysis in other papers from the series.

“Renormalizable” gravity. It was argued in [11] that in the so-called Plebanski formulation [10] the behavior of gravity under renormalization is more transparent and simpler than in the usual metric based scheme. In particular, the renormalization group flow turns out to be a flow in the space of scalar functions of two complex variables. Thus, using certain field redefinition arguments as well as direct computation it was shown that the class of theories defined by the action:

\[
S[B, A, \Psi] = \frac{1}{8\pi G} \int_M B^i \wedge F^i + \frac{1}{2} \left( \Lambda \delta^{ij} + \Psi^{ij} + \phi \delta^{ij} \right) B^i \wedge B^j, \tag{3}
\]

\( \phi = \phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3)) \)

is closed under the renormalization group flow. It is in this sense that the theory is referred to as “renormalizable”. As explained above, for any \( \phi \) the theory is actually equivalent to a metric theory given by an infinite expansion in powers of the curvature of this metric. Thus, the statement that is closed under the renormalization group flow does not contradict what is known from the usual perturbative quantum gravity. In this action \( B^i \) is a (complex) \( su(2) \) Lie-algebra valued 2-form (indices \( i, j, \ldots = 1, 2, 3 \) are \( su(2) \) Lie-algebra ones), \( F^i = dA^i + (1/2)[A, A]^i \) is the curvature of an \( su(2) \) Lie-algebra valued connection \( A^i \), \( \Psi^{ij} \) is a traceless symmetric “Lagrange multiplier” field, that on shell gets related to the “curvature”, see more on this below, \( \Lambda \) is a multiple of the cosmological constant, the “usual” cosmological constant \( \Lambda \) appearing in the EH action in the combination \( R - 2\Lambda \) is related to the one in \( \Psi \) via \( \Lambda = -3\Lambda \), and \( G \) is the Newton’s constant. Finally, the function \( \phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3)) \) is (as its arguments indicate) a function of two scalars that can be constructed from \( \Psi^{ij} \). This function is zero for Plebanski formulation of GR [10]. However, as is shown in [11], such an additional term is generated by quantum corrections, and, importantly, this is the only term that gets generated. The renormalization group flow is a flow in the space of functions \( \phi \), which makes this function scale dependent: \( \phi = \phi_\mu \), where \( \mu \) is energy scale. The asymptotic safety scenario of Weinberg [12] can be reformulated in this context as a conjecture that there exists a non-trivial limit \( \phi^* = \lim_{\mu \to \infty} \phi_\mu \), and that the theory that describes
gravity at our energy scales is on a renormalization group flow trajectory that leads to $\phi^*$. However, gravity being a diffeomorphism invariant theory, and there being no way to define what “energy scale” means in a diffeomorphism invariant context, it is not unreasonable to suppose that the theory that describes gravity at all scales is the one with $\phi = \phi^*$, i.e. that the theory at all scales coincides with the UV fixed point one. The main aim of this paper is to study the theory (3) assuming that $\phi$ is a given function (equal to the unknown $\phi^*$). The asymptotic safety conjecture and the problem of determining $\phi^*$ is not addressed here.

Quantum corrections or a new scale? Before we embark on a systematic study of the theory defined by (3) it is worth emphasizing one important conceptual difference between (3) and the usual “quantum corrected” Einstein-Hilbert action. Thus, let us remind the reader that Einstein-Hilbert action receives quantum corrections from counterterms necessary to cancel the divergences arising in perturbation theory. This counterterms have the form of various invariants constructed from Riemann curvature tensor integrated over the spacetime. At leading (first) order the quantum corrections are of the form $(\text{curvature})^2$ integrated over the spacetime. As curvature has dimensions $1/L$, $L$ being length, such terms are dimensionless. To give them the dimension $M \cdot L$, $M$ being mass, required from the action one has to multiply these terms by a dimensionfull parameter - the Planck constant $\hbar$. Thus, the terms in the quantum corrected action that are of second order in curvature have a multiple of $\hbar$ in front of them and thus are quantum corrections. This power of $\hbar$ agrees with the fact that these terms arise at one loop order of perturbation theory. Containing a prefactor of $\hbar$, these terms should be ignored when considering the classical gravity theory, as this is obtained via $\hbar \to 0$ limit. This nicely complements the fact that, had we considered such terms in the classical action, the resulting theory would have fourth derivatives of the dynamical fields, and as such would have various unpleasant problems. This provides us with an “explanation” of why Einstein-Hilbert action is the correct action for the classical theory of gravity.

Let us now consider the theory defined by (3). In spite of the seeming similarity to the usual metric based quantum gravity in that all powers of the “curvature field” $\Psi$ arise in counterterms (in (3) the function $\phi$ may be thought of as a power series expansion in both of its arguments), the theory (3) is quite different. Indeed, in the usual metric based quantum gravity higher powers of the curvature field imply higher derivatives in the equations of motion. In the case of (3) the field $\Psi$ becomes related to the curvature only on-shell, and the action is second-order for any choice of the function $\phi$. This suggests that the $\phi$-term of the action (3) should be considered not a quantum correction, but instead an additional term in the classical action. Note, also that the two expansions: one in powers of the curvature in the usual metric based gravity, and the other in powers of $\Psi$ in theory (3) are quite different. Indeed, as is clear from (1) even a seemingly innocuous modification with $\phi = q \text{Tr}(\Psi)^2$ leads to an infinite expansion in powers of the curvature when interpreted in metric terms!

Let us supplement this “second derivatives only” argument with dimensional analysis. The function $\phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3))$ has the dimension of the other terms in the brackets in the second term of (3), i.e. $1/L^2$. This term becomes equally important as the term $\Psi^{ij}$ when the curvature field $\Psi^{ij}$ is of the same order as $\phi$. This can be phrased in a more meaningful way by saying that the effects of the $\phi$-term become significant for curvatures $\Psi^{ij}$ such that the eigenvalues of the matrix $\Psi^{ij}$ are of the same order as the function $\phi$ of these eigenvalues: $\lambda_{1,2} \sim q \phi(\lambda_1, \lambda_2)$. This equation defines a new length scale $l_\phi : \lambda_{1,2} \sim 1/l_\phi^2$.

For “small” curvatures the term proportional to $\text{Tr}(\Psi)^2$ in the expansion of $\phi$ is dominant. The coefficient in front of this term has dimensions $L^2$. It is this coefficient that can be used as a definition of the length scale $l_\phi$:

$$\phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3)) \sim l_\phi^2 \text{Tr}(\Psi)^2 + O(\Psi^3).$$

Now, to determine whether the $\phi$-term in the action (3) is a classically important term or just a quantum correction one must ask whether $l_\phi$ goes to zero when $\hbar \to 0$. In the usual metric based scheme $l_\phi \sim l_p$, the Planck length and does go to zero in the classical limit. The fact that (3) is second-order in derivatives suggests that this does not have to happen in this theory and that $l_\phi$ may remain finite even when $\hbar$ is sent to zero. An attempt to verify whether this is the case would be beyond the scope of this paper. We hope, however, that the reader will find the presented arguments sufficiently motivating to consider (3) at least as an interesting “quantum modified” theory of gravity.

“Non-metric” modified gravity theory. Thus, if one is to take the quantum gravity scenario reviewed above seriously, one is led to consider the theory (3) (with some yet unknown function $\phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3))$) as the classical theory of gravity, i.e. as a gravitational theory not just at scales where quantum gravity effects are

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1There is another, more standard, and actually related explanation of this that uses the Wilsonian renormalization group flow arguments, see e.g. [12]. We will not use such Wilsonian way of reasoning in the present “classical” paper.
important, but at our scales as well. Apart from the already emphasized difference with the quantum corrected Einstein-Hilbert action, the major difference between \( g \) as a classical theory of gravitation and GR is that the theory \( g \) is no longer about metric in any obvious way. Indeed, as in Plebanski formulation of GR, the gravity theory \( g \) is formulated in such a way that the metric never appears. However, in Plebanski gravity one of the equation implies “metricity”, which then in particular implies that \( A \) in \( g \) coincides with the metric-compatible connection for some metric \( g \). In contrast, when \( \phi \neq 0 \) it is no longer true that \( A \) is metric-compatible. It is for this reason that we propose to refer to this theory (or, rather to the class of theories defined by \( g \)) as non-metric gravity.

As it should have become clear from arguments above, Einstein’s GR is a very good approximation to theories \( g \) when curvatures are smaller than \( 1/l^2 \). However, when curvatures are large so are deviations from general relativity. In particular, deviations from Einstein’s GR are expected to be large near spacetime singularities. As we shall see in one of the subsequent papers, the behavior of all the fields of theory \( g \) near e.g. a would be singularity inside a black hole is much less dramatic than that in Einstein’s GR. This gives yet another motivation to take the theories \( g \) seriously.

Thus, the main aim of this and other papers from the series is to study the theory \( g \) (with the function \( \phi \) assumed given) as a classical gravitational theory. Our main goal will be to describe those new effects that arise when switching from description of gravity based on GR to the one based on \( g \). We will also be interested in corrections to GR predictions that arise from \( g \). In this first paper of the series one of our aims is to describe the consequences of the “modified” metricity conditions that follow from \( g \) when \( \phi \neq 0 \). As we shall see, in spite of the fact that we are working with non-metric gravity theory, certain metric, or rather a quadruple of one-forms (that can be used to define a metric), does arise naturally. Once this is understood, we will obtain field equations in their general form. These field equations will be used as the starting point of discussion in subsequent papers.

The nature of “modification”. It is worth explaining the main results of our analysis in simple terms, so that the reader is not lost in the rather technical discussion of the main body of the paper. In order to explain what is the main feature that makes gravity \( g \) different from the usual GR let us recall some basic facts about the self-dual formulation of gravity due to Plebanski \([10]\). In this approach, the main field that replaces the usual metric of GR is a 2-form \( B \) with values in the Lie algebra of (complexified) \( su(2) \).

One of the equations of the theory states:

\[
B^i \wedge B^j = \frac{1}{3} \delta^{ij} B^k \wedge B_k,
\]

or, when written in the spinor form

\[
B^{AB} \wedge B^{CD} = \frac{1}{4} \epsilon^{AC} \epsilon^{BD} B^{EF} \wedge B_{EF}.
\]

It is not hard to show that this equation implies that there exists a quadruple of one-forms \( \theta^{AA'} \) (\( A, A' \) are spinor indices, see the main text for more details) such that \( B^{AB} \) is given by:

\[
B^{AB} = \theta^{AA'} \wedge \theta^{A'}_B.
\]

In turn, the tetrad \( \theta^{AA'} \) allows one to construct a metric \( ds^2 = \theta^{AA'} \otimes \theta_{AA'} \). It is in this sense that the equation \( g \) implies “metricity” of \( B \). One can also show that \( B^{AB} \) given by \( g \) is self-dual as a 2-form, where the self-duality is defined with respect to the above metric. Thus, after the “metricity” equation \( g \) of Plebanski formulation of GR is solved, the field \( B^{AB} \) becomes a self-dual 2-form, valued in \( su(2) \). As such it can be thought of as a map identifying the space of self-dual 2-forms (at a point of spacetime) with the Lie algebra \( su(2) \). The spinor formalism, which is used to convert spacetime indices into spinor ones, then interprets \( B^{AB} \) as the identity map.

In gravity theory \( g \) most of the above is still true. The “modified” metricity equations that follow from the action still imply existence of a quadruple of one-forms \( \theta^{AA'} \) (tetrad), such that \( B^{AB} \) is a self-dual 2-form with respect to the metric defined by the tetrad. This allows one to interpret \( B^{AB} \) as a map from self-dual 2-forms to \( su(2) \). However, this map is no longer the identity, see formula \( g \) for the corresponding expression. The action of the metric-compatible derivative operator \( \nabla \) on 2-forms and the A-compatible derivative operator \( D \) on \( su(2) \)-valued quantities is different, see formulas \( g \) for the relations between two derivative operators. This difference is quantified by departure of \( B^{AB} \) from the identity map, which is in turn related to certain derivatives of the field \( \phi \), see relations \( g \).
The content of field equations of the theory is also similar to that in GR. The equations are given by $\mathcal{S}_0 \mathcal{S}_1$ and, similarly to the usual GR case, state that the anti-self-dual part of the curvature 2-form $F(A)$ is proportional to the “stress-energy” 2-form $T$, while the self-dual part of $F(A)$ is related to the Lagrange multiplier field $\Psi$ in a certain way. What is different is that one cannot anymore simply solve for the components of $\Psi$ in terms of the second derivatives of the tetrad, as the corresponding equations become more involved and contain second derivatives of $\Psi$. One now has to solve the combined system of equations for both the tetrad and the components of $\Psi$ simultaneously.

**A purely metric formulation?** Given the fact that a certain metric (defined by the tetrad) does appear in the theory one may question if its legitimate to refer to the theory as “non-metric”. Indeed, as we have sketched above, one might imagine solving for the components of the field $\Psi$ in terms of the derivatives of the tetrad (one will generate an infinite expansion in derivatives this way, as a derivative operator will have to be inverted), and then substituting the solution back into the action. One would obtain a generally covariant action for the tetrad-defined metric, which would have the form of an infinite expansion in terms of curvature invariants, with the first term being the usual Einstein-Hilbert action. This is indeed possible in principle, even though probably rather hard in practice. It would be of interest to see which exactly subclass of such infinite expansions gets produced by actions of the form $\mathcal{S}_3$ when $\phi$ is varied. This procedure seems to suggest that the formulation $\mathcal{S}_3$ is just a rather compact way of re-writing a certain class of non-local (containing all powers of the curvature) gravity actions. The critic may argue that this may be interesting by itself, but probably does not call for a name “non-metric” gravity, which suggests a much more profound change of conceptual framework.

What we think makes this purely metric interpretation misleading are two things: (i) the metric (or the tetrad) that is obtained in the process of solving the equations for $B$ is defined only up to a conformal factor. The reason for this is that in the general case of non-vanishing $\phi$ it is a matter of choice how to normalize the tetrad one-forms, there is no canonical normalization anymore. We shall return to this point in the main text. (ii) It is both possible and natural to couple matter fields not to metric (which only arises when one of the equations of motion is solved) but directly to the $B$ field. For example, the action describing a coupling of Maxwell field to non-metric $B$ is given by $\mathcal{S}_3$:

$$S_{EM}[B, a, \phi] = \int_M f(a) \wedge \phi^i B^j - \frac{1}{2} \phi^i \phi^j B^i \wedge B^j,$$

where $a$ is the electromagnetic potential, $f(a) = da$, and $\phi^i$ is a new 0-form field that is introduced to make the coupling directly to $B$ possible, see the above cited paper for more details on this action. Note that this action was introduced in $\mathcal{S}_3$ to describe the coupling of Maxwell field to the usual metric gravity, just in the 2-form formalism. However, this action does make sense even in the non-metric case, as we shall demonstrate in another paper from the series. One could, of course, couple the electromagnetic field directly to the metric arising from $B$ in the usual way, but this seems highly unnatural given the possibility of direct coupling $\mathcal{S}_3$. Once this is done, the electromagnetic field becomes aware of the non-metric character of $B$, as we hope to demonstrate in another paper of the series.

Thus, to summarize, in spite of a possibility of (a very complicated) purely metric formulation of the theory $\mathcal{S}_3$, the fact that matter fields can be coupled directly to $B$ justifies the non-metric approach we take.

**Energy conservation.** Another important difference from the usual case is that the energy conservation equation (obtained from an analog of the Bianchi identity by a certain contraction) gets modified in the non-metric case. The precise statement is given in the equation $\mathcal{S}_1$ below. Let us just emphasize here that the energy conservation equation is aware of the non-metric character of the $B$ field, both because of the fact that the covariant derivative $\mathcal{D}$ is no longer the self-dual part of the metric-compatible spin connection, as well as due to an explicit presence of non-metric terms in the equation. The departure of energy conservation condition from its usual metric form should be of experimental significance. We hope to discuss possible experimental predictions in another paper from the series.

**Organization of the paper.** We start in the next section by discussing what the modified “metricity” equations imply for the $B$ field. Then, in section $\mathcal{S}_3$ we develop spinor techniques for dealing with spacetime forms, which will be of immense value for us in the following sections. Section $\mathcal{S}_4$ solves the equations for the gauge field $A$, with the result being expressed in terms of the usual metric-compatible spin-connection as well as the derivatives of the functions controlling non-metricity. We discuss field equations in section $\mathcal{S}_5$. Finally, section $\mathcal{S}_6$ derives and analyzes the “Bianchi” identity. Of particular importance is the result on the contracted “Bianchi” identity which implies modifications in the usual energy-momentum conservation law.
2 Modified “metricity” equations

Our first goal is to study what the presence of the $\phi$-term in (3) implies for the field $B^i$. We remind the reader that in the absence of this term the equation one obtains by varying (3) with respect to $\Psi^{ij}$ guarantees “metricity”, something that we will also demonstrate below. When $\phi$-term is present, this “metricity” equation gets modified, and it is our goal in this section to study what this modification implies.

The “metricity” condition. Thus, let us consider the equation one gets by varying (3) with respect to $\Psi^{ij}$. Special care should be taken in view of the fact that $\Psi^{ij}$ is traceless is equivalent to the condition that $\Psi^{ij}$ is traceless. Thus, let us consider the equation one gets by varying (3) with respect to $\Psi^{ij}$.

\[ B^i \wedge B^j + \left( 2\partial_1 \phi \Psi^{ij} + 3\partial_2 \phi((\Psi^2)_{ij} - \frac{1}{3}\delta^{ij}\text{Tr}((\Psi^2))) \right) B^k \wedge B_k = \frac{1}{3}\delta^{ij}B^k \wedge B_k, \]  

where $\partial_1, \partial_2 \phi$ are the partial derivatives of $\phi$ with respect to the first and second arguments correspondingly.

When $\phi = 0$ we get the “usual” metricity equations of Plebanski formulation of GR. Before we can continue with our quest on what this equation implies we need to review some facts about the field $\Psi^{ij}$. The simplest way to describe this field, and relations satisfied by it is via the so-called spinor techniques, which we therefore have to review.

Spinors. We will not attempt a comprehensive introduction to spinors, describing only what is relevant for us here. There are many excellent sources, see e.g. [14], on the subject, to which we refer the reader for more details.

The action (3), as well as the metricity equation (9) has been written using $su(2)$ Lie-algebra valued fields. One can pass to the spinorial language by considering the fundamental representation of this Lie-algebra in terms of traceless $2 \times 2$ matrices. Thus, each $su(2)$ index $i, j, \ldots$ gets replaced by a pair of spinor indices $AB$ where $A, B, \ldots = 1, 2$. These indices are raised-lowered using the tensor $\epsilon^{AB}$, which is skew, i.e. $\epsilon^{BA} = -\epsilon^{AB}$.

We will use conventions of [14]:

\[ \psi^A = \epsilon^{AB}\psi_B, \quad \psi_B = \psi^A\epsilon_{AB}, \]  

i.e. the index that is contracted is always located on top in the first spinor and at the bottom in the second spinor. Due to the fact that $\epsilon^{AB}$ is skew, the condition that a second rank spinor $X^{AB}$ is traceless is equivalent to the condition that it is symmetric. Thus, each $su(2)$ index $i, j, \ldots$ is equivalent to a symmetric pair of spinor indices $AB$. Note that the fact that $\epsilon^{AB}$ is skew also implies that the product of any spinor with itself is zero: $\lambda^A\lambda_A = 0$.

Spinorial representation of the curvature field $\Psi$. The field $\Psi^{ij}$ is represented in the spinor form by a rank 4 spinor $\Psi^{ABCD}$. This spinor is symmetric in each of the two pairs of indices $AB$ and $CD$ and is antisymmetric under the exchange of these two pairs (as $\Psi^{ij}$ is a symmetric matrix). It is easy to see that the condition that $\Psi^{ij}$ is traceless is equivalent to the condition that $\Psi^{ABCD}$ is completely symmetric.

It is often convenient to choose a basis in the space of spinors. Let the basic spinors be denoted by $\tilde{i}^A, \tilde{\sigma}^A$. The reason why we put a tilde over the basic spinors will become clear in the next section, where we contrast the “internal” spinors that we are dealing with in this section with “spacetime” spinors. The untilded notation is reserved for the “spacetime” spinor basis.

It is customary to normalize the basic spinors as:

\[ \tilde{i}^A\tilde{\sigma}_A = 1. \]  

The curvature field $\Psi$ can then be decomposed as follows.

\[ \Psi^{ABCD} = \Psi_0\tilde{i}^{AB}\tilde{i}^{CD} + \Psi_1\tilde{i}^{AB}\tilde{\sigma}^C\tilde{\sigma}^D + \Psi_2\tilde{i}^{AB}\tilde{\sigma}^C\tilde{\sigma}^D + \Psi_3\tilde{i}^{AB}\tilde{\sigma}^C\tilde{\sigma}^D + \Psi_4\tilde{\sigma}^A\tilde{\sigma}^B\tilde{\sigma}^C\tilde{\sigma}^D, \]  

where the five quantities $\Psi_0, \ldots, \Psi_4$ are the (basis dependent) spinor curvature components.

Basis of rank 2 spinors. It is convenient to introduce the following 3 basic rank 2 spinors:

\[ \tilde{i}^{AB} = \tilde{X}^{AB}, \quad \tilde{\sigma}^A\tilde{\sigma}^B = \tilde{X}^{AB}, \quad \tilde{i}^{AB} = \tilde{X}^{AB}. \]  

The following commutational relations are easy to obtain:

\[ [\tilde{X}_-, \tilde{X}_+] = 2\tilde{X}, \quad [\tilde{X}, \tilde{X}_+] = \tilde{X}_+, \quad [\tilde{X}, \tilde{X}_-] = -\tilde{X}_-, \]  

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where our convention for the commutator is \([X,Y]^B_A = X^C Y^B_A - Y^C X^B_A\). This allows us to identify:
\[
\tilde{X}_- = \left( \begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array} \right), \quad \tilde{X}_+ = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad \tilde{X} = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\] (15)
The rank 2 spinors \(\tilde{X}_\pm, \tilde{X}\) thus span the \(su(2)\) Lie algebra. The Killing-Cartan form evaluated on the basic rank 2 spinors is:
\[
(\tilde{X}_+, \tilde{X}_-) = 1, \quad (\tilde{X}, \tilde{X}) = -\frac{1}{2}
\] (16)
where \((X,Y) := -\text{Tr}(XY) = -X^B Y^A = X^A Y^B\).

**Field \(\Psi\) in terms of the rank 2 spinors.** The following representation of the field \(\Psi^{ABCD}\) in terms of the rank 2 spinors \(\tilde{X}_\pm, \tilde{X}\) turns out to be convenient:
\[
\Psi = \Psi_0 \tilde{X}_- \otimes \tilde{X}_- + \frac{\Psi_1}{2}(\tilde{X}_- \otimes \tilde{X}_+ + \tilde{X}_+ \otimes \tilde{X}_-) + \frac{\Psi_2}{6}(\tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \tilde{X}) + \frac{\Psi_3}{2}(\tilde{X}_+ \otimes \tilde{X}_+ + \tilde{X}_- \otimes \tilde{X}_- + 4\tilde{X} \otimes \tilde{X} + \tilde{X}_- \otimes \tilde{X}_-).
\] (17)
A proof is by an elementary computation.

**Principal spinors and Petrov classification.** Any completely symmetric rank 4 spinor \(\Psi^{ABCD}\) can be represented as:
\[
\Psi^{ABCD} = k_1^{A} k_2^{B} k_3^{C} k_4^{D},
\] (18)
where \(k_1^A, \ldots, k_4^A\) are referred to as principal spinors. The Petrov type of the curvature field \(\Psi^{ABCD}\) is determined according to coincidence relations between the principal spinors. This goes from type I - general, all 4 principal spinors are distinct to type \(N\) - all 4 spinors coincide.

**A convenient gauge.** An elementary computation shows that by an SU(2) rotation of the basis \(\tilde{X}^A, \tilde{\partial}^A\) the curvature field \(\Psi^{ABCD}\) can be brought into the form:
\[
\Psi = \alpha(\tilde{X}_- \otimes \tilde{X}_+ + \tilde{X}_+ \otimes \tilde{X}_-) + \beta(\tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \tilde{X}),
\] (19)
i.e. the spinor curvature components \(\Psi_1, \Psi_3\) can be eliminated and the curvature components \(\Psi_0, \Psi_4\) can be made equal. This is possible always except in the case when \(\Psi\) is of type \(III\), i.e. when 3 of the principal spinors coincide and do not coincide with the fourth principal spinor. A local SU(2) transformation allows to bring the field \(\Psi\) into the form (19) in any region of spacetime where the Petrov type of \(\Psi\) does not change. Let us start by discussing the general case. The other (algebraically special) cases (except type \(III\)) can be obtained from the general case sending one of the parameters to zero, or making them equal. The type \(III\) case will be treated below separately.

**Matrix products.** Using the gauge (19) it is easy to evaluate the quantities \(\text{Tr}(\Psi)^2, \text{Tr}(\Psi)^3\) that we need as arguments of the function \(\phi\) in terms of the quantities \(\alpha, \beta\). We have:
\[
\text{Tr}(\Psi)^2 = 6\beta^2 + 2\alpha^2, \quad \text{Tr}(\Psi)^3 = -6\beta^3 + 6\beta\alpha^2.
\] (20)
Note that these formulas imply that \(\phi\) is a function of \(\alpha^2\) only. Therefore, \(\phi_{\alpha}|_{\alpha=0} = 0\), where \(\phi_{\alpha}\) is a partial derivative with respect to \(\alpha\). This fact will be important when we consider (in another paper) the spherically symmetric static solution.

We will also need the traceless part of the matrix \(\Psi^2\), as it appears in the metricity equation (19). We have:
\[
\Psi^2 - \frac{1}{3} \text{Id} \text{Tr}(\Psi)^2 = 2\alpha\beta(\tilde{X}_- \otimes \tilde{X}_+ + \tilde{X}_+ \otimes \tilde{X}_-) + \frac{1}{3}(\alpha^2 - 3\beta^2)(\tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \tilde{X}).
\] (21)
Here \(\text{Id}\) is the identity tensor, which has the expression
\[
\text{Id} = \tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ - 2\tilde{X} \otimes \tilde{X}.
\] (22)
The tensor that appears in brackets on the left-hand-side of (26) is then given by:

\[(2\alpha \partial_1 \phi + 6\alpha \beta \partial_2 \phi)(\bar{X}_- \otimes \tilde{X}_- + \tilde{X}_+ \otimes \bar{X}_+) + (2\beta \partial_1 \phi + (\alpha^2 - 3\beta^2)\partial_2 \phi)(\tilde{X}_+ \otimes \bar{X}_- + \bar{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \bar{X}).\]  

(T23)

**Change of coordinates.** It is convenient to use \(\alpha, \gamma\) instead of the quantities \(\text{Tr}(\Psi)^2, \text{Tr}(\Psi)^3\) as arguments of the function \(\phi\). The change of coordinates is elementary. Thus, we replace

\[\partial_1 \phi = \partial_\alpha \phi \partial_1 \phi + \partial_\beta \phi \partial_1 \beta, \quad \partial_2 \phi = \partial_\alpha \phi \partial_2 \phi + \partial_\beta \phi \partial_2 \beta,\]  

(T24)

The following identities are then checked by an elementary computation:

\[2\alpha \partial_1 \alpha + 6\alpha \beta \partial_2 \alpha = 1/2, \quad 2\beta \partial_1 \beta + (\alpha^2 - 3\beta^2)\partial_2 \beta = 1/6\]  

(T25)

\[2\alpha \partial_1 \beta + 6\alpha \beta \partial_2 \beta = 0, \quad 2\beta \partial_1 \alpha (\alpha^2 - 3\beta^2)\partial_2 \alpha = 0.\]

Thus, we get:

\[\frac{\partial \phi}{\partial \Psi} = \frac{\phi_\alpha}{2}(\tilde{X}_- \otimes \bar{X}_- + \tilde{X}_+ \otimes \bar{X}_+) + \frac{\phi_\beta}{6}(\tilde{X}_+ \otimes \bar{X}_- + \bar{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \bar{X}).\]  

(T26)

where \(\phi_\alpha, \phi_\beta\) are the partial derivatives of the function \(\phi = \phi(\alpha, \beta)\) viewed as a function of \(\alpha, \beta\).

**Metricity equation.** Having obtained a convenient representation (26) for the derivative of the function \(\phi\) with respect to the curvature field \(\Psi\) we are ready to write the metricity equations (9) in a simple form. Let us decompose the 2-form \(B^i = B^{AB}\) into a basis of rank 2 spinors:

\[B = B_+ \tilde{X}_+ + B_- \bar{X}_- + B \tilde{X},\]  

(T27)

where \(B_\pm, B\) are some 2-forms. The trace of the product of two of these 2-forms is easy to compute:

\[(B \wedge B) = 2B_+ \wedge B_- - \frac{1}{2} B \wedge B.\]  

(T28)

Taking into account (26), after some algebra, one obtains for the non-trivial part of the metricity equations (9):

\[B_+ \wedge B_+ = B_- \wedge B_- = -\phi_\alpha \left(B_+ \wedge B_- - \frac{1}{4} B \wedge B\right),\]  

(T29)

\[B_+ \wedge B = B_- \wedge B = 0,\]

\[2B_+ \wedge B_- + B \wedge B = -\phi_\beta \left(B_+ \wedge B_- - \frac{1}{4} B \wedge B\right).\]

In the metric case (usual GR) the right hand side of all these equations is zero.

**A solution.** The problem of finding a solution to (29) is complicated by the fact that one is free to apply GL(4) transformations and change the basis of one forms that is used to decompose the 2-forms \(B_\pm, B\). At the same time, the availability of this rather large gauge freedom allows one to choose a simple convenient gauge. As is shown in the appendix, one can always choose a quadruple of one-forms \(l, n, m, \bar{m}\) such that all equations (29) are satisfied by:

\[B = \tilde{X}_-(m \wedge l + a n \wedge \bar{m}) + \tilde{X}_+(n \wedge \bar{m} + a m \wedge l) + \tilde{X}c(l \wedge n - m \wedge \bar{m}),\]  

(T30)

with \(a, c\) related to \(\phi_\alpha, \phi_\gamma\) in a certain way. To find these relations we substitute the expression (30) for \(B\) into the equations (29) and get:

\[2a + \phi_\alpha(1 + a^2 + c^2/2) = 0, \quad (1 + a^2 - c^2) + \phi_\gamma (1 + a^2 + c^2/2) = 0.\]  

(T31)

Thus, we get quadratic equations for \(a, c\). We will not need an explicit solution in this paper. Let us also note that when \(\phi = 0\) the solution to (31) is \(a = 0, c = 1\), which, when substituted into (30), gives the usual “metric” 2-form:

\[B = \tilde{X}_+ m \wedge l + \tilde{X}_- n \wedge \bar{m} + \tilde{X}(l \wedge n - m \wedge \bar{m}).\]  

(T32)
We will also need the expression for its square:

\[ \Psi = \alpha (\bar{X} \otimes \bar{X} + \bar{X} \otimes \bar{X}). \]  

(35)

We will also need the expression for its square:

\[ \Psi^2 = -\frac{\alpha^2}{2} \bar{X} \otimes \bar{X}. \]  

(36)

Note that this is traceless, so no need to subtract the trace part.

The metricity equations become:

\[ B_- \wedge B_- = B \wedge B_- = 2B_+ \wedge B_- + B \wedge B = 0, \]  

(37)

\[ B_+ \wedge B \sim \alpha (B_+ \wedge B_- - (1/4)B \wedge B), \quad B_+ \wedge B_- \sim \alpha^2 (B_+ \wedge B_- - (1/4)B \wedge B). \]

A solution of these equations is:

\[ B = \bar{X}_-(m \wedge l) + \bar{X}_+(n \wedge \bar{m} + a(l \wedge n - m \wedge \bar{m}) \wedge \bar{m}) + \bar{X}(l \wedge n - m \wedge \bar{m}), \]  

(38)

where \( a \) is proportional to \( \alpha \). Thus, a “canonical” quadruple of one-forms appears even in the type III case, and the field \( B \) is still purely self-dual. We will not consider the Petrov type III case further in the present paper.
3 Spinor techniques

So far we only dealt with algebraic equations for $B$ involving $\mathfrak{su}(2)$ matrices, and we saw that these are much easier to solve if one employs spinor notations. The spinors we have introduced above are “internal” ones, in that they simply replace the internal $\mathfrak{su}(2)$ indices by a pair of spinor ones. In this section we will introduce different spinors, the ones that will allow us to simply computations with forms and vector fields considerably. In usual gravity, the “internal” and “spacetime” spinors coincide, and this statement is equivalent to the statement that $B$ has the form (32). In our more general case we will need to introduce two different types of spinors, one for dealing with “internal” indices, one for spacetime indices. In this section we will remind the reader how the spacetime indices can be converted into spinor ones, and how this simplifies certain computations. The material reviewed here is standard, see e.g. [14]. The only non-standard point is that the spinors we are going to deal with in this section are different from the ones considered above. For this reason, we have denoted the corresponding “spacetime” spinor basis will be denoted by $i^A, o^A$.

There are several different ways to introduce spinor techniques. We will first present what we think is a more geometric way, and then give the usual one of e.g. [14].

Two-forms as maps. It will be extremely useful to think of the 2-forms from spaces $\mathcal{S}, \mathcal{S'}$ as maps acting on one-forms, and sending them to vector fields. This way of thinking about 2-forms will allow us to develop a spinor approach to forms. What we are about to describe is standard in the usual GR.

Let us consider a given 2-form $B$. Any one-form $k$ can be exterior multiplied by $B$ with the result being a 3-form $k \wedge B$. Now, given a basis $l, n, m, \bar{m}$ in the space of one-forms we can use this basis to produce 4 numbers out of the 3-form $k \wedge B$. Indeed, multiplying $k \wedge B$ (from say the right) by each of the basis one-forms one obtains a 4-form that must be proportional to the “volume” 4-form $l \wedge n \wedge m \wedge \bar{m}$. Let us introduce a special notation for this proportionality coefficient. Thus, we write:

$$k \wedge B \wedge e^l := (k \wedge B \wedge e^l) l \wedge n \wedge m \wedge \bar{m},$$

which defines the quantity $(k \wedge B \wedge e^l)$. Here $e^l$ is any of the basis one-forms. One can use the 4 quantities $(k \wedge B \wedge e^l)$ to form a vector field $e^a := \sum_l (k \wedge B \wedge e^l)e^{l,a}$. Here $e^{l,a}$ denotes the basis of vector fields dual to $e^l = e^l_a$ one forms. By definition: $l^a n_b + n^a l_b - m^a \bar{m}_b - \bar{m}^a m_b = \delta^a_b$, where $\delta^a_b$ is the Kronecker delta - a mixed tensor of rank 2, and $l^a n_a = 1, m^a \bar{m}_a = -1$, while all other scalar products are zero.

The above discussion allows us to interpret any 2-form $B$ as a map from the space of one-forms to the space of vector fields. It is useful to work out this map for the self-dual 2-forms forming a basis in $\mathcal{S}$. An elementary computation gives:

$$m \wedge l : n_a \rightarrow -m^a, \quad \bar{m}_a \rightarrow -l^a, \quad n \wedge \bar{m} : m_a \rightarrow n^a, \quad \bar{m}_a \rightarrow \bar{m}^a, \quad l \wedge n - m \wedge \bar{m} : l_a \rightarrow -l^a, \quad \bar{m}_a \rightarrow \bar{m}^a, \quad m_a \rightarrow -m^a, \quad n_a \rightarrow n^a. \quad (40)$$

A similar map for the anti-self-dual forms is given by:

$$l \wedge \bar{m} : n_a \rightarrow -\bar{m}^a, \quad \bar{m}_a \rightarrow -l^a, \quad n \wedge m : \bar{m}_a \rightarrow -m^a, \quad m_a \rightarrow -\bar{m}^a, \quad l \wedge n + m \wedge \bar{m} : l_a \rightarrow \bar{m}^a, \quad \bar{m}_a \rightarrow \bar{m}^a, \quad m_a \rightarrow -m^a, \quad n_a \rightarrow -n^a. \quad (41)$$

Spinor interpretation. The above described way of viewing two-forms as maps from one-forms to vector fields will allow us to develop a completely form-free approach, in which forms will get replaced by spinorial objects. Let us first deal with the basic one-forms (tetrad-forming) $l, n, m, \bar{m}$. To replace them by spinor quantities we will have to introduce the so-called primed spinors, in addition to the unprimed spinors we have considered so far. Primed spinors form a different representation of the Lorentz group. The vector representation is obtained by tensoring together the primed and unprimed representations. This is the reason why the primed spinors are necessary to describe one-forms.

The primed spinors are denoted by a symbol with a primed spinorial index next to it, e.g. $\lambda^A'$ is a rank one primed spinor. Primed indices are similarly raised and lowered with their corresponding $e^A B^\lambda$ anti-symmetric rank 2-spinor. The raising-lowering conventions are the same (10) as for the unprimed spinors. It is convenient to introduce a basis in the space of primed spinors. We get the primed spinor basis $i^A', o^A'$. 

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Let us now introduce an important set of relations which are to provide the identification between rank 2 mixed primed-unprimed spinors and one-forms. We write:

\[ l_a = o_A o_{A'}, \quad n_a = i_A i_{A'}, \quad m_a = o_A i_{A'}, \quad \tilde{m}_a = i_A o_{A'} \]  \hspace{2cm} (42)

As it is clear, these relations depend both on the spinor basis chosen as well as on the basis in the space of one-forms.

Using the above relations we can give a description of self-dual 2-forms in terms of symmetric rank 2 unprimed spinors. Indeed, let us identify:

\[ m \land l = X_+, \quad n \land \tilde{m} = X_, \quad l \land n - m \land \tilde{m} = -2X \]  \hspace{2cm} (43)

for the self-dual 2-forms and

\[ l \land \tilde{m} = X_+, \quad n \land m = -X_-, \quad l \land n + m \land \tilde{m} = 2X' \]  \hspace{2cm} (44)

for the anti-self-dual ones. Here the unprimed rank 2 spinors are essentially the same to the ones we already mixed primed-unprimed spinors and one-forms. We write:

\[ v \]

where the quantities \( B \) as a rank 4 spinor. Indeed, let us identify:

\[ i^{A'B'} := X^{AB}_{-}, \quad o^{A'B'} := X^{AB}_{+}, \quad \frac{1}{2}(i^{A}o^{B} + o^{A}i^{B}) = X^{AB}, \]

\[ i^{A}i^{B'} := X_{-}^{AB}, \quad o^{A'}o^{B'} := X_{+}^{AB}, \quad \frac{1}{2}(i^{A'}o^{B'} + o^{A'}i^{B'}) = X^{A'B'}. \]  \hspace{2cm} (45)

It is then easy to check that the action of the rank 2 spinors on the mixed spinors \( (\mathbf{12}) \) is exactly as in \( (\mathbf{10}) \). Indeed, let us consider e.g. \((m \land l)(n)\). The corresponding spinor quantity is \( X^{AB}_{+}i^{A}i^{B} = -o^{A}i^{A'} = -m^a \), as in \( (\mathbf{10}) \). Similarly, \((l \land \tilde{m})(n)\) is given by \( X^{AB}_{+}i^{A}'i^{B'} = -i^{A}o^{A'} = -\tilde{m}^a \) as in \( (\mathbf{11}) \).

**Another way to describe two-forms as rank 2-spinors.** The following is a more standard approach to spinors, see e.g. [14]. One starts by postulating the relations \((\mathbf{42})\) between the 4 basic one-forms and spinors of the spinor basis. This relation allows one to replace any spacetime index by a mixed pair of spinor indices. It is then easy to see that e.g. self-dual two forms becomes rank 2 unprimed spinors. Indeed, let us consider a 2 form \( B = v \land w \) obtained by an exterior product of two one-forms \( v, w \). Introducing indices this reads: 

\[ B_{ab} = 2v_{[a}w_{b]} \]  \hspace{2cm} (46)

where the quantities \((v \land w)_{A'B'}\) are defined as:

\[ (v \land w)_{A'B'} = v_{(A'}w_{B')}^{A'B'}, \quad (v \land w)^{A'B'} = -v_{A'}^{(A}w_{B')}. \]  \hspace{2cm} (47)

The choice of minus signs in these formulas is so that they agree with \((\mathbf{13})\), \((\mathbf{14})\).

**B as a rank 4 spinor.** The above description of self-dual two-forms as rank 2 unprimed symmetric spinors allows us to interpret the \( B \) field \((\mathbf{30})\) as a rank 4 unprimed spinor. Indeed, using \((\mathbf{13})\) we write:

\[ B = a(X_+ \otimes X_- + \bar{X}_+ \otimes \bar{X}_-) + \bar{X}_- \otimes X_+ + \bar{X}_+ \otimes X_- - 2c\bar{X} \otimes X. \]  \hspace{2cm} (48)

We see, therefore, that the \( B \) field interpreted as a map is the identity map \((\mathbf{22})\) in the metric case \(a = 0, c = 1\) (in this case the “internal” and “spacetime” spinors coincide, as we shall see below), and deviates from identity in the non-metric case. This gives one way to understand what the non-metric deformation does to \( B \). For completeness, let us also give the expression for \( B \) as a rank 4 spinor in the type \( III \) case:

\[ B = -2a\bar{X}_+ \otimes X + Id. \]  \hspace{2cm} (49)

**Metric compatible derivative operator.** Having a basis of one-forms \( l, n, m, \tilde{m} \) one has a metric (the one with respect to which these basic one-forms are orthonormal), and can consider the derivative operator compatible with this metric. We will denote this operator by \( \nabla \). Spinor representation \( \nabla_{A'A'} \) of this derivative operator will be very convenient for us in what follows. In order to describe the action of this operator on forms,
we just have to understand its action on basic spinors, as we have already translated one- and two-forms that are of interest for us into spinor form. Thus, we introduce what is known as spin-coefficients according to the following formulas:

\[
\gamma^+_{AA} := -i^B \nabla_{AA} i^B, \quad \gamma^-_{AA} := -o^B \nabla_{AA} o_B, \quad \frac{1}{2} \gamma_{AA'} := i^B \nabla_{AA'} o_B = o^B \nabla_{AA'} i_B.
\] (50)

The derivatives of the basic spinors are then given by the following formulas:

\[
\nabla_{AA'} i_B = -\frac{1}{2} \gamma_{AA'} i_B - \frac{1}{2} \gamma_{AA'} o_B, \quad \nabla_{AA'} o_B = \gamma^+_{AA'} i_B + \frac{1}{2} \gamma^-_{AA'} o_B.
\] (51)

Let us note that the anti-symmetric tensor \(\epsilon_{AB} = o_A i_B - i_A o_B\) that serves the role of the metric in the spinor space is preserved by the derivative operator \(\nabla : \nabla \epsilon_{AB} = 0\), which is quite easy to check using (51). This fact is related to the fact that \(\nabla\) preserves the metric constructed from the basis of one-forms.

Using (51) it is easy to work out the action of the derivative operator \(\nabla\) on the basic two-forms. We will only need these results for the self-dual 2-forms which appear in the decomposition of \(B\). Thus, using (43) and (51) we get:

\[
\nabla_{AA'} X^B_{BC} = -\gamma_{AA'} X^-_{BC} - 2\gamma_{AA'}^+ X^B_{BC}, \quad \nabla_{AA'} X^B_{+BC} = \gamma_{AA'} X^B_{+BC} + 2\gamma_{AA'}^+ X^B_{BC},
\] (52)

\[
\nabla_{AA'} X^-_{BC} = \gamma_{AA'} X^-_{BC} - \gamma_{AA'}^+ X^B_{BC}.
\]

We now note that these derivatives of self-dual 2-forms (which are 3-forms) can be converted into vector fields. This is done simply by contracting the unprimed index of \(\nabla_{AA'}\) with an unprimed index of the tensors \(X_{\pm}, X\). Thus, we introduce the vector fields \((\nabla X_{\pm}), (\nabla X)\) corresponding to \(\nabla X_{\pm}, \nabla X\), and get for the corresponding mixed spinors:

\[
(\nabla X_\pm)^{AA'} = -X^-_{\pm} \gamma_B^{A'} - 2X^{AB} \gamma_B^{A'} + 2X^{AB} \gamma^-_{B}^{A'}, \quad (\nabla X_+)^{AA'} = X^B_{+} \gamma_B^{A'} + 2X^B_{+} \gamma^-_{B}^{A'},
\] (53)

\[
(\nabla X)^{AA'} = X^-_{\pm} \gamma_B^{A'} - X^{AB} \gamma_B^{A'},
\]

These formulas can be rewritten much more compactly by suppressing the spinor indices. Thus, implying matrix multiplication, we can re-write:

\[
(\nabla X_\pm) = -X_\pm \gamma - 2X^+ \gamma^+, \quad (\nabla X_+) = X_+ \gamma + 2X \gamma^-, \quad (\nabla X) = X_- \gamma^- - X_+ \gamma^+.
\] (54)

4 Compatibility equations

Having understood what the modified metricity equations imply, and developed spinor techniques for dealing with forms and their derivatives, we are ready to analyze another set of equations that follow from the action (3) - the compatibility equations between the 2-form \(B\) and the connection \(A\). As we shall see in this section, these equations allow one to find the connection \(A\) in terms of the derivatives of the quantities that appear in the expression (30) for \(B\). Spinor techniques developed in the previous section will be of great help here.

The compatibility condition. The condition relating \(B\) and \(A\) is obtained by varying the action (3) with respect to the connection \(A\). It reads \(DB^2 = 0\), where \(D\) is the covariant derivative with respect to the connection \(A\). In spinorial notations this equation reads \(dB + [A, B] = 0\), or, with indices:

\[
dB^{AB} + A^{AC} \wedge B^B_C - B^{AC} \wedge A^B_C = 0.
\] (55)

To rewrite this in a more manageable form let us decompose the connection into the basis of “internal” rank 2 spinors:

\[
A = \tilde{X}_- A_- + \tilde{X}_+ A_+ + \tilde{X} A.
\] (56)

The commutator present in (55) is then easy to compute:

\[
[A, B] = \tilde{X}_- (A_- \wedge B - A \wedge B_-) + \tilde{X}_+ (A \wedge B_+ - A_+ \wedge B) + 2\tilde{X} (A_\wedge B_+ - A_+ \wedge B_-).
\] (57)

Derivatives of the basis spinors. To compute the derivative \(dB\) we need to act on the spinors \(\tilde{v}^A, \tilde{o}^A\). We could have chosen a constant basis such that the basis spinors do not depend on a point in spacetime. However,
to write the field $\Psi$ is its most convenient for computations form we have chosen to adapt the spinor basis at every point to the field $\Psi$ and its proper spinors. Thus, the basis in which the field $\Psi$ has the simple form (19) is generally not constant. Thus, we have to allow for no-vanishing derivatives of $i^A, \delta^A$. Let us proceed similarly to what was done in the previous section and decompose these derivatives into the basic spinors:

$$d\tilde{i}^A := h\tilde{i}^A + g\tilde{\alpha}^A, \quad d\tilde{\alpha}^A := f\tilde{i}^A - h\tilde{\alpha}^A. \quad (58)$$

Here, $h, g, f$ are one-forms, and to write the second relation we have used the normalization condition $\tilde{i}^A\tilde{\alpha}_A = 1$, which implies that the one-form coefficient in front of $\tilde{\alpha}^A$ in the second relation is minus the one-form coefficient in front of $\tilde{i}^A$ in the first.

**Gauge transformations.** It is useful to discuss what gauge transformations that act on the basis $\tilde{i}^A, \tilde{\alpha}^A$ translate to when they act on the one-forms $h, g, f$. Choosing directions of the spinors $i^A, \alpha^A$ and requiring them to be normalized does not fix them completely. Indeed, there is still a freedom of rescaling: $i^a \rightarrow \alpha i^A, \alpha^A \rightarrow (1/\alpha)\alpha^A$, where $\alpha$ is a complex number different from zero. It is easy to see that under this gauge transformation:

$$h \rightarrow h + \alpha^{-1} d\alpha, \quad g \rightarrow \alpha^2 g, \quad f \rightarrow \alpha^{-2} f, \quad (59)$$

and so $h$ transforms as a U(1) connection, while the other one-forms $g, f$ transform as Higgs fields.

**Derivatives of the basis rank 2 spinors.** Using (58) it is easy to compute the derivatives of the basic rank 2 spinors $\tilde{X}^{\pm}, \tilde{X}$. We get:

$$dX_- = 2h\tilde{X} - 2g\tilde{X}, \quad d\tilde{X} = 2f\tilde{X} - 2h\tilde{X} + 2g\tilde{X}. \quad (60)$$

These formulas are quite similar to (52) except that “internal” spinors are considered.

**The compatibility equations.** Combining (57) and (60) we can write down the compatibility equations that follow from (59):

$$2h \wedge B_- + dB_- + f \wedge B + A_- \wedge B - A \wedge B_- = 0,$$

$$g \wedge B + dB_+ - 2h \wedge B_+ + A \wedge B_+ - A_+ \wedge B = 0,$$

$$2g \wedge B_- + dB_+ + 2f \wedge B_+ + (A_- \wedge B_+ - A_+ \wedge B_-) = 0. \quad (61)$$

Each of these equations is a condition that a certain 3-form vanishes. Therefore, each gives rise to 4 equations when projected onto a basis of 3-forms. Thus, the compatibility equations are $3 \times 4 = 12$ equations for 3 one-forms $A_{\pm}, A$. We see that the number of equations matches the number of unknowns. Below we will write down an explicit solution for $A$ in terms of the $B$ field of the form (30).

**Solving the compatibility equations.** To solve the equations (61) we decompose the one-forms $A_{\pm}, A$ into the same basis $l, n, m, \tilde{m}$ that was used to write down the expression (30) for $B$. We then multiply each of the equations (61) by one-forms $l, n, m, \tilde{m}$ in turn, and thus extract components of these equations. This procedure allows us to find $A$ in terms of $B$.

A very convenient way of doing this is to use the spinor method developed in the previous section. In view of (62) decomposing $A_{\pm}, A$ into basic one-forms is equivalent to introducing the rank 2 mixed spinors $A_{\pm AA'}, A_{AA'}$. The field $B$ can in turn be represented by a rank 4 unprimed spinor (13), or, equivalently, each of the 2-forms $B_{\pm}, B$ can be represented by a rank 2 unprimed spinor. We have:

$$B_- = X_+ + aX_-, \quad B_+ = X_- + aX_+, \quad B = -2cX, \quad (62)$$

where we have omitted spinor indices for compactness. Note that it is the “spacetime” rank 2 spinors $X_{\pm}, X$ that are used here. These should not be confused with the “internal” ones. To solve compatibility conditions we will need the inverses of these matrices, where the inverse $B^{-1}$ of $B$ is defined so that $(B^{-1})^{AC}B^B_C = \delta^{AB}$. These are easy to find, we get:

$$B^{-1} = -X_+ - (1/a)X_+, \quad B_+^{-1} = -X_- - (1/a)X_-, \quad B^{-1} = -(2/c)X. \quad (63)$$

In preparation for solving the equations (61), let us introduce special notations for the commutators that appear in (61):

$$C_- = A_- \wedge B - A \wedge B_-, \quad C_+ = A \wedge B_+ - A_+ \wedge B, \quad C = A_- \wedge B_+ - A_+ \wedge B_. \quad (64)$$

13
Using the spinorial notations for all the quantities, and converting the 3-forms that appear here into vector fields, these expressions can be rewritten as:

\[ C^{EF'}_+ = B^{FE}A^{E'}_+ - B^{FE}A^{E'}_-, \quad C^{EF'}_- = B^{FE}A^{E'}_- - B^{FE}A^{E'}_+, \quad C^{FF'}_- = B^{FE}A^{E'}_- - B^{FE}A^{E'}_+, \quad (65) \]

or, in a more compact form, implying matrix multiplication:

\[ C_- = BA_ - B_ A, \quad C_+ = B_ A - BA_+, \quad C = B_ A - B_ A+. \quad (66) \]

Now, assuming \( C \pm, C \) are known, it is easy to solve for \( A \pm, A \). Thus, for example, multiplying the first equation by \( B_ A^{-1} \) and the second by \( B_ A^{-1} \), adding the results and then subtracting the third equation we get:

\[ (B_ A^{-1}B_- - B_ A^{-1}B_+)A = C - B_ A^{-1}C_- - B_ A^{-1}C_. \quad (67) \]

One then checks that

\[ B_ A^{-1}B_- - B_ A^{-1}B_+ = \frac{1-a^2}{c} \text{Id}, \quad (68) \]

and therefore

\[ A = \frac{c}{1-a^2} (C - B_ A^{-1}C_- - B_ A^{-1}C_+). \quad (69) \]

By similar manipulations

\[ A_- = \frac{a}{c(1-a^2)} (-BB_-^{-1}C + B_ A^{-1}C_- + C_+), \quad A_+ = \frac{a}{c(1-a^2)} (-BB_+^{-1}C + C_- + B_ A^{-1}C_+). \quad (70) \]

**Dependence on one-forms** \( h, g, f \). One should now simply take the expressions for \( C \pm, C \) from the compatibility conditions \( (61) \) and substitute these into the above expressions to find \( A \). It can now be checked that the one-forms \( h, g, f \) appear in the connection \( A \) in the following simple way:

\[ A = -\tilde{X}_- f + \tilde{X}_+ g + 2\tilde{X} h + \tilde{A}, \quad (71) \]

where \( \tilde{A} \) is independent of \( h, g, f \). This is as expected, for

\[ -\tilde{X}_- f + \tilde{X}_+ g + 2\tilde{X} h = \begin{pmatrix} h & g \\ f & -h \end{pmatrix} = dG^{-1} \cdot G, \quad (72) \]

where \( G \) is an \( \text{SL}(2) \) transformation that sends the basis \( \tilde{\gamma}^A, \tilde{\delta}^A \) into some constant basis.

We note that the components of \( \tilde{A} \) have a very simple meaning:

\[ \tilde{\gamma}^A D\tilde{\gamma}_A = -\tilde{A}_+, \quad \tilde{\delta}^A D\tilde{\delta}_A = -\tilde{A}_-, \quad \tilde{\delta}^A D\tilde{\gamma}_A = \tilde{\delta}^A D\tilde{\delta}_A = \frac{1}{2} \tilde{A}, \quad (73) \]

where \( D \) is the covariant derivative with respect to the connection \( A \). These relations are easy to check using \( (71) \). Thus, the components of \( \tilde{A} \) are the spin-coefficients, but for the “internal” spinor basis instead of the “spacetime” one. As we shall see very soon, in the metric case the two sets of spin-coefficients coincide. However, in a more general non-metric situation the two sets are different. Correspondingly, the covariant derivatives that act on “internal” and “spacetime” spinor indices are different.

**Spin-coefficients.** To compute \( \tilde{A} \) we have to substitute

\[ \tilde{C}_- = -(\nabla B_-), \quad \tilde{C}_+ = -(\nabla B_+), \quad \tilde{C} = -({1/2})(\nabla B), \quad (74) \]

where \( B_+, B \) are given by \( (62) \) into the formulas \( (70), (69) \). Note that we are free to replace the usual derivatives here by the covariant ones, as their (exterior product) action on forms is the same. Thus, using \( (54) \) we have:

\[ \tilde{C}_- = X_-(a\gamma - \nabla a) - X_+\gamma + 2X(a\gamma^+ - \gamma^-), \quad \tilde{C}_+ = X_+\gamma - X_-(a\gamma + \nabla a) + 2X(\gamma^+ - a\gamma^-), \quad \tilde{C} = cX_+\gamma^- - cX_+\gamma^+ + X\nabla e. \quad (75) \]
This gives:
\[
\hat{A}_- = \frac{1}{2c(1-a^2)} \left( c(X_+ + aX_-)\nabla c - 2(X_- + aX_+)\nabla a + (1 + c^2 - a^2)\gamma^- - 2(1 - c^2 + 3a^2)X\gamma^- \\
- (1 - c^2 - a^2)\alpha\gamma^+ + 2a(3 - c^2 + a^2)X\gamma^+ + 4a(X_- - aX_+)\gamma\right),
\]
\[
\hat{A}_+ = \frac{1}{2c(1-a^2)} \left( c(X_- + aX_+)\nabla c - 2(X_+ + aX_-)\nabla a + (1 + c^2 - a^2)\gamma^+ + 2(1 - c^2 + 3a^2)X\gamma^+ \\
- (1 - c^2 - a^2)\alpha\gamma^- - 2a(3 - c^2 + a^2)X\gamma^- + 4a(X_- - aX_+)\gamma\right),
\]
\[
\hat{A} = \frac{1}{1-a^2} \left( cX\nabla c - 2aX\nabla a + (1 + a^2)\gamma + (1 - c^2 + a^2)(X_+\gamma^+ - X_-\gamma^-) + 2a(X_-\gamma_+ - X_+\gamma^-)\right).
\]

As a check of these results, let us note that in the metric case \(a = 0, c = 1\) and the computed “internal” spin-coefficients coincide with the “spacetime” ones.

For practical computations, e.g. in the case of spherical symmetry, many of the quantities we have been discussing simplify, and the spinor method is an overkill. A different method of computing the spin-coefficients that avoids spinors and uses only exterior product of forms is given in the Appendix.

5 Field equations

In this section, using the results we have obtained above, we write down the field equations for theory (3). As we shall see, these become equations for the derivatives of the one-forms \(l, n, m, \bar{m}\) as well as for the curvature functions \(\alpha, \gamma\) and their derivatives.

The field equations. The equations we are to discuss in this section are obtained by varying the action with respect to \(B^i\). We get:
\[
F^i + (\Lambda \delta^{ij} + \Psi^{ij} + \phi \delta^{ij})B^j = 8\pi GT^i, \tag{77}
\]
where the right hand side is obtained by varying the matter part of the action with respect to \(B^i\). Note that in the discussion above we have tacitly assumed that matter only couples to the \(B\) field and not to \(A, \Psi\). This is true for the gauge fields, Maxwell field in particular, which will serve as a principal example of test matter in the subsequent papers of the series.

Spinor form. As before, it is useful to write these equations using spinors. We will make use of the gauge in which \(\Psi\) has the form (19) and \(B\) has the form (30). Let us first compute the matrix that appears in front of \(B^i\) in the second term of (77):
\[
\begin{align*}
(\Lambda + \Phi)\text{Id} + \Psi &= \alpha(\hat{X}_- \otimes \hat{X}_- + \hat{X}_+ \otimes \hat{X}_+) + \\
(\gamma + \Lambda + \Phi)(\hat{X}_- \otimes \hat{X}_- + \hat{X}_+ \otimes \hat{X}_+) + 2(2\gamma - (\Lambda + \Phi))\hat{X} \otimes \hat{X}.
\end{align*}
\tag{78}
\]

The second term on the left hand-side of (77) can now be easily computed:
\[
((\Lambda + \Phi)\text{Id} + \Psi)B = \hat{X}_-(\alpha B_+ + (\beta + \Lambda + \Phi)B_-) + \hat{X}_+(\alpha B_- + (\beta + \Lambda + \Phi)B_+) = (\Lambda + \Phi - 2\beta)\hat{X}B. \tag{79}
\]

To get the field equations it remains to split each of the 2-forms \(F^{AB}, T^{AB}\) into their self-dual and anti-self-dual parts. In the decomposition of \(F^{AB}\) both self-dual and anti-self dual 2-forms appear. For the models we have considered (Maxwell theory coupled to gravity) \(T^{AB}\) happens to be anti-self-dual only as a 2-form. Thus, for anti-self-dual \(T = T'\) the field equations can be split as:
\[
F' = 8\pi GT', \tag{80}
\]
\[
F + \hat{X}_-(\alpha B_+ + (\beta + \Lambda + \Phi)B_-) + \hat{X}_+(\alpha B_- + (\beta + \Lambda + \Phi)B_+) = (\Lambda + \Phi - 2\beta)\hat{X}B = 0. \tag{81}
\]

We thus see that field equations for theory (3) have interpretation quite similar to that in usual GR. The curvature 2-form \(F^{AB}\) splits into two parts: “Weyl” part \(F\) and “Ricci” part \(F'\). The first “Weyl” part gets related to the “curvature field” \(\Psi\) components \(\alpha, \gamma\) via equation (81), while the second “Weyl” part is proportional to the “stress-energy” tensor \(T'\) as required by the “Einstein” equation (80).
Thus, modifications due to non-metricity are not so dramatic after all. The main difference with the usual GR case is that the connection components (and thus curvature $F$ components) depend not only on the frame $l, n, m, \bar{m}$ and its derivatives but also on the curvature field $\Psi$ components $\alpha, \gamma$ and its derivatives. Also, these curvature components $\alpha, \gamma$ appear in the second term of the “Weyl” field equations non-linearly (via function $\phi$). This gives us the schematic structure described by equation (1).

**Curvature.** To compute the curvature of $A$ we notice that only the $A$ part contributes. We get:

$$F_- = d\hat{A}_- + \hat{A}_- \wedge \hat{A}, \quad F_+ = d\hat{A}_+ + \hat{A} \wedge \hat{A}_+, \quad F = d\hat{A} + 2\hat{A}_- \wedge \hat{A}_+. \quad (82)$$

One should now substitute (70), (69) into these expressions. The result is not very illuminating, and we shall refrain from giving it here.

### 6 Bianchi identity

In this section we derive Bianchi identities for theory (3) and discuss the energy conservation. Spinor method is most effective for this purpose, and will be used heavily in this section.

**“Bianchi” identity.** A very important identity, analogous to the Bianchi identity in usual GR, is obtained by applying the operator of covariant derivative with respect to $A$ to the equation (77). In view of $DF = 0$ identically, we get

$$D(\Lambda \delta^{ij} + \Psi^{ij} + \phi \delta^{ij})B^i = 8\pi G DT^i, \quad (83)$$

In usual GR, contracting this equation in a certain way gets rid of the left-hand-side, and gives the equation of energy conservation. We would like to obtain an analog of this equation for theory (3).

**Computation.** To compute the covariant derivative of the quantity on the left-hand-side of (83) we need to know how $D$ acts on the basic rank 2 internal spinors $\tilde{X}_\pm, \tilde{X}$ as well as on the 2-forms $B_\pm, B$. The action of $D$ on tilded “internal” rank 2 spinors is given by:

$$D\tilde{X}_- = -\tilde{A}\tilde{X}_- - 2\tilde{A}_+\tilde{X}, \quad D\tilde{X}_+ = \tilde{A}\tilde{X}_+ + 2\tilde{A}_-\tilde{X}, \quad D\tilde{X} = \tilde{A}_-\tilde{X}_- - \tilde{A}_+\tilde{X}_+, \quad (84)$$

which is easy to show using the formulas (73) for covariant derivatives of the basic spinors.

The action of $D$ on 2-forms $B_\pm, B$ is that of the derivative operator $\nabla$ compatible with the metric defined by $l, n, m, \bar{m}$. The most economic way to compute this action is using spinor representation. In this representation the self-dual 2-forms $B_\pm, B$ become the matrices (92). The action of the operator $\nabla$ on them can be obtained from relations (66) and (74). We have the following formulas:

$$(\nabla B_-) = B_- \tilde{A} - B \tilde{A}_, \quad (\nabla B_+) = B \tilde{A}_+ - B_+ \tilde{A}, \quad (\nabla B) = 2(B_- \tilde{A}_+ - B_+ \tilde{A}_-), \quad (85)$$

where matrix multiplication on the right-hand-side is implied. Using these formulas, and passing into the spinor representation by replacing 3-forms of the type $A \wedge B$ by the spinor quantity $BA$ (matrix multiplication implied) we get the following 3 components of the “Bianchi” identities:

$$B_+\nabla \alpha + B_-\nabla (\beta + \phi) + \alpha(B \tilde{A}_+ - 2B_+ \tilde{A}) - 3\beta B \tilde{A}_- = 8\pi G(DT)_-, \quad (86)$$

$$B_-\nabla \alpha + B_+\nabla (\beta + \phi) + \alpha(2B_- \tilde{A} - B \tilde{A}_-) + 3\beta B \tilde{A}_+ = 8\pi G(DT)_+, \quad (86)$$

$$B\nabla (\phi - 2\beta) + 2\alpha(B_- \tilde{A}_+ - B_+ \tilde{A}_-) - 6\beta(B_- \tilde{A}_- - B_+ \tilde{A}_+) = 8\pi G(DT),$$

where on the right hand side of these equations one finds the projections of the quantity $(DT)^{AB}$ onto the 3 rank 2 spinors $\tilde{X}_\pm, \tilde{X}$.

**Contracted “Bianchi” identity.** Let us now multiply the first of the equations in (86) by $B_+$, the second of them by $B_-$ and add the results. We get:

$$(B_+B_+ + B_-B_-)\nabla \alpha + (B_+B_- + B_-B_+)\nabla (\beta + \phi) + \alpha(B_+B \tilde{A}_+ - B_-B \tilde{A}_-) - 3\beta(B_+B \tilde{A}_+ - B_-B \tilde{A}_+) = 8\pi G(B_+(DT)_+ + B_-(DT)_+). \quad (87)$$
Let us multiply the third equation in (86) by $B/2$. We get:

$$(1/2)BB\nabla(\phi - 2\beta) + \alpha(BB_+ A_- - BB_- A_+) - 3\beta(BB_+ A_+ - BB_- A_-) = 8\pi G((1/2) B(DT)).$$

(88)

Now, subtracting this equation from (87), and using the fact that $BB_+ + B_+ B = 0, BB_- + B_- B = 0$ we get:

$$(B_+ B_+ + B_- B_-)\nabla\alpha + (B_+ B_- + B_- B_+)\nabla(\beta + \phi) - (1/2)BB\nabla(\phi - 2\beta) = 8\pi G(B_+(DT)_- + B_-(DT)_+ - (1/2) B(DT)).$$

(89)

It is now elementary to check that the quantity on the left-hand-side of this equation is zero. Indeed, using $B_+ B_+ + B_- B_- = -2a, B_+ B_- + B_- B_+ = -(1 + a^2), BB = c^2$ we get for the left-hand-side of (89):

$$-2a\nabla\alpha - (1 + a^2)\nabla(\beta + \phi) - (c^2/2)\nabla(\phi - 2\beta) = -2a\nabla\alpha - (1 + a^2 - c^2)\nabla\beta - (1 + a^2 - c^2/2)\nabla\phi.$$

(90)

However, this obviously vanishes in view of (31). Thus, we have established the following of “energy-momentum” conservation law for theory (3):

$$B_+(DT)_- + B_-(DT)_+ - (1/2) B(DT) = 0.$$  

(91)

In view of the significance of this result let us write it down re-introducing all indices:

$$(X_{AB}^A + aX_{AB}^B)(D_{B'}^B T_{-A'}) + (X_{AB}^A + aX_{AB}^B)(D_{B'}^B T_+)^{A'} + cX_{AB}(D_{B'}^B T_{B'}^B) = 0.$$

(92)

In the metric case $a = 0, c = 1$ one can recognize in this the usual contracted Bianchi identity $D_{B'}^A T^{ABA'B'} = 0$ albeit written in a slightly unusual way.

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**Appendix: Solving the “metricity” equations**

In this appendix we prove that using the GL(4) freedom in choosing a basis $l, n, m, \bar{m}$, any solution of the equations (29) can be brought into the form (30). We first note that decomposing the three 2-forms $B_{\pm}, B$ into some basis of 2-forms (obtained by choosing a basis of one-forms $l, n, m, \bar{m}$) we get $3 \times 6 = 18$ quantities. The dimension of the available group GL(4) of transformations is $4^2 = 16$. Thus, we must seek for a parameterization of $B_{\pm}, B$ by 2 quantities. Expression (30) is such a parameterization by $a, c$. All the available gauge freedom has been used to put $B$ into this form. Of course, one does not expect any $B$ to be transformable into this form, only those fields satisfying the set of equations (29).

Let us start by considering a general 2-form $C$ and decompose it into a basis $m \wedge l, n \wedge \bar{m}, l \wedge \bar{m}, n \wedge m, l \wedge n - m \wedge \bar{m}, l \wedge n + m \wedge \bar{m}$ of two forms, where $l, n, m, \bar{m}$ is some basis of one-forms. We have:

$$C = c_1 l \wedge m + c_2 n \wedge \bar{m} + c_3 (l \wedge n - m \wedge \bar{m}) + c_4 l \wedge \bar{m} + c_5 n \wedge m + c_6 (l \wedge n + m \wedge \bar{m}).$$

(93)

We would like to see how much this 2-form can be simplified by applying the transformations of the one-forms $l, m$ only. First, consider the GL(2) rotations:

$$l \rightarrow \alpha l + \beta m, \quad m \rightarrow \gamma l + \delta m.$$  

(94)

The form $C$ changes to:

$$C \rightarrow c_1 (\alpha \delta - \gamma \beta) m \wedge l + c_2 n \wedge \bar{m} + (1/2)(c_1' \beta + c_2' \gamma - (c_3 - c_4') \delta - (c_3 + c_4') \alpha)(l \wedge n - m \wedge \bar{m})$$

(95)

$$+ (c_1' \alpha - (c_3 - c_4') \gamma) l \wedge \bar{m} + (c_2' \delta - (c_3 + c_4') \beta) n \wedge m$$

$$+ (1/2)(c_1' \beta - c_2' \gamma - (c_3 - c_4') \delta + (c_3 + c_4') \alpha)(l \wedge n + m \wedge \bar{m}).$$
It is now easy to see that for \( c_3 \neq c_3' \) by choosing the parameters of (94) to be given by

\[
\beta = \frac{c_2' \alpha}{c_3 - c_3'}, \quad \gamma = \frac{c_1' \alpha}{c_3 - c_3'}, \quad \delta = \frac{(c_3 + c_3') \alpha}{c_3 - c_3'}
\]

one can eliminate the anti-self-dual part of \( C \) altogether. When \( c_3 = c_3' \) the formulas are bit different, but also easily obtainable from (95). Choosing, in addition, the parameter \( \alpha \) to be such that \( \alpha \delta - \gamma \beta = 1/c_1 \) makes the coefficient in front of the first term \( m \wedge l \) unity. Thus, using the \( \text{GL}(2) \) freedom in choosing \( l, m \) one can bring \( C \) into its “canonical” form:

\[
C = m \wedge l + c_2 n \wedge \bar{m} + c_3 (l \wedge n - m \wedge \bar{m}).
\]

(97)

Note that this completely fixes the \( \text{GL}(2) \) freedom.

Let us now consider the transformations of \( l, m \) which involve the other one-forms \( n, \bar{m} \):

\[
l \rightarrow l + \alpha n + \beta \bar{m}, \quad m \rightarrow m + \gamma n + \delta \bar{m}.
\]

(98)

Simple analysis shows that there is a way to perform this transformation that no anti-self-dual part is introduced. This requires: \( \alpha = \delta, \beta = \gamma = 0 \). Under such a transformation the 2-form \( C \) changes from its form (97) to

\[
C \rightarrow m \wedge l + (\alpha^2 + c_2 - 2ac_3)n \wedge \bar{m} + (c_3 - \alpha)(l \wedge n - m \wedge \bar{m}).
\]

(99)

Thus, choosing \( \alpha = c_3 \) we can eliminate the last term. To summarize, using the \( \text{GL}(2) \) freedom as well as inhomogeneous transformations (98), any 2-form \( C \) can be brought to the form:

\[
C = m \wedge l + cn \wedge \bar{m},
\]

(100)

We emphasize that in order to achieve this form only the transformations of the one-forms \( l, m \) were used. One can show by a similar argument that any 2-form \( C' \) can be brought into the form: \( C' = n \wedge \bar{m} + c'm \wedge l \) by acting on the one-forms \( n, \bar{m} \) only.

After this preliminary considerations we are ready to consider the problem at hand. In the first step of the proof we would like to show that the metricity equations (29) imply that there exists a basis of one-forms \( l, n, m, \bar{m} \) such that the 2-forms \( B_{\pm}, B \) admit a decomposition in which they are purely self-dual. A simple way to do this is to go to a different spinor basis. Recall that we have chosen the basis \( i^A, \bar{o}^A \) so that the components \( \Psi_1, \Psi_3 \) vanish and components \( \Psi_0, \Psi_4 \) are equal. Let us choose a different basis and point \( \bar{i}^A, \bar{o}^A \) along two of the principal spinors of \( \Phi^{ABCD} = (\delta \bar{o}/\delta \Psi)^{ABCD} \). By doing this we achieve that the components \( \Phi_0, \Phi_4 \) vanish. By further rescaling \( \bar{i}^A \rightarrow \alpha \bar{i}^A, \bar{o}^A \rightarrow (1/\alpha) \bar{o}^A \) one can achieve that \( \Phi_1 = \Phi_3 \). Thus, the rank 4 spinor \( \Phi \) that appears in the metricity equations becomes:

\[
\Phi = (\xi/2)(\bar{X}_- \otimes \bar{X} + \bar{X} \otimes \bar{X}_-) + (\xi/2)(\bar{X}_+ \otimes \bar{X} + \bar{X} \otimes \bar{X}_+) + (\chi/6)(\bar{X}_- \otimes \bar{X}_+ + \bar{X}_+ \otimes \bar{X}_- + 4\bar{X} \otimes \bar{X}).
\]

(101)

The metricity equations in this basis become:

\[
B_+ \wedge B_+ = B_- \wedge B_- = 0,
\]

(102)

\[
B_+ \wedge B = B_- \wedge B = -\xi(B_+ \wedge B_- - \frac{1}{4} B \wedge B),
\]

\[
2B_+ \wedge B_- + B \wedge B = -\chi(B_+ \wedge B_- - \frac{1}{4} B \wedge B).
\]

The first two of these equations imply that \( B_+, B_- \) are simple. Therefore, there exist such one-forms \( l, n, m, \bar{m} \) that

\[
B_+ = l \wedge m, \quad B_- = n \wedge \bar{m}.
\]

(103)

As we have already discussed, one can use the \( \text{SL}(2) \) freedom in choosing \( l, m \) to transform the 2-form \( B \) (without changing neither \( B_+ \) nor \( B_- \)) into a purely self-dual 2-form:

\[
B = am \wedge l + bn \wedge \bar{m} + c(l \wedge n - m \wedge \bar{m}).
\]

(104)

After this, the equations (102) become equations determining \( a, b, c \) in terms of \( \xi, \chi \). They imply, in particular, that \( a = b \). The main point for us here is that in the basis \( i^A, \bar{o}^A \) in which the metricity equations become
the solution for $B_\pm$, $B$ is a set of purely self-dual 2-forms. Transformation of the basis back to the basis in which the metricity equations are (29) only mixes the 2-forms $B_\pm$, $B$. But it cannot change the fact that the all these three 2-forms are self-dual. This proves that there exists a basis $l, n, m, \bar{m}$ in which the 2-forms solving (29) are purely self-dual - it is the same basis as the one that solves (102).

In the last step of the proof we are given three self-dual 2-forms $B_\pm$, $B$ satisfying (29). We can now use the available freedom in choosing the basis $l, n, m, \bar{m}$ to bring these 3-forms into the desired form (30). To achieve this we first bring the self-dual 2-forms $B_\pm$ into their “canonical” forms (as discussed above) by acting on the pairs $l, m$ via $l \rightarrow l + \alpha n, m \rightarrow m + \alpha \bar{m}$ and on $n, \bar{m}$ via $n \rightarrow n + \alpha' l, \bar{m} \rightarrow \bar{m} + \alpha' m$ correspondingly, as well as rescaling $l, n, m, \bar{m}$. This way we achieve

$$B_+ = m \wedge l + a n \wedge \bar{m}, \quad B_- = n \wedge \bar{m} + a' m \wedge l.$$  (105)

Importantly, under these transformations the self-dual 2-form $B$ remains self-dual, as is clear from (59). Given the 2-forms (105) together with a general self-dual 2-form $B$ it is easy to see that the metricity equations of the second line of (29) imply that $B$ is of the form

$$B = c(l \wedge n - m \wedge \bar{m}).$$  (106)

The two equations of the first line of (29) imply, in particular, that $a = a'$. This finishes our proof of the validity of the representation (30) for $B$. The above discussion also illustrates the fact that it is only the conformal class of the metric $l \otimes n - m \otimes \bar{m}$ that is determined by $B$ satisfying the metricity equations, not the metric itself. Indeed, in passing from the basis in which the metricity equations are (102) to the one in which they have the form (29) one has to change the basis of one-forms $l, n, m, \bar{m}$. This induces a conformal rescaling of the metric $l \otimes n - m \otimes \bar{m}$ by the factor $1 - \alpha a'$.

We note that the basis corresponding to (29) and not to (102) was used throughout the paper as it is much easier to relate the quantities appearing in the rank 4 spinor $\Phi$ to the derivatives of the field $\phi$ precisely in this basis. The reason for this is that the rank 4 spinor of the form (19) preserves its form when taken to any power. On the contrary, the rank 4 spinor of the form (101) does not preserve its form. This is why it is much better to deal with (19) for questions that require computation of functions of $\Psi$.

Appendix: Connection $A$ without spinors

Sometimes it is more convenient to avoid spinors, and do computations directly with forms. Here we give the expressions for the spin-coefficients that are obtained this way. Thus, using the expression (30) for $B$ and decomposing $A$ into the basis of one-forms, we can easily compute the projections of $C_\pm, C$ onto the basis one-forms. We have:

$$ \begin{align*}
(C_- \wedge l) &= cA_{-n} - aA_m, & (C_- \wedge n) &= -cA_{-l} + A_{\bar{m}}, \\
(C_- \wedge m) &= -cA_{-\bar{m}} + aA_l, & (C_- \wedge \bar{m}) &= cA_{-m} - A_n.
\end{align*}$$  (107)

$$ \begin{align*}
(C_+ \wedge l) &= A_m - cA_{+n}, & (C_+ \wedge n) &= -aA_{\bar{m}} + cA_{+l}, \\
(C_+ \wedge m) &= -A_l + cA_{+\bar{m}}, & (C_+ \wedge \bar{m}) &= aA_n - cA_{+m}.
\end{align*}$$  (108)

$$ \begin{align*}
(C \wedge l) &= A_{-m} - aA_{+m}, & (C \wedge n) &= A_{+\bar{m}} - A_{-\bar{m}}, \\
(C \wedge m) &= -A_{-l} + aA_{+l}, & (C \wedge \bar{m}) &= -A_n + aA_{-n}.
\end{align*}$$  (109)

Assuming $C_\pm, C$ are given, it is easy to write down the solution for $A$. We get:

$$\begin{align*}
A_{-l} &= -\frac{1}{1 - a^2} \left( (C \wedge m) - \frac{a}{c}((C_+ + aC_-) \wedge n) \right), & A_{-n} &= -\frac{1}{1 - a^2} \left( a(C \wedge \bar{m}) - \frac{1}{c}((C_- + aC_+) \wedge l) \right), \\
A_{-m} &= \frac{1}{1 - a^2} \left( (C \wedge l) - \frac{a}{c}((C_+ + aC_-) \wedge \bar{m}) \right), & A_{-\bar{m}} &= \frac{1}{1 - a^2} \left( a(C \wedge n) - \frac{1}{c}((C_- + aC_+) \wedge m) \right), \\
A_{+l} &= -\frac{1}{1 - a^2} \left( a(C \wedge m) - \frac{1}{c}((C_+ + aC_-) \wedge n) \right), & A_{+n} &= -\frac{1}{1 - a^2} \left( (C \wedge \bar{m}) - \frac{a}{c}((C_- + aC_+) \wedge l) \right), \\
A_{+m} &= \frac{1}{1 - a^2} \left( a(C \wedge l) - \frac{1}{c}((C_+ + aC_-) \wedge \bar{m}) \right), & A_{+\bar{m}} &= \frac{1}{1 - a^2} \left( (C \wedge n) - \frac{a}{c}((C_- + aC_+) \wedge m) \right).
\end{align*}$$  (110-111)
\[ A_t = \frac{1}{1 - a^2} (c(C \wedge n) - ((C_+ + aC_-) \wedge m)), \quad A_n = \frac{1}{1 - a^2} (c(C \wedge l) - ((C_- + aC_+) \wedge \bar{m})), \]  
\[ A_m = -\frac{1}{1 - a^2} (c(C \wedge \bar{m}) - ((C_+ + aC_-) \wedge l)), \quad A_{\bar{m}} = -\frac{1}{1 - a^2} (c(C \wedge m) - ((C_- + aC_+) \wedge n)). \] 

These relations should be compared to (70), (80).

To obtain the spin coefficients \( \tilde{A} \) we can use the following identities:

\[-(\tilde{C}_+ + a\tilde{C}_-) = (1 + a^2)d(n \wedge \bar{m}) + 2ad(m \wedge l) + da \wedge B_-,
-(\tilde{C}_- + a\tilde{C}_+) = (1 + a^2)d(m \wedge l) + 2ad(n \wedge \bar{m}) + da \wedge B_+.\] 

This gives:

\[-(\tilde{C}_+ + a\tilde{C}_-) \wedge l) = (1 + a^2)((d\bar{m})_{nm} - (dn)_{nm}) - 2a(dl)_{nm} + (da)_m,\]
\[-(\tilde{C}_+ + a\tilde{C}_-) \wedge n) = -(1 + a^2)((dn)_{ml} + 2a((dm)_{mn} + (dl)_{lm}) - (da)\bar{n},\]
\[-(\tilde{C}_+ + a\tilde{C}_-) \wedge m) = (1 + a^2)((-d\bar{m})_{lm} - (dn)_{ln}) - 2a(dm)_{nl} - a(da)_l,\]
\[-(\tilde{C}_+ + a\tilde{C}_-) \wedge \bar{m}) = -(1 + a^2)((d\bar{m})_{ml} + 2a((dm)_{nm} - (dl)_{ln}) + (da)_n.\]

The notation that we used here is almost self-explanatory. Thus, e.g. \( dm = (dm)_{nm} n \wedge m + \ldots \), where the dots stand for the other terms. Finally, we get for the projections of the 3-form \( \tilde{C} \):

\[-2(\tilde{C} \wedge l) = -c(dl)_{mn} + c(dm)_{nm} + c(d\bar{m})_{nm} + (dc)_n,\]
\[-2(\tilde{C} \wedge n) = -c(dn)_{mn} + c(dm)_{ml} - c(d\bar{m})_{\bar{m}n} - (dc)_l,\]
\[-2(\tilde{C} \wedge m) = c(dl)_{ln} + c(dm)_{mn} + c(dm)_{ln} - (dc)_{\bar{m}},\]
\[-2(\tilde{C} \wedge \bar{m}) = c(dm)_{ml} - c(dm)_{nm} + c(d\bar{m})_{ln} + (dc)_m.\]

References


