de Sitter space and the equivalence between \( f(R) \) and scalar-tensor gravity

Valerio Faraoni

Physics Department, Bishop’s University
Sherbrooke, Québec, Canada J1M 0C8

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It is shown that, when \( f'' \neq 0 \), metric \( f(R) \) gravity is completely equivalent to an \( \omega = 0 \) scalar-tensor theory with respect to perturbations of de Sitter space, contrary to previous expectations. Moreover, the stability conditions of de Sitter space with respect to homogeneous and inhomogeneous perturbations coincide in most scalar-tensor theories, as is the case in metric \( f(R) \) gravity.

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The acceleration of the cosmic expansion discovered using high redshift supernovae of type Ia [2] has been the subject of many attempts to understand its causes, and various models have been proposed for this phenomenon. As an alternative to dark energy models, modified gravity theories described by the action \( [31] \)

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(m)}
\]

have been studied, where \( f(R) \) is a non-linear function of the Ricci curvature \( R \) that incorporates corrections to the Einstein-Hilbert action which is instead described by a linear function \( f \). These modified gravity theories have been studied in the “metric formalism”, in which the action \( [1] \) is varied with respect to the (inverse) metric \( g^{ab} \), in the “Palatini formalism”, in which a Palatini variation with respect to both \( g^{ab} \) and the (non-metric) connection \( \Gamma^a_{bc} \) is performed [3]; and in the “metric-affine” context, in which also the matter part of the action \( S^{(m)} \) is allowed to depend on (and is varied with respect to) the connection \( \Gamma^a_{bc} \) [4]. Recently, it has been shown that most models proposed thus far in the metric formalism violate the weak-field constraints coming from Solar System experiments [2, 6, 7, 8]. However, there are still viable models [9]. Earlier, and independent, interest in \( f(R) \) gravity was motivated by scenarios of the early universe, such as Starobinsky inflation with \( f(R) = R + aR^2 - \Lambda \) and no scalar field [10].

A common tool in the study of \( f(R) \) gravity is its equivalence with a special scalar-tensor theory [2, 11]. We restrict ourselves to the metric formalism in what follows. Then, one introduces the scalar field \( \phi = R \) and the action \( [1] \) can be rewritten as

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\psi(\phi) R - V(\phi)] + S^{(m)}
\]

when \( f''(R) \neq 0 \), where a prime denotes differentiation with respect to \( \phi \),

\[
\psi(\phi) \equiv f'(\phi)
\]

and

\[
V(\phi) = \phi f'(\phi) - f(\phi).
\]

It is obvious that \( [1] \) implies \( [2] \) when \( \phi = R \). Vice-versa, by varying \( [2] \) with respect to \( \phi \) one obtains

\[
f''(R)(\phi - R) = 0 ,
\]

which yields \( \phi = R \) if \( f''(R) \neq 0 \). The action \( [2] \) describes a scalar-tensor theory with Brans-Dicke parameter \( \omega = 0 \); similarly, Palatini \( f(R) \) gravity can be reduced to a Brans-Dicke theory with parameter \( \omega = -3/2 \) [11].

In a previous paper [12], we raised doubts about the complete physical equivalence of the actions \( [1] \) and \( [2] \) (when \( f'' \neq 0 \), in the context of perturbations of de Sitter space. de Sitter solutions are important because they are found very often to be late time attractors in the dynamics of the accelerating universe, or inflationary attractors in the early universe, and because the weak-field limit of \( f(R) \) gravity is obtained by expanding the relevant field equations around de Sitter space with a spherical symmetric perturbation [2, 5, 8, 13]. In [14, 15] we pointed out that the stability condition for de Sitter space with respect to inhomogeneous perturbations in \( f(R) \) gravity, which we derived using a gauge-invariant formalism [15, 17], coincides with the corresponding stability condition with respect to homogeneous perturbations, and is

\[
\frac{(f'_0)^2 - 2f_0f''_0}{f'_0 f''_0} \geq 0 .
\]

Here, and in the following, a zero subscript denotes quantities evaluated in the de Sitter background. As a consequence of this result, one can restrict oneself to the much simpler homogeneous perturbations of de Sitter space, which depend only on time and do not suffer from the notorious gauge-dependence problems. By contrast, it appears that in the scalar-tensor gravity theory \( [2] \) the stability conditions with respect to homogeneous and inhomogeneous perturbations do not coincide [12, 14, 15, 18]. The theory \( [2] \) with \( \omega = 0 \) is a very peculiar scalar-tensor theory, for which linear stability with respect to inhomogeneous perturbations corresponds again to

\[
\frac{(f'_0)^2 - 2f_0f''_0}{f'_0 f''_0} \geq 0.
\]
as in \( f(R) \) gravity, while stability with respect to homogeneous perturbations is equivalent to

\[
\frac{f''_0}{f'_0} - \frac{2f_0f''_0}{f'_0} \geq 0 \tag{8}
\]

(beware of a typographical error in eq. (33) of \([12]\)). Here the subscript 0 denotes a quantity evaluated in the background de Sitter space \((H_0, \omega_0)\). Although the difference between \([7]\) and \([8]\) consists only of the factor \(f''_0\) in the denominator and appears to be minimal, it is by no means trivial because one can not \textit{a priori} decide that \(f''_0\) should be positive. Based on this difference, doubts were raised in \([12]\) on whether the two actions \([1]\) and \([2]\) are always physically equivalent. Complete physical equivalence was questioned also in \([30]\). On the other hand, in the weak-field limit and in other studies of \(f(R)\) gravity \([5, 6, 7, 8]\), it is found that this theory produces the same results as the \(\omega = 0\) scalar-tensor theory \([2]\). Here we show how these two viewpoints can be reconciled in the light of recent results. The sign of \(f''(R)\) \textit{must} be positive: in Ref. \([19]\) we studied the Dolgov-Kawasaki instability \([20]\) originally discovered in the special model \(f(R) = R - \mu^4/R\) (the prototype of corrections to the Einstein-Hilbert action aimed at explaining the present cosmic acceleration \([21, 22]\). Since the field equations of the metric \(f(R)\) formalism are the fourth order equations in the metric components

\[
f'(R)R_{ab} - \frac{f(R)}{2} g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \Box f'(R) + \kappa T_{ab} ,
\]

by taking the trace one obtains the dynamical equation for the Ricci scalar

\[
3f''(R) \Box R + 3f'''(R) \nabla^c R \nabla_c R + f'(R)R - 2f(R) = \kappa T .
\tag{9}
\]

This shows that \(R\) (or \(\phi\) in the scalar-tensor description \([2]\)) is a truly dynamical field, as opposed to the case of general relativity in which it satisfies the well-known algebraic equation \(R = -\kappa T\) obtained from the trace of the Einstein equations. Therefore, in metric \(f(R)\) gravity there is room for an instability in \(R\) (which can also be seen as an instability in the matter sector \([20]\)). This instability was studied in \([19]\) for a general form of \(f(R)\) and it was found that it is avoided if and only if \(f''(R) \geq 0\). This condition must be satisfied, in particular, for de Sitter spaces with constant curvature \(R_0\) and this is all is needed to show the equivalence of the inequalities \([7]\) and \([8]\). Therefore, homogeneous and inhomogeneous perturbations of de Sitter space become equivalent both in metric \(f(R)\) gravity and in its \(\omega = 0\) scalar-tensor formulation. The doubts raised in \([12]\) about their equivalence are therefore dissipated.

Note that setting \(f''_0 > 0\) is not justified \textit{a priori}: the Ricci instability that plagues metric \(f(R)\) gravity when \(f''(R) < 0\) is by all means non-trivial. It arises because, contrary to the Einstein equations, eq. \([9]\) is of fourth order and its trace gives a dynamical equation for \(R\). In the Palatini formalism, in which the field equations are only of second order, \(R\) satisfies an algebraic equation as in general relativity, it is not a dynamical field, and there is no such instability \([23]\). The stability condition \(f''(R) \geq 0\) can be given a simple physical interpretation. Assume that the effective gravitational coupling \(G_{eff}(R) \equiv G/f'(R)\) is positive; then, if \(G_{eff}\) increases with the curvature, i.e.,

\[
\frac{dG_{eff}}{dR} = -\frac{f''(R)G}{(f'(R))^2} > 0 ,
\tag{11}
\]

at large curvature the effect of gravity becomes stronger, and since \(R\) itself generates larger and larger curvature via eq. \([10]\), the effect of which becomes stronger and stronger because of an increased \(G_{eff}(R)\), a positive feedback mechanism acts to destabilize the theory. If a small curvature grows and grows without limit the system runs away. If instead the effective gravitational coupling decreases when \(R\) increases, which is achieved when \(f''(R) > 0\), a negative feedback mechanism operates which compensates for the increase in \(R\) and there is no running away of the solutions. It is curious that general relativity, with \(f''(R) = 0\) and \(G_{eff} = \text{constant}\), is the borderline case between stable behaviour \((f'' > 0)\) and instability \((f'' < 0)\).

At this point, we want to correct a mistake in the literature about homogeneous perturbations of de Sitter space in scalar-tensor gravity. It is stated in Ref. \([24]\) that there are no stable de Sitter spaces in scalar-tensor gravity: this is clearly wrong as such examples abound in the literature (see, e.g., \([25]\) and the references in \([26, 27]\), as the condition \([25]\) for stability shows. The error in the conclusion of \([24]\) seems to originate from incorrect signs in eqs. (2.7)-(2.9) of that paper, which rule the evolution of homogeneous perturbations. In fact, de Sitter spaces can be stable with respect to more general inhomogeneous perturbations in general gravity theories described by an action of the form

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ f(\phi) + \omega(\phi)g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi) \right] \tag{12}
\]

which contains both \(f(R)\) and scalar-tensor gravity as special cases. The gauge-invariant linear stability condition with respect to inhomogeneous perturbations is \([18]\)

\[
\frac{\partial^2 f}{\partial x^2} \left|_0 \right. - \frac{\partial^2 \psi}{\partial x^2} \left|_0 \right. + \frac{R_0 f''_{R_0}}{f'_0} \leq 0 ,
\tag{13}
\]

where \(F \equiv \partial f/\partial R\) and \(f_{\phi R} \equiv \partial^2 f/\partial \phi \partial R\). Analogous stability conditions with respect to various kinds of classical and semiclassical instabilities were established in Refs. \([25, 28]\).
It is straightforward to show that this inequality coincides with the stability condition (13) by using \( f(\phi, R) = \phi R \), \( R_0 = 12H_0^2 \), \( F = \phi \), \( f_R = 1 \), and replacing \( \omega_0 \) with \( \omega_0/\phi_0 \) in the denominator of (13) to account for the different form of the coefficient of the kinetic term of \( \phi \) in the actions (12) and (14). There are, of course, de Sitter spaces which are stable with respect to homogeneous perturbations.

Now, a general scalar-tensor theory of the form

\[
S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[ \psi(\phi) R - \omega(\phi) g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right]
\]

(24)

can be recast in the form (14) by introducing the new scalar field \( \varphi = \psi(\phi) \), obtaining

\[
S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[ \varphi R - \bar{\omega}(\varphi) \left( \frac{d\varphi}{d\phi} \right)^2 + \Delta \varphi - U(\varphi) \right]
\]

(25)

where

\[
\bar{\omega}(\varphi) = \frac{\varphi \omega(\psi^{-1}(\varphi)) \left( \frac{d\psi}{d\phi} \right)^2}{\psi^{-1}(\varphi)}, \quad \Delta \varphi = V(\psi^{-1}(\varphi))
\]

(26)

(27)

The action (25) coincides with the action (14) that we have just studied. (14) is equivalent to (24) provided that the function \( \varphi = \psi(\phi) \) is invertible and the derivatives of \( \psi \) and its inverse \( \psi^{-1} \) are well-defined. Under this assumption (which is not always satisfied for the choices of \( \psi(\phi) \) found in the literature — see, e.g., [29]), we have shown above that the stability conditions of de Sitter space with respect to homogeneous and inhomogeneous perturbations coincide. Therefore, one can restrict oneself to considering the much simpler homogeneous perturbations.

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