Small Shear Viscosity of a Quark-Gluon Plasma Implies Strong Jet Quenching

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We derive an expression relating the transport parameter \( \hat{q} \) for a parton jet in the QCD radiative energy loss and the shear viscosity \( \eta \) of a quark-gluon plasma. We find that the value for \( \hat{q} \) derived from data on jet quenching is consistent with a shear viscosity-to-entropy density ratio of the matter formed in nuclear collisions that is close to the conjectured lower bound \( \eta/s \geq 1/4\pi \). We also argue that the ratio \( T^3/\hat{q} \), where \( T \) is the temperature, is a more broadly valid measure of the coupling strength of the medium than \( \eta/s \).

The highly excited, strongly interacting matter formed in collisions of large nuclei at the Relativistic Heavy Ion Collider (RHIC) exhibits two unusual properties which have received considerable attention recently: (1) The emission of hadrons with large transverse momentum is strongly suppressed in central collisions [1], and (2) the collective flow of the matter is well described by relativistic hydrodynamics with a negligible shear viscosity [2].

In collisions of large nuclei at the Relativistic Heavy Ion Collider (RHIC) we can then estimate the range of the ratio of the shear viscosity to the entropy density of the matter

\[
\eta/s \geq 1/4\pi
\]

to be caused by gluon radiation induced by multiple collisions of the leading parton with color charges in the quasi-thermal medium [3, 4, 5]. The intensity of medium induced gluon emission is governed by a transport parameter \( \hat{q} \), which is the squared average transverse momentum transfer from the medium to the fast parton per unit path length [6]. In this letter, an expression relating the parameter \( \hat{q} \) and the shear viscosity \( \eta \) of the partonic medium will be derived. Employing the values of \( \hat{q} \) extracted from the phenomenological study of jet quenching at RHIC we can then estimate the range of the ratio of the shear viscosity to the entropy density of the matter produced in central heavy-ion collisions.

In the framework of perturbation theory, the transport parameter

\[
\hat{q}_R = \rho \int dq_{\perp}^2 \frac{d\sigma_R}{dq_{\perp}^2},
\]

governing the radiative energy loss of a propagating parton in a SU(3)-color representation \( R \) can be expressed in terms of the light-cone gluon correlation function

\[
\hat{q}_R = \frac{4\pi C_R\alpha_s}{N_c^2 - 1} \int dq_{\perp} (F_{ai}^{+}(0) F_{ai}^{+}(y^-)) e^{i\xi p^+ y^-},
\]

In Eq. (2), \( \langle \mathcal{O} \rangle = (2\pi)^{-3} \int d^3p/2p^+ f(p)\langle \mathcal{O} \rangle |p\rangle \) denotes the ensemble average of an operator \( \mathcal{O} \) in the medium composed of states \( |p\rangle \) with occupation probability \( f(p) \), \( \xi = (k_T^2)/(2E(p^+)) \), \( (k_T^2) \) is the average transverse momentum carried by the gluons in \( |p\rangle \), and the index \( i \) sums over the two transverse directions. \( \rho = \int d^3p f(p)/(2\pi)^3 \) denotes the density of scattering centers, mainly gluons, in the matter. \( d\sigma_R/dq_{\perp}^2 \) is the differential cross section for a parton on a scattering center. A nonperturbative definition of \( \hat{q}_R \) in terms of a Wilson loop along the light-cone has been proposed in ref. [6].

In principle, \( \hat{q}_R(E) \) depends on the energy \( E \) and virtuality of the leading parton in the jet. In the limit of a thick (large opacity) medium of thickness \( L \), the virtuality of the parton is determined by the saturation scale \( Q_s \) of the medium [8], and the transport parameter in parton energy loss takes on the universal form \( \hat{q} = Q_s^2/L \), leading to a weak energy dependence. We will neglect such energy dependence here. For a massless parton in a weakly coupled gluon plasma:

\[
\hat{q}_R = \frac{8\zeta(3)}{\pi} N_c C_R \alpha_s^2 T^3 \ln \frac{1}{\alpha_s},
\]

in the leading log approximation [8]. In the remainder of this paper, we will omit the subscript in the notation for the transport parameter \( \hat{q} \equiv \hat{q}_A \) for a gluon jet.

The shear viscosity of a fluid is defined as the coefficient of the contribution to its stress tensor, which is proportional to the divergence-free part of the velocity gradient. In the framework of kinetic theory, the shear viscosity \( \eta \) is determined by the mean-free path \( \lambda_f(p) \) of a constituent particle of momentum \( p \) in the medium:

\[
\eta \sim C \rho(p)\lambda_f,
\]

with \( C \approx 1/3 \) [8, 9]. A heuristic connection between \( \eta \) and \( \hat{q} \) can be established by the observation that the mean-free path is related to the average transport cross section of a particle in the medium:

\[
\lambda_f = (\rho \sigma_t)^{-1}.
\]

When soft scattering dominates, as in the case of perturbative QCD, the transport cross section is related to the differential cross section by the relation:

\[
\sigma_t \approx \frac{4}{s} \int dq_{\perp}^2 \frac{d\sigma}{dq_{\perp}^2} \equiv \frac{4\hat{q}}{s \rho},
\]

where \( \sqrt{s} \) is the center-of-mass energy. For a thermal ensemble of massless particles, \( \langle p \rangle \approx 3T \) and \( \langle s \rangle \approx 18T^2 \),
and thus:

$$\eta \approx 13.5 C \frac{T^3}{\bar{q}}.$$  \hspace{1cm} (6)

Using the relation $s \approx 3.6 \rho$ for the entropy density of a gas of free, massless bosons, the following heuristic equation relating the ratio of the shear viscosity to the entropy density with the transport coefficient may be derived:

$$\frac{\eta}{s} \approx 3.75 C \frac{T^3}{\bar{q}}.$$  \hspace{1cm} (7)

This relation shows that a large value of $\bar{q}$ implies a small value for the ratio $\eta/s$, which is thought to be bounded by the quantum limit $\eta/s \geq (4\pi)^{-1}$ [10].

For a more rigorous derivation, we need to consider a specific physical origin of the shear viscosity. We start with the shear viscosity due to parton collisions in the quark-gluon plasma, following Arnold et al. [11]. In the Chapman-Enskog approach to linearized transport theory, one parametrizes the deviation $f_1(p)$ of the parton distribution from equilibrium due to the shear by a function $\bar{\Delta}(p)$ in the form (using the notation of ref. [12])

$$f_1(p) = -\frac{\bar{\Delta}(p)}{E \tau^2} p_i p_j (\nabla u)_{ij},$$  \hspace{1cm} (8)

where $(\nabla u)_{ij}$ denotes the traceless velocity gradient. One can then derive an integral equation for $\bar{\Delta}(p)$ from the linearized Boltzmann equation. An analytic estimate can be obtained by restricting $\bar{\Delta}$ to the functional form $\bar{\Delta}(p) = Ap/T$. The shear viscosity is expressed in terms of $\bar{\Delta}(p)$ as

$$\eta_c = -\frac{1}{15 T} \int \frac{d^3 p}{(2\pi)^3} p^4 \bar{\Delta}(p) \frac{\partial f_0}{\partial E},$$  \hspace{1cm} (9)

where $f_0(p)$ denotes the equilibrium distribution of partons. Assuming that collisions in the medium are dominated by soft scattering, the integral over the differential scattering cross section can be extracted from the collision integral, yielding a factor $\bar{q}/\rho$, where $\bar{q}$ is defined by Eq. (11) for a gluon jet. After a lengthy calculation one finds that in this limit the shear viscosity for a thermal gluon plasma can be expressed in the form:

$$\eta_c \approx C \frac{T^3}{\bar{q}}.$$  \hspace{1cm} (10)

In order to extract the constant, we can make use of the known result for the shear viscosity of a pure gluon gas in the leading logarithmic approximation [11]:

$$\eta_{LL} = \frac{3.81 N_c^2 - 1}{\pi^2 N_c^2} \frac{T^3}{\alpha_s^2 \ln(1/\alpha_s)}.$$  \hspace{1cm} (11)

Inserting the jet quenching parameter in the weak coupling limit [3] and using the expression for the density of a thermal gas of free gluons, we obtain $C' \approx 4.85$. Comparing with Eq. (10), we find that this corresponds to $C \approx 0.36$.

We next consider the case of anomalous shear viscosity, which is generated by dynamically created turbulent color fields in a rapidly expanding quark-gluon plasma [13]. The nearly boost invariant longitudinal expansion imprinted on the matter produced in relativistic heavy ion collisions necessarily induces an oblate momentum distribution of partons with respect to the beam axis after a short time $\tau > Q^{-1}_c \approx 0.2 \text{ fm}/c$. The anisotropic momentum distribution leads to the formation of unstable collective plasma modes [14]. The plasma instabilities result in the exponential growth of long wave length modes of the glue fields, which ultimately saturate due to their nonlinear self-interaction [13, 10]. The expanding quark-gluon plasma will then settle into a turbulent state characterized by a non-vanishing expectation value of the gluon correlator $\langle F_{a+}(0) F_{a+}^\dagger(y^-) \rangle$ dominated by soft color fields.

The presence of localized domains of color fields in a turbulent quark-gluon plasma induces an anomalous contribution to the shear viscosity of the matter [12, 13]. This contribution is derived by considering the effect of random color fields on the propagation of quasi-thermal partons, which is described by a Fokker-Planck equation of the form:

$$\left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \nabla p \frac{D_R}{D_0} (f, \mathbf{v}) \right] f(p, \mathbf{v}, t) = 0$$  \hspace{1cm} (12)

with the average parton phase space distribution $f$ and the diffusion tensor

$$D_{ij} = \int_{-\infty}^t dt' \langle F_i(t') F_j(t') \rangle .$$  \hspace{1cm} (13)

Here $\mathbf{F} = gQ^a_n (\mathbf{E}^a + \mathbf{v} \times \mathbf{B}^a)$ is the color Lorentz force generated by the turbulent color fields and $Q^R$ is color charge of the parton in the given representation. The shear viscosity implied by Eq. (12) can be evaluated, e.g., for randomly distributed color fields satisfying

$$\langle B_{i}^a B_{j}^b \rangle = \langle E_{i}^a E_{j}^b \rangle = \delta_{ij} \delta^{ab} (E^2 + B^2);$$

$$\langle E_{i}^a B_{j}^b \rangle = 0;$$

$$\langle F_{i}^a F_{j}^{a+} \rangle = \frac{1}{6} (E^2 + B^2).$$  \hspace{1cm} (14, 15)

with a correlation time $\tau_m$ along the light-cone. The anomalous shear viscosity for gluons in this field configuration is [12]

$$\eta_A = \frac{16 \zeta(6)(N_c^2 - 1)^2 T^6}{\pi^2 N_c g^2 (E^2 + B^2)^2 \tau_m}.$$  \hspace{1cm} (16)

The diffusion coefficient $D_{ij}$ in Eq. (13), is closely related to the transport parameter for radiative energy loss,
Eq. (2). Applying the definition of \( \hat{q} \) as rate of growth of the transverse momentum fluctuations of a fast gluon to the random turbulent color field ensemble in Eq. (15), one finds (for partons in the adjoint color representation):

\[
\hat{q} = \frac{8\pi\alpha_s N_c}{3(N_c^2 - 1)} \langle E^2 + B^2 \rangle_{\tau m}.
\]

(17)

Combining this expression with Eq. (16) we obtain a relation between the contributions of turbulent color fields to the transport parameter in gluon energy loss and the anomalous shear viscosity of gluons:

\[
\eta_{\lambda} = \frac{32\zeta(6)(N_c^2 - 1) T^6}{3\pi^2} \hat{q} \approx 4.51 \frac{T^3 \rho}{\hat{q}}.
\]

(18)

where again we inserted the expression for the density of a free gluon gas. The numerical coefficient here is very close to the one, \( C' \), obtained for the collisional viscosity. Alternatively, we can calculate the dimensionless ratio of the shear viscosity and the entropy density. For a free thermal gluon gas, the entropy density is given by

\[
s = 2(N_c^2 - 1) \frac{2\pi^2}{45} T^3,
\]

(19)

which implies the following result for the shear viscosity-to-entropy ratio:

\[
\frac{\eta_{\lambda}}{s} = \frac{8\pi^2 T^3}{63} \frac{T^3}{\hat{q}} \approx 1.25 \frac{T^3}{\hat{q}},
\]

(20)

again in agreement with (7) for \( C \approx 1/3 \). We thus conclude that Eq. (7) with \( C \approx 1/3 \) should be valid if the transport properties of a thermal gauge theory are described by kinetic perturbation theory.

In order to explore the opposite limit, we consider a strongly coupled theory, \( N = 4 \) supersymmetric Yang-Mills (SYM) theory. Analytical results for \( \eta/s \) and \( \hat{q} \) have been derived for this theory in the limit of large \( \hat{q} \) Hooft coupling \( \lambda = g^2 N_c \). The strong coupling expression for the ratio of shear viscosity to entropy density is well established [18, 19]:

\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \frac{135}{(8\lambda)^{3/2}} + \cdots \right].
\]

(21)

There is not yet universal agreement about the correct generalization of the jet quenching parameter. Here we follow Liu et al. [6] who defined \( \hat{q} \) by means of an adjacent Wilson loop along the light-cone and found the result

\[
\hat{q} = \frac{\pi^{3/2} \Gamma(3/4)}{\Gamma(5/4)} \sqrt{\lambda} T^3 \approx 7.53 \sqrt{\lambda} T^3.
\]

(22)

Inserting this expression into the right-hand side of (7) for \( C \approx 1/3 \) we obtain

\[
1.25 \frac{T^3}{\hat{q}} \approx 0.166 \frac{\sqrt{\lambda}}{\hat{q}}.
\]

(23)

which violates the bound \( \eta/s > 1/4\pi \) for \( \lambda > 4.35 \), showing that the relation between \( \eta \) and \( \hat{q} \) does not hold in the strong coupling limit of this theory.

That the heuristic relation (4) derived for a weakly coupled gauge theory does not hold in the strong coupling limit of the SYM theory, is not surprising: In a strongly coupled system, thermal modes are not approximately described as elementary quasi-particles [20]. On the other hand, highly energetic excitations can retain a quasi-particular nature, and their interaction with the medium continues to increase with the strength of the coupling, e. g. as \( \sqrt{\lambda} \) in the N=4 SYM theory [21]. The violation of the relation (16) may thus be considered as a criterion for whether a gauge theory is strongly coupled.

We now proceed to explore whether the relation (7) applies to the matter produced at RHIC, on the basis of the phenomenological analysis of jet quenching measurements. Within the framework provided by the twist expansion, the total radiative energy loss for a propagating parton may be expressed as

\[
\frac{\Delta E}{E} = \int dy \int dz \, z K(y, 1 - z),
\]

(24)

where

\[
K(y, z) = \pi \alpha_s \frac{C_R}{C_F} \rho(y) \xi G(\xi) \int \frac{dt^2}{t^4} \frac{1 + z^2}{(1 - z)^2} \sqrt{\frac{2(1 - z)}{E y}} \left[ 1 - \cos \left( \frac{\ell^2 y}{2E z(1 - z)} \right) \right] + \delta(1 - z) \Delta K(y).
\]

(25)

Here \( \Delta K(y) \) is the virtual correction to gluon bremsstrahlung, and \( \rho(y) \) denotes the local gluon density in the medium which is related to the transport parameter via

\[
\rho(y) \xi G(\xi) = \frac{2}{\pi} \int dy^{-} \text{tr} \left[ F_{i+}^+ (0) F_{i+} (y^{-}) \right] e^{i p^+ y^{-}} = \frac{N_c^2 - 1}{4\pi^2} \frac{\hat{q} \mu(E, y)}{\alpha_s C_F}.
\]

(26)

The medium modification of the vacuum fragmentation function \( D(z_h) \) of the scattered parton can be expressed in terms of the function \( K(y, z) \):

\[
\Delta D(z_h) = \int dz \int_{z_h}^{1} \frac{dz}{z} K(y, z) D(z_h/z).
\]

(27)

We note that for a medium of limited thickness, as it is encountered by a scattered parton in the final state of a relativistic heavy ion collision, \( \hat{q} \) may possess a mild (logarithmic) energy dependence [22]. A nontrivial energy dependence is also expected when the propagating parton is a massive quark [24]. We here disregard such effects, because they are not yet clearly established on the basis of the available data.

A recent phenomenological study [25] using next-to-leading order (NLO) jet cross sections together with
medium modified fragmentation functions conducted a simultaneous fit to the experimental data on single hadron and di-hadron suppression in the most central Au+Au collisions at RHIC. Such a fit yields the following for the transport parameter at an initial time $\tau_0 \approx 1 \text{ fm}/c$:

$$q_{R0} \tau_0 \alpha_s \frac{C_F}{C_R} = 0.35 - 0.41 \text{ GeV}^2.$$  

In the phenomenological analysis, the fractions of quark and gluon jets are given by NLO pQCD and parton energy loss or the transport parameter is set to be proportional to $C_R$. With $\alpha_s = 0.3$, one obtains for the gluon quenching parameter:

$$\hat{q}_0 = 2.6 - 3.1 \text{ GeV}^2/\text{fm}.$$  

One can also estimate the average initial entropy density from the measured charged hadron multiplicity of the most central Au+Au collisions as $s_0 = (33 \pm 3) \text{ fm}^{-3}$ at $\tau_0 = 1 \text{ fm}/c$. Assuming a weakly coupled pure gluon plasma, this will give an initial temperature $T_{0} = (337 \pm 10) \text{ MeV}$. Using Eq. (7) with $C = 1/3$ one then finds

$$\frac{\eta_0}{s_0} \approx 1.25 \frac{T_0^3}{q_0} = 0.08 - 0.10.$$  

This result is close to the conjectured lower bound for $\eta/s$, and certainly lies well within the range of values $(\eta/s < 0.3)$ which were found to be compatible with the measured hadron spectra from Au+Au collisions at RHIC in a recent model study [27].

In summary, we have derived a general relation between the shear viscosity $\eta$ and the jet quenching parameter $q$ for a weakly coupled quark-gluon plasma, which relates a small ratio of shear viscosity to entropy density to a large value of the jet quenching parameter. For the value of $\hat{q}$ deduced from jet quenching data obtained at RHIC, the implied ratio $\eta/s$ is close to the conjectured absolute lower bound.

The fact that $\eta/s$ saturates in the limit of strong coupling of the SYM theory, but $q$ continues to increase, suggests that the ratio $T_q^3/q$ may serve as a more broadly applicable measure of the coupling strength of a quark-gluon plasma. An improved, independent determination of both sides of Eq. (6) from experimental data would thus permit a model independent, quantitative assessment of the strongly coupled nature of the quark-gluon plasma produced in heavy ion collisions [28].

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[21] Even the life-time of very energetic quasi-particles (measured in the rest frame of the medium) may be limited in the strongly coupled SYM theory, because the rate of energy loss grows in proportion with their energy [22].