Non-Gaussianities in N-flation

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Abstract

We compute non-Gaussianities in $\mathcal{N}$-flation, a string motivated model of assisted inflation with quadratic, separable potentials and masses given by the Marčenko-Pastur distribution. After estimating parameters characterizing the bi- and trispectrum in the horizon crossing approximation, we focus on the non-linearity parameter $f_{NL}$, a measure of the bispectrum; we compute its magnitude for narrow and broad spreads of masses, including the evolution of modes after horizon crossing. We identify additional contributions due to said evolution and show that they are suppressed as long as the fields are evolving slowly. This renders $\mathcal{N}$-flation indistinguishable from simple single-field models in this regime. Larger non-Gaussianities are expected to arise for fields that start to evolve faster, and we suggest an analytic technique to estimate their contribution. However, such fast roll during inflation is not expected in $\mathcal{N}$-flation, leaving (p)re-heating as the main additional candidate for generating non-Gaussianities.

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I. INTRODUCTION

The generation of a nearly scale invariant spectrum of primordial perturbations, as observed in the cosmic microwave background radiation (CMBR) \([1, 2, 3, 4]\) or large scale structure (LSS) surveys \([5, 6]\), is a decisive indicator of an inflationary epoch in the early universe. During inflation \([7]\), an accelerated expansion of the early universe, perturbations are stretched beyond the Hubble radius and re-enter the late universe, making them observable to us, e.g. in the CMBR.

A potential approach to discriminating between inflationary models consist of measuring deviations from purely Gaussian statistics; Most simple models of inflation predict quasi scale invariant, nearly Gaussian, adiabatic perturbations. However, more intricate models, such as hybrid \([8, 9]\) or multi-field inflationary ones (see e.g. \([10]\) for a review), are expected to deviate from Gaussianity. Henceforth, a consideration of higher order correlation functions, such as the bispectrum or trispectrum, could potentially shed light into the fundamental physics responsible for generating primordial fluctuations \([11, 12]\).
A rough estimate of non-Gaussianities, as measured by the bispectrum, is the non-linearity parameter $f_{NL}$, properly defined later in the text. Currently, primordial non-Gaussian features in the cosmic microwave background radiation have not been detected. For instance, the observational bound on the non-linearity parameter obtained from the WMAP 3 data set alone is $-54 < f_{NL} < 114$ [1, 13, 14], and future experiments will improve upon it [15, 16, 17, 18]. Hence, theoretical predictions for $f_{NL}$ in well motivated inflationary models should be made before the increasingly improved observational data is available.

All models of inflation predict, to a certain extent, some level of primordial non-Gaussianity. In single-field models, non-Gaussianities are small, usually of the order of the slow roll parameters [19, 20, 21, 22, 23, 24, 25], because fluctuations freeze out once their wavelength crosses the Hubble radius. In multi-field models, however, the presence of multiple light degrees of freedom perpendicular to the adiabatic direction leads to the generation of isocurvature perturbations, which in turn allow the comoving curvature perturbation $\zeta$ to evolve after horizon crossing (HC). This leads to an additional source of potentially detectable non-Gaussianity [24].

One such multi-field model is assisted inflation, which was originally proposed to relax fine tuning of potentials, (see e.g. [26, 27, 28, 29]); it relies on $\mathcal{N}$ scalar fields, preferably uncoupled, which assist each other in driving an inflationary phase. Even though each individual field may not be able to generate an extended period of inflation on its own, they can do so cooperatively. This phenomenological model is attractive, since super Planckian initial values of the fields can be avoided [26, 27, 28] if the number of fields is large. A string motivated implementation of assisted inflation is $\mathcal{N}$-flation $^1$ [31, 32, 33, 34, 35, 36, 37]: here the many fields are identified with axions arising from some KKL T compactification of type IIB string theory [37].

Although the existence of $\mathcal{N}$-flation via a concrete construction has not been proven, it provides a test-bed for multi-field inflation. After an expansion around the minima in the axion potentials and a redefinition of fields, an effective quadratic potential for each field results. Furthermore, due to shift symmetries one can argue that any cross coupling between the fields is heavily suppressed [37]. The masses for the $\mathcal{N}$ fields, originally considered identical in [34], can be evaluated more accurately by means of random-matrix theory [37], without further knowledge of the underlying construction; the resulting distribution or spectrum of masses conforms to the Marčenko-Pastur (MP) distribution [38], controlled by only two constants: the average mass and a parameter fixing the width/shape of the spectrum, which can be identified with the ratio of the number of axions

$^1$ See also [26] for another implementation of assisted inflation within M-theory, making use of multiple M5-branes.
to the total dimension of the moduli space in a given construction. The effect of this distribution on the power-spectrum has been computed in \[31, 37\], yielding a slightly redder spectral index.

Non-Gaussianities arising from assisted inflation and $\mathcal{N}$-flation have also been considered in the past \[33, 39\], but studies were usually confined to the horizon crossing approximation or very few fields. In the HC-approximation, such models become indistinguishable from their single-field analogs. This is easily understood since, by discarding the evolution of modes after horizon crossing, the main distinguishing feature of multi-field models is neglected. As a result, the spectrum of scalar and tensor perturbations is exactly the same \[2\] to the one generated by an effective single-field model \[40\].

In this paper we focus on $\mathcal{N}$-flation with an arbitrary, but large number of fields, so that the Marčenko-Pastur distribution can be used. We consider narrow and broad mass spectra, extending the study of \[41\] where the formalism employed in this study was developed; its application, however, was limited to simple toy models. Here, we analytically compute the non-linearity parameter $f_{NL}$ without assuming the horizon crossing approximation, but within the slow roll approximation. A comparison with simple estimates as well as the HC-limit, which is re-derived for completeness, reveals that additional contributions due to the evolution of modes after horizon crossing are present, but their magnitude is limited to a few percent of the HC result. Hereafter, we relax the slow roll approximation and argue that large, but possibly transient contributions to $f_{NL}$ should be expected from faster rolling fields, which are however not expected during $\mathcal{N}$-flation, since heavy fields should evolve slowly up until (p)re-heating commences. Nevertheless, an analytic method to retain the main physical effect of fields starting to roll faster during inflation is suggested: an effective single-field model with steps in its potential.

The concrete outline is as follows: in Section II we review $\mathcal{N}$-flation, focusing on the evolution during slow roll. After that, we introduce non-Gaussianities in Section III and provide, for later reference, the horizon crossing results of the non-linearity parameters characterizing the bi- and trispectrum, Section III A. Based on the formalism developed in \[41\], we estimate $f_{NL}$ for narrow mass spectra in Section III B 1 providing a simple approximation. Then, we take general broad mass spectra, properly described by the MP-distribution, to compute $f_{NL}$ without any additional approximations, Section III B 2. The resulting additional contributions to $f_{NL}$ due to the evolution outside the horizon are discussed and compared to previous analytic approximations, Section

\[2\] There is a subtlety regarding initial conditions in $\mathcal{N}$-flation, since there is no attractor solution for quadratic potentials, making $\mathcal{N}$-flation sensitive to the initial field values – we come back to this issue later on.
Finally, we consider relaxing the slow roll condition in Section III C, before we conclude in Section IV.

II. $\mathcal{N}$-FLATION AND SLOW ROLL

We start by considering the action for $\mathcal{N}$ scalar fields,

$$ S = \frac{m^2}{2} \int d^4x \sqrt{-g} \left( \frac{1}{2} \sum_{A=1}^{\mathcal{N}} \partial^\mu \varphi_A \partial_\mu \varphi_A + W(\varphi_1, \varphi_2, ...) \right) $$

(1)

which we assume to be responsible for driving an inflationary phase (see e.g. [10] for a review on multi-field inflation). The unperturbed volume expansion rate from an initial flat hypersurface at $t^*$ to a final uniform density hypersurface at $t^c$ is given by

$$ N(t_c, t_*) \equiv \int_{t_*}^{t_c} H dt , $$

(2)

where $H$ is the Hubble parameter.

In $\mathcal{N}$-flation [34], the $\mathcal{N} \sim 1000$ scalar fields that drive inflation are identified with axion fields. If one expands the periodic axion potentials around their minima all cross-couplings vanish [37]. Hence, in the vicinity of their minima, the fields have a potential of the form

$$ W(\varphi_1, \varphi_2, ..., \varphi_\mathcal{N}) = \sum_{A=1}^{\mathcal{N}} V_A(\varphi_A) $$

(3)

$$ = \sum_{A=1}^{\mathcal{N}} \frac{1}{2} m^2_{A} \varphi^2_A . $$

(4)

We arranged the fields according to the magnitude of their masses, that is $m_A > m_B$ if $A > B$. It should be noted that $\mathcal{N}$-flation is a specific realization of assisted inflation [26, 27, 28, 29, 4], where the many scalar fields assist each other in driving an inflationary phase, so that no single scalar field needs to traverse a super-Planckian stretch in field space.

Further, the spectrum of masses in (4), which were assumed to be equal in [34], can be evaluated more accurately by means of random matrix theory and was found by Easther and McAllister to conform to the Marčenko-Pastur (MP) law [37]. This results in a probability for a given mass of

$$ p(m^2) = \frac{1}{2\pi \beta m^2 \sigma^2} \sqrt{(b - m^2)(m^2 - a)}, $$

(5)

$3 \ \mathcal{N}$ refers to the number of scalar fields, whereas $N$ is the number of e-folds.

$4 \ \text{See also [30] for another realization of assisted inflation based on M-theory from multiple M5-branes.}$
where \( \beta \) and \( \sigma \) completely describe the distribution: \( \sigma \) is the average mass squared and \( \beta \) controls the width and shape of the spectrum (see Fig. [1]). As a consequence, the smallest and largest mass are given by

\[
m_1^2 = a \equiv \sigma^2(1 - \sqrt{\beta}),
\]

\[
m_N^2 = b \equiv \sigma^2(1 + \sqrt{\beta}).
\]

In \( \mathcal{N} \)-flation, \( \beta \) can be identified with the number of axions contributing to inflation divided by the total dimension of the moduli space (Kähler, complex structure and dilaton) in a given KKLT compactification of type IIB string theory [37]. We carry out the remainder of this analysis by treating \( \beta \) as a free parameter, but keeping in mind that \( \beta \sim 1/2 \) is preferred, due to constraints arising from the renormalization of Newton’s constant [34]. We will not need to specify \( \sigma \), since it will cancel out; however, its magnitude is of course constrained by the COBE normalization just as \( m^2 \) in chaotic inflation [31].

At this point, we introduce a convenient dimensionless mass parameter

\[
x_A \equiv \frac{m_A^2}{m_1^2},
\]

as well as the suitable short-hand notation

\[
z \equiv \sqrt{\beta},
\]

\[
\xi \equiv \frac{m_N^2}{m_1^2} = \frac{(1 + z)^2}{(1 - z)^2}.
\]

Expectation values with respect to the MP-distribution can then be evaluated via

\[
\langle f(x) \rangle \equiv \frac{1}{N} \sum_{A=1}^{\mathcal{N}} f(x_i)
\]

\[
= \frac{(1 - z)^2}{2\pi z^2} \int_1^\xi \frac{\sqrt{(\xi - x)(x - 1)}f(x)}{x} \, dx.
\]

Since \( f(x) = x^{\alpha+1}y^\lambda x \) will appear frequently in our analysis, we introduce a more convenient notation and define functions \( \mathcal{F}_\alpha^\lambda \), namely

\[
\mathcal{F}_\alpha^\lambda(y) \equiv \mathcal{F}_\alpha(y^\lambda x),
\]

where

\[
\mathcal{F}_\alpha(\omega) \equiv \int_{1/\xi}^{1} \sqrt{(1 - s)(s - \xi^{-1})}s^\alpha \omega^s \, ds.
\]
FIG. 1: Probability of a given mass according to the Marčenko-Pastur distribution from (5), depending on \( \beta \) and the dimensionless square mass \( x = m^2/m_0^2 \), rescaled with the expectation value \( \langle x \rangle \) (also dependent on \( \beta \)): (a) 3D-plot for \( 0.1 < \beta < 1 \), (b) slices for \( \beta_1 = 1/4, \beta_2 = 1/2 \) and \( \beta_3 = 3/4 \); the closer \( \beta \) is to one, the broader the mass spectrum becomes.

so that the expectation values become

\[
\langle x^{\alpha+1}y^\lambda \rangle = \frac{(1-z)^2}{2\pi z^2} x^{\alpha+2} F_\alpha(y).
\] (15)

A few useful properties of the \( F_\alpha \)-functions and analytic approximations can be found in appendix A.

Throughout most of the analysis, we restrict ourselves to the slow roll approximation. As explained above, \( \mathcal{N} \) fields contribute to the energy density of the universe through a separable potential. In this regime, the dynamics of \( \mathcal{N} \)-flation are as follows: firstly, note that the field equations and Friedman equations can be written as

\[
3H \dot{\varphi}_A \approx \frac{\partial V_A}{\partial \varphi_A} = -V_A',
\] (16)

\[
3H^2 \approx W.
\] (17)

Here and in the following we set the reduced Planck mass to \( m_p = (8\pi G)^{-1/2} \equiv 1 \). Here and in the following we set the reduced Planck mass to \( m_p = (8\pi G)^{-1/2} \equiv 1 \). This approximation is valid if the slow roll parameters

\[
\varepsilon_A \equiv \frac{1}{2} \frac{V_A'^2}{W^2} , \quad \eta_A \equiv \frac{V_A''}{W} ,
\] (18)

are small \( (\varepsilon_i \ll 1, \eta_i \ll 1) \) and

\[
\varepsilon \equiv \sum_{A=1}^{\mathcal{N}} \varepsilon_A \ll 1
\] (19)
holds. The number of e-folds of inflation becomes

\[ N(t_c, t_\ast) = - \int_{t_\ast}^{t_c} \sum_{i=1}^{N} \frac{V_A}{V_A'} d\varphi_A , \]  

(20)

and the field equations can then be integrated to yield

\[ \frac{\varphi_A^c}{\varphi_A^\ast} = \left( \frac{\varphi_B^c}{\varphi_B^\ast} \right)^{m_A^2/m_B^2} . \]

(21)

Notice that this relationship between fields does not correspond to an attractor solution. As a result, predictions of \( N \)-flation can depend on initial conditions – admittedly, a less attractive feature of the proposal.

There is another subtlety of \( N \)-flation: if the mass spectrum is broad, that is \( \xi \gg 1 \) corresponding to the limit \( \beta \rightarrow 1 \), the heavier fields will drop out of slow roll early, even as inflation continues (this is the case for the preferred value of \( \beta \sim 1/2 \), corresponding to \( \xi \sim 34 \)). The effect of these fields on the non-Gaussianities, which might be crucial, cannot be estimated properly by using the \( \delta N \)-formalism, which we employ in Section III A and III B. We come back to this issue in Section III C.

### III. NON-GAUSSIANITIES

Recent observations of e.g. the cosmic microwave background radiation (CMBR) [1, 2, 3, 4] or the large scale structure of the Universe [5, 6] made it possible to measure two point statistics of scalar perturbations to higher accuracy. Existing measurements are consistent with a Gaussian spectrum, for which all odd correlation functions vanish, while the even ones can be expressed in terms of the two point function. For instance, the non-linearity parameter \( f_{NL} \), a measure of the three point function or bispectrum, is constraint to lie between \(-54 < f_{NL} < 114 \) by the WMAP3 data alone [1]. Future CMBR measurements, e.g via the Planck satellite [15], are expected to tighten this bound up to \( | f_{NL} | \sim 5 \) (See eg. [42] for a review, and [43] to relate a given primordial bispectrum to the one imprinted onto the CMBR). Furthermore, an all sky survey of the the 21-cm background in the frequency range from 14 MHz to 40 MHz with multipoles up to \( 10^5 \) could potentially limit this parameter down to \( | f_{NL} | \sim 0.01 \) [18]; but it should be noted that this proposal is highly optimistic. If such an observation is indeed possible, higher order correlation functions might also be within reach of observation.

It is expected that more complicated models of the early Universe, such as \( N \)-flation, lead to larger contributions than simple single field models. The physical reason for this expectation is the
presence of isocurvature modes, which in turn cause evolution of fluctuations even after crossing
of the Hubble radius (referred to as horizon crossing in the following), see e.g. [44]. Therefore, we
will give a thorough examination of non-Gaussianities in models of $N$-flation, only restricted by
the slow roll approximation.

A. Using Slow Roll and the Horizon Crossing Approximation

To estimate the magnitude of the three and four point functions we use the $\delta N$-formalism, first
proposed by Starobinsky in [45] and extended by Sasaki and Stewart [46] and others [39, 47, 48, 49].
In this approach one relates the perturbation of the volume expansion rate $\delta N$ to the curvature
perturbation $\zeta$, which is possible if the initial hypersurface is flat and the final one is a uniform
density hypersurface [46]. Notice that this quantity is conserved on large scales in simple models,
even beyond linear order [50, 51]. Given this relationship between the curvature perturbation and
the volume expansion rate, one can evaluate the momentum independent pieces of non-linearity
parameters, which are related to higher order correlation functions, in terms of the change in $N$
during the evolution of the Universe, see e.g. [52].

Following this approach, the power-spectrum $P_\zeta$ and the bispectrum $B_\zeta$ can be computed.
The ratio or $B_\zeta$ to $P^2_\zeta$ is proportional to the non-linearity parameter $f_{NL}$, modulo a momentum
dependent prefactor and factors of $2\pi$. This computation was performed in [39, 48, 49] with the
result

$$-\frac{6}{5}f_{NL} = \frac{r}{16}(1 + f) + \sum_{A,B=1}^{N} N_A N_B N_{AB},$$
\begin{equation}
\equiv \frac{r}{16}(1 + f) - \frac{6}{5}f^{(4)}_{NL} \tag{22}
\end{equation}

where we introduced the short hand notation

$$N_A \equiv \frac{\partial N}{\partial \varphi_A^*},$$
\begin{equation}
N_{AB} \equiv \frac{\partial^2 N}{\partial \varphi_A^* \partial \varphi_B^*}, \tag{24}
\end{equation}

$$\vdots$$

The separate Universe formalism developed by Rigopoulos and Shellard in e.g. [51] is equivalent to the $\delta N$-
formalism.
and we refer the interested reader to [39, 48] for details. We shall focus on the second term in (22) since the first one is known and rather small. If we further use the summation convention for capital indices, we arrive at

$$-\frac{6}{5} f_{NL}^{(4)} = \frac{N_A N_B N^{AB}}{(N_D N^D)^2}. \tag{26}$$

To estimate the magnitude of the four point function, we use the momentum independent parameters $\tau_{NL}$ and $g_{NL}$ as introduced in [52] (see also [54]),

$$\tau_{NL} = \frac{N_{AB} N^{AC} N^B N_C}{(N_D N^D)^3}, \tag{27}$$

$$g_{NL} = \frac{25 N_{ABC} N^A N^B N_C}{54 (N_D N^D)^3}. \tag{28}$$

As mentioned above, we use the horizon crossing approximation in this section in order to compute the derivatives of the volume expansion rate; that is, we assume that modes do not evolve once they cross the Hubble radius at $t^\ast$. With this in mind, we can set $\varphi_A^c = 0$. Since we also use the slow roll approximation, we can write (2) as

$$N(t_c, t_\ast) = -\int_c^{t_\ast} \sum_{A=1}^{N} \frac{V_A}{V_A'} d\varphi_A. \tag{29}$$

Confining ourselves to $N$-flation, that is, if we make use of $V_A = m_A^2 \varphi_A^2 / 2$, and employing equal energy initial conditions, $m_A^2 \varphi_A^2 = m_B^2 \varphi_B^2$, we arrive at

$$N_A = \frac{V_A}{\sqrt{2\varepsilon_A W}}, \tag{30}$$

$$N_{AB} = \delta_{AB} \left( 1 - \frac{\eta_A V_A}{2\varepsilon_A W} \right), \tag{31}$$

$$N_{ABC} = -\frac{\delta_{AB}\delta_{AC}\eta_A \sqrt{2\varepsilon_A}}{2\varepsilon_A^2} \left( -\frac{V_A}{W} \eta_A + \varepsilon_A \right), \tag{32}$$

where all potentials and slow roll parameters have to be evaluated at $t^\ast$, and we replaced first derivatives with respect to $\varphi_A$ by $V_A' = \sqrt{2\varepsilon_A W}$.

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6 On geometrical grounds, we know $0 \leq f \leq 5/6 \ [20, 48]$ ($f$ characterizes the shape of the momentum triangle and is largest for an equilateral triangle), while $r$ is the usual tensor:scalar ratio. The observational upper limit on this quantity depends on the priors used in the fitting process, but we can reliably estimate that $r/16 < 0.1 \ [1]$. This bound will be improved by future experiments, for example Clover $[53]$, among others.

7 Note that $f_{NL}$ was already computed in $[53]$ within the horizon crossing approximation, but we repeat it here for completeness.

8 Note that some authors prefer to set $\varphi_A^c = \varphi_A^*$ instead – the different choices lead to the same results if modes are indeed frozen after horizon crossing.
It is now straightforward to evaluate the non-Gaussianity parameters to

$$\frac{6}{5}f_{NL}^{(4)} = \frac{1}{2N},$$  \hspace{1cm} (33)

$$\tau_{NL} = \frac{1}{(2N)^2},$$  \hspace{1cm} (34)

$$g_{NL} = 0.$$  \hspace{1cm} (35)

These parameters are leading order in the slow roll approximation, independent of the mass spectrum of $N$-flation and indeed indistinguishable from a single-field model with quadratic potential [52]. This is expected, since we neglected the main feature distinguishing multi-field models from single-field ones: the evolution of perturbations after horizon crossing due to the presence of isocurvature modes. The vanishing of $g_{NL}$ is due to the use of quadratic potentials so third derivatives of the potentials vanish.

**B. Beyond the Horizon Crossing Approximation**

We saw in the previous section that $N$-flation is indistinguishable from single-field models with respect to non-Gaussianities if the horizon crossing approximation is used. Consequently, we must incorporate the discriminating feature – the evolution of modes after HC. To simplify matters, we make further use of the slow roll approximation, and restrict ourselves to computing $f_{NL}^{(4)}$, which is the most observationally constrained parameter.

The general expression of $f_{NL}^{(4)}$ was computed in [41] to

$$-\frac{6}{5}f_{NL}^{(4)} = 2 \sum_{A=1}^{N} \frac{u_A^2}{\epsilon_A^2} \left( 1 - \eta_A \frac{u_A}{2} \right) + \sum_{B,C=1}^{N} \frac{u_B u_C A_{BC}}{\left( \sum_{D=1}^{N} \frac{u_D^2}{\epsilon_D^2} \right)^2},$$  \hspace{1cm} (36)

where $u_A$ is given by

$$u_A \equiv \frac{\Delta V_A}{W^*} + \frac{W^c \epsilon_A^c}{W^* \epsilon^c},$$  \hspace{1cm} (37)

with $\Delta V_A \equiv V_A^* - V_A^c > 0$, and the symmetric $A$-matrix

$$A_{BC} = -\frac{W^2}{W^*} \sum_{A=1}^{N} \epsilon_A \left( \frac{\epsilon_C}{\epsilon} - \delta_{CA} \right) \left( \frac{\epsilon_B}{\epsilon} - \delta_{BA} \right) \left( 1 - \frac{\eta_A}{\epsilon} \right).$$  \hspace{1cm} (38)

To evaluate this expression we first compute the field values at $t^*$ and $t_c$ (see [41] for details). To do so, we need $2N$ conditions, given by the $N - 1$ dynamical relations between the fields from [21]

$$\frac{\varphi_c}{\varphi_1^c} = \left( \frac{\varphi_A}{\varphi_A^c} \right)^{m_A^2/m_c^2},$$  \hspace{1cm} (39)
\( \mathcal{N} - 1 \) initial conditions, chosen for simplicity to be equal energy initial conditions\(^9\)

\[
\varphi^*_i = \frac{m^2_i}{m^2_N} \varphi^*_1, \quad (40)
\]
a condition that stems from (2) using the requirement that \( t_\ast \) be \( \mathcal{N} \) e-folds before \( t_c \)

\[
4N = \sum_{i=1}^{\mathcal{N}} \left[ (\varphi^*_i)^2 - (\varphi^*_c)^2 \right], \quad (41)
\]
and the last one by demanding that slow roll ends for at least one field at \( t_c \). In the present case, the field with the largest mass leaves slow roll first \[^{41}\] when

\[
\eta_{\mathcal{N}}^c = 1. \quad (42)
\]

Once the masses are specified, solving these conditions is usually possible. We distinguish two cases: the first one involves narrow mass spectra that have \( \beta \ll 1 \). This will result in a simple, analytic expression; the second one deals with more realistic broad mass spectra, e.g. for \( \beta \sim 1/2 \).

To simplify our notation, we suppress the superscript \( c \) from here on.

1. **Narrow Mass Spectra**

By a narrow mass spectra we mean

\[
\delta_A \equiv 1 - \frac{m^2_i}{m^2_{\mathcal{N}}} \ll 1, \quad (43)
\]

\[
\delta \equiv \frac{1}{\mathcal{N}} \sum_{A=1}^{\mathcal{N}} \delta_A \ll 1. \quad (44)
\]

Obviously, this case corresponds to the limit \( \beta \to 0 \), which is not theoretically favored, but is, nevertheless, the easiest one to consider. Using the Marčenko-Pastur distribution, we arrive at the relation

\[
\delta = 1 - \langle x^{-1} \rangle \quad (45)
\]

\[
= 1 - \frac{(1 - z)^2}{1 - z^2}, \quad (46)
\]

between \( \delta \) and \( z = \sqrt{\beta} \), where we made use of appendix A in the last step.

\(^9\) Since there is no attractor solution for quadratic potentials, there is an unavoidable dependence on initial conditions in \( \mathcal{N} \)-flation.
Using \((42)-(46)\) one can evaluate the field values \(\varphi^*_A\) and \(\varphi^c_A\) and the corresponding slow roll parameters, which can be found in \([41]\). There, it was also shown that the second term in \((36)\) becomes
\[
\sum_{B,C}^{N} \frac{u_B u_C}{\epsilon^*_B \epsilon^*_C} A_{BC} \left( \sum_{D=1}^{N} \frac{u^2_D}{\epsilon^*_D} \right)^2 = O(\delta^2/N^2),
\]
that is, it is second order in the slow roll parameters, as well as second order in \(\delta\).

Using the same method, the first term in \((36)\) becomes
\[
\sum_{A=1}^{N} \frac{u^2_A}{\epsilon^*_A} \epsilon^*_A \left( 1- \eta^*_A \frac{u_A}{2\epsilon^*_A} \right) \left( \sum_{D=1}^{N} \frac{u^2_D}{\epsilon^*_D} \right)^2 = \frac{1}{2(2N+1)} \left( 1 - \frac{\delta_N - \delta}{2N+1} \right) + O(\delta^2),
\]
which includes, naturally, the contribution proportional to \(1/N\), already present in the horizon crossing approximation. Hence we arrive at
\[
-\frac{6}{5} f^{(4)}_{NL} = \frac{1}{(2N+1)} \left( 1 - \frac{\delta_N - \delta}{2N+1} \right) + O(\delta^2),
\]
where we only kept the leading order contribution in \(\delta\). If we now use \(\delta\) from \((46)\) and
\[
\delta_N = \frac{(1-z)^2}{(1+z)^2},
\]
we get after expanding in \(z = \sqrt{\beta}\)
\[
-\frac{6}{5} f^{(4)}_{NL} = \frac{1}{(2N+1)} \left( 1 - \frac{2\sqrt{\beta}}{2N+1} \right) + O(\beta).
\]

This is our first major result: if we compare the above with \((33)\), we observe an additional term proportional to \(\sqrt{\beta}\), which vanishes if all the masses coincide. This is expected since, in the equal mass case, there is no evolution of modes after they cross the horizon. However, isocurvature modes get sourced for a non-zero width of the mass distribution. This in turn causes modes to evolve even after their wavelength becomes larger than the Hubble radius, leaving an imprint onto \(f^{(4)}_{NL}\). For simplicity, we restricted ourselves to small \(\beta\), so that expanding in terms of this small parameter is possible; hence, the resulting correction is even more suppressed by \(\sqrt{\beta}\). Now, considering that the preferred mass spectrum in \(\mathcal{N}\)-flation corresponds to \(\beta \sim 1/2\),
which is indeed broad, we must compute the exact\(^{10}\) expression for \(f_{NL}^{(4)}\) in the next section. Since the resulting, cumbersome expression is valid even for small values of \(\beta\), we can compare it with the straightforward analytic expression in (51) (see Fig. 3 and 4). It should be noted that the contribution due to (47) will also be negligible for broad mass spectra, in agreement with the conclusions of [41].

2. Broad Mass Spectra

In \(\mathcal{N}\)-flation the mass spectrum is known to conform to the Marčenko-Pastur distribution (5). Armed with the expectations values introduced in (11), along with (15) we can evaluate all sums in (36). However, before we can perform this replacement, we need to proceed in a similar manner as we saw in the previous section, where we evaluated the narrow mass spectrum. Strictly speaking, we have to calculate the field values at \(t_c\) and \(t^*\) from (39)-(42) in order to compute the potentials and slow roll parameters that appear in (36)-(38).

Before we continue, we should mention another subtlety in our analysis: we take \(t_c\) to be the time when the field with the largest mass leaves slow roll (42); this time does not correspond to the end of inflation if the mass spectrum is stretched out considerably; since there is no cross coupling between the fields, the remaining fields can successfully continue to drive inflation even if a heavy field leaves slow roll. Consequently, several e-folds of inflation should be expected to follow after \(t_c\). Henceforth, the volume expansion rate appearing in our expressions may be smaller than the usual \(N \approx 60\). We will discuss the effects of fields which leave slow roll while inflation continues in Section III C.

Given this caveat, let us now evaluate the field values \(\varphi_A^*\) and \(\varphi_A^-\): plugging (39) and (40) in (41) we arrive at

\[
\varphi_A^{*2} = \frac{4N}{N} \frac{1}{\langle x^{-1} \rangle - \langle x^{-1} y^2 \rangle}
\]

(52)

where we replaced the sums by expectation values as introduced in (11) and defined

\[
y \equiv \frac{\varphi_A^2}{\varphi_A^2}.
\]

(53)

On the other hand, (42) becomes

\[
\frac{\xi}{2N} = \frac{\langle y^2 \rangle}{\langle x^{-1} \rangle - \langle x^{-1} y^2 \rangle},
\]

(54)

\(^{10}\) Exact in the statistical sense using the large \(\mathcal{N}\) limit, since we are going to use the Marčenko-Pastur mass distribution in order to evaluate expectation values.
FIG. 2: Solving \( (55) \) numerically leads to \( \varphi_1^2/\varphi_2^2 \equiv \bar{y}(\beta) \) for (a) \(-9 \leq \log_{10}(\beta) \leq -1\), (b) \(0.1 \leq \beta \leq 0.9\). We took \( N = 60 \) in all plots.

after using \( (39) \) and \( (40) \) as well as \( (52) \). Equation \( (54) \) is now uncoupled and needs to be solved for \( y \). Using the definitions of the \( F \)-functions in \( (13) \), we can write \( (54) \) also as

\[
0 = 1 - 2N \frac{F^{-1}_1(y)}{F^{-1}_0 - F^{-1}_2(y)} .
\]

Unfortunately, one cannot solve \( (55) \) analytically, but it is easily done with standard numerical routines as implemented in e.g. MAPLE. If we denote the solution to \( (55) \) by \( \bar{y}(\beta) \) (see Fig. 2 for a plot of \( \bar{y} \) over \( \beta \) for \( N = 60 \)), we arrive from \( (39)-(42) \) at the desired field values

\[
\varphi_{\ast A} = \frac{1}{x_A aN \left( F^{-1}_0 - F^{-1}_2(\bar{y}) \right)} ,
\]

\[
\varphi_A = \frac{\bar{y} x_A}{x_A aN \left( F^{-1}_0 - F^{-1}_2(\bar{y}) \right)} ,
\]

where we defined

\[
a \equiv \frac{(1 - z)^2}{2\pi z^2} .
\]

In the following, we suppress the argument of the \( F \)-functions, since it is always given by \( \bar{y} \).

It is now straightforward to evaluate the slow roll parameters appearing in \( (18) \) to

\[
\eta_{\ast A} = x_A a \frac{2N}{\xi} \left( F^{-1}_0 - F^{-1}_2 \bar{y} \right) ,
\]

\[
\eta_A = x_A \frac{\eta_{\ast A}}{\xi} ,
\]

\[
\varepsilon_{\ast A} = \frac{\eta_{\ast A}}{N} ,
\]

\[
\varepsilon_A = x_A \bar{y} x_A \frac{2N}{N \xi^2 a \left( F^{-1}_0 - F^{-1}_2 \bar{y} \right)} ,
\]

\[
\phi_{\ast A} = \frac{1}{x_A aN \left( F^{-1}_0 - F^{-1}_2(\bar{y}) \right)} ,
\]

\[
\phi_A = \frac{\bar{y} x_A}{x_A aN \left( F^{-1}_0 - F^{-1}_2(\bar{y}) \right)} .
\]
and $u_A$ from (37) to
\[
 u_A = \frac{1}{\mathcal{N}} (1 - \bar{y}^x A + \bar{c} x A \bar{y}^x) ,
\]
where we defined
\[
 \bar{c} \equiv \frac{\mathcal{F}_0 - \mathcal{F}_1}{2N \xi \mathcal{F}_0} .
\]

After some more algebra, we can evaluate the two components of $f_{NL}^{(4)}$ in (36) to
\[
 f(\beta) \equiv \sum_{A=1}^{N} \frac{u_A^2}{\xi A} \left( 1 - \eta_A A \frac{u_A}{2x_A} \right) \left( \sum_{B=1}^{N} \frac{u_B^2}{\xi B} \right)^2
\]
\[
 = \frac{G}{4Na} \frac{\mathcal{F}_0^2 - \mathcal{F}_1^2 - \mathcal{F}_2^2 + \mathcal{F}_3^2 + \bar{c} \xi \left[ \mathcal{F}_1^2 + 2 \mathcal{F}_2^2 - 3 \mathcal{F}_3^2 \right] + \bar{c}^2 \xi^2 \left[ -\mathcal{F}_0^2 + 3 \mathcal{F}_3^2 \right] - \bar{c}^2 \xi^3 \mathcal{F}_1^2}{(\mathcal{F}_1^2 - 2 \mathcal{F}_2^2 + \mathcal{F}_3^2 - 2 \bar{c} \xi \left[ \mathcal{F}_1^2 - \mathcal{F}_0^2 \right] + \bar{c}^2 \xi^2 \mathcal{F}_0^2)^2}
\]
\[
\text{and}
\]
\[
 F(\beta) \equiv \sum_{B,C=1}^{N} \frac{u_B u_C A_{BC}}{\xi B^2 C^2} \left( \sum_{D=1}^{N} \frac{u_D^2}{\xi D} \right)^2 \left( \sum_{D=1}^{N} \frac{u_E^2}{\xi E} \right)
\]
\[
 = - \frac{W^2}{W^\ast H^2} \left( C - \frac{D}{\varepsilon} - \frac{AB^2}{\varepsilon^3} + \frac{2BE}{\varepsilon^2} - \frac{B^2}{\varepsilon} \right)
\]
with
\[
 G \equiv a \left( \mathcal{F}_0^2 - \mathcal{F}_1^2 \right) ,
\]
\[
 \frac{W^2}{W^\ast} = \frac{\xi^2 G^2}{4N^2} ,
\]
\[
 \varepsilon = \frac{2Na F_0^1}{G} ,
\]
\[
 A \equiv \frac{2Na}{G} F_1^1 ,
\]
\[
 B \equiv \frac{4Na}{G} \left( \left[ \mathcal{F}_1^1 - \mathcal{F}_0^0 \right] + \bar{c} \xi \mathcal{F}_0^0 \right) ,
\]
\[
 C \equiv \frac{8N^3 a}{\xi^2 G^3} \left( \mathcal{F}_0^0 - 2 \mathcal{F}_1^2 + \mathcal{F}_2^2 + 2 \bar{c} \xi \left[ \mathcal{F}_2^2 - \mathcal{F}_3^2 \right] + \bar{c}^2 \xi^2 \mathcal{F}_3^3 \right) ,
\]
\[
 D \equiv \frac{8N^3 a}{\xi^2 G^3} \left( \left[ \mathcal{F}_1^1 - 2 \mathcal{F}_2^2 - \mathcal{F}_3^3 \right] + 2 \bar{c} \xi \left[ \mathcal{F}_0^2 - \mathcal{F}_3^3 \right] + \bar{c}^2 \xi^2 \mathcal{F}_3^3 \right) ,
\]
\[
 E \equiv \frac{4N^2 a}{\xi G^2} \left( \mathcal{F}_0^0 - \mathcal{F}_1^2 + \bar{c} \xi \mathcal{F}_1^2 \right) ,
\]
\[
 H \equiv \frac{2Na}{G} \left( \mathcal{F}_0^0 - 2 \mathcal{F}_1^2 + \mathcal{F}_2^2 + 2 \bar{c} \xi \left[ \mathcal{F}_1^1 - \mathcal{F}_2^2 \right] + \bar{c}^2 \xi^2 \mathcal{F}_3^3 \right) ,
\]
FIG. 3: $-f_{NL}^{(4)}(2N+1)6/5$ over $\log_{10}(\beta)$ computed using: a. the horizon crossing approximation $-\frac{6}{5}f_{NL}^{(4)}(2N+1)6/5 = 1$, b. the $\delta$-expansion from (51), c. the "exact" expression from (78) and d. the approximation from (80). We took $N = 60$ in all plots. Note that b. and d. are both good approximations up until $\beta \sim 0.1$.

so that

$$\frac{-6}{5}f_{NL}^{(4)}(\beta) = 2 (f(\beta) + F(\beta)).$$

(78)

This is our second major result and will be discussed in the next section.

3. Discussion

A plot of $f_{NL}^{(4)}$ over $\beta$ can be found in Figures 3 and 4, where it is also compared with the analytic approximation in (51), the horizon crossing approximation $-f_{NL}^{(4)}6/5 = 1/(2N + 1)$ and the approximation in (80). Throughout the analysis, we took $N = 60$ as a rough estimate for the number of e-folds. It is evident that the approximation in (51) is good up to $\beta \leq \bar{\beta} \sim 1/10$. In this region, the leading order contribution to the exact expression in (78) stems from the prefactor in (67), which includes a dependence on $\beta$ via $G$ defined in (69), and the first summands, so that one may also use

$$\frac{-6}{5}f_{NL}^{(4)}(\beta) \approx \frac{G}{2Na} \frac{1}{F_{-2}^0 - F_{-2}^{-1}}$$

(79)

$$= \frac{1}{2N} \frac{F_{-2}^0 - F_{-2}^{-1}}{F_{-2}^0}$$

(80)

as an approximation for small $\beta$. Here $F_{-2}^0 = (1 - z)^2/(1 - z^2)$ from (A3) and $F_{-2}^{-1}(\bar{y})$ is defined in (13) where $\bar{y}(\beta)$ is the solution to (55). Naturally, we recover the horizon crossing result (33) in the limit that $\beta \to 0$. 

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FIG. 4: $-\frac{f_{NL}^{(4)}}{5}6/5$ over $\beta$ computed using: a. the horizon crossing approximation $-f_{NL}^{(4)}(2N+1)6/5 = 1$, b. the $\delta$-expansion from (51), c. the exact expression from (78) and d. the approximation from (80). We took $N = 60$ in all plots. Note that both approximations fail to recover the turn of $f_{NL}^{(4)}$ observable in Figure (b).

Both, the $\delta$-expansion and the above approximation in (80), fail for $\beta \sim 1/2$, see Fig. 4. However, the contribution due to $F$ defined in (67) is negligible even for very broad spectra (e.g. up to $\beta = 9/10$ in Fig. 4b), in agreement with the conclusions of [41]: there, two-field models with a large ratio of the two masses were solved analytically and an additional slow roll suppression was found for $F$. Hence we may use

$$f_{NL}^{(4)} \approx -\frac{5 \mathcal{G}}{6 \sqrt{4Na}} \frac{\mathcal{F}^0_2 - \mathcal{F}^1_2 - \mathcal{F}^2_2 + \mathcal{F}^3_2 + \tilde{c}_\xi \left[ \mathcal{F}^1_{-1} + 2 \mathcal{F}^2_{-1} - 3 \mathcal{F}^3_{-1} \right] + \tilde{c}_\xi^2 \left[ -\mathcal{F}^2_0 + 3 \mathcal{F}^3_0 \right] - \tilde{c}_\xi^3 \mathcal{F}^3_1}{(\mathcal{F}^0_{-2} - 2 \mathcal{F}^1_{-2} + \mathcal{F}^2_{-2} - 2 \tilde{c}_\xi \left[ \mathcal{F}^2_{-1} - \mathcal{F}^1_{-1} \right] + \tilde{c}_\xi^2 \mathcal{F}^3_0)^2}$$

(81)

as an approximation in the region that is preferred in $N$-flation.

Around the preferred value of $\beta = 1/2$ the magnitude of $-f_{NL}^{(4)}$ is smaller than the horizon crossing result, but only by a few percent (see Fig. 4). Such a deviation will never be observable. The minimum is reached for $\beta \approx 0.74$ and $-f_{NL}^{(4)}$ increases for larger values of $\beta$ so that it catches up with the horizon crossing result around $\beta \approx 0.88$. For even larger values of $\beta$ the magnitude of the non-linearity parameter increases more and more, seemingly becoming significantly large. However, one should take this result with caution, specifically the limit $\beta \rightarrow 1$. Here, we took the final time $t_c$ to be the time at which the heaviest field leaves slow roll. In addition, we assumed

\[11\] Remember that the limit $\beta \rightarrow 1$ corresponds to an infinitely broad mass spectrum, so $\beta < 1$ always.
that sixty e-folds of inflation occurred between $t_s$ and $t_c$. However, if the spectrum of masses is indeed very broad, there will be a considerable amount of inflation even after the heaviest fields leave/left slow roll; hence, the potentially large value for $-f_{NL}^{(4)}$ at $t_c$ might very well be a transient phenomenon, due to a few heavy fields. Note that according to the Marčenko-Pastur distribution the majority of fields will have relatively light masses in the broad spectrum case, see Figure 1 and that the majority of the masses are smaller than the average one for $\beta$ close to one. As a result, one might actually neglect the few heavy fields altogether, meaning, one might want to truncate the mass spectrum, since heavy fields will rapidly settle in their minimum.

Of course, once a field leaves slow roll, our formalism is not applicable any more up until the field settled in its minimum. We propose a few methods to estimate the production of non-Gaussianities analytically during these intervals in the next section and conjecture that these instances should be the dominant sources of non-Gaussianities in multi-field inflationary models (See also [44, 55] for recent numerical work).

Before we continue, we would like to remind the reader of the limitations of our approach: first, we focused on potentials without any cross coupling between the fields. One can argue in favor of vanishing couplings in the case of $N$-flation if the fields stay close to the minima of their potential [37], but in general such an assumption is rather artificial 12. If such couplings are present, one should expect an enhanced production of non-Gaussianities. We also considered only quadratic potentials with mass spectra that conform to the Marčenko-Pastur distribution. We focused on this class, since the MP-distribution is expected to properly describe the spread of masses in $N$-flation in the large $N$ limit. We do not expect qualitative differences for other spectra. Nevertheless, if the potentials are not quadratic but quartic or exponential, we expect an additional suppression: since an attractor solution is present for potentials of these types (see e.g. [29]), isocurvature perturbations will be suppressed and in turn any evolution of modes after horizon crossing will also be suppressed, resulting in an additional reduction of non-Gaussianities. Lastly, we considered equal energy initial conditions, mainly to simplify the computations. Since there is no attractor solution for quadratic potentials, there is a dependence on the chosen initial state. This unavoidable sensitivity to the initial configuration of fields is a flaw of $N$-flation, since the model becomes less predictive. However, the evident slow roll suppression of non-Gaussianities is insensitive to the chosen initial state, see e.g. the two-field cases studied in [41].

12 We thank F. Quevedo for useful comments regarding this point. Naturally, this means that the conditions for successful assisted inflation are hard to satisfy.
C. Beyond the Slow Roll Approximation

We saw in the previous sections that multi-field inflationary models like $\mathcal{N}$-flation do not generate large non-Gaussianities during slow roll. Henceforth, one can hardly discern them from simple single-field models. We expect our conclusions to be quite general, applicable to any kind of multi-field model during slow roll, given that the potential is separable and the kinetic terms of the scalar fields are canonical. However, if fields are evolving faster, one might still produce measurable contributions to the non-linearity parameter $f_{NL}$ or higher order parameters such as $g_{NL}$ or $\tau_{NL}$.

Consider for instance the case where a few fields start to evolve faster while inflation still continues. Whenever a field behaves that way, the trajectory in field space will make a sharp turn and consequently, isocurvature perturbations will cause the adiabatic mode to evolve so that non-Gaussianities are generated. This additional source of non-Gaussianity could be recaptured by an effective one-field model: first, replace the multiple inflaton fields by one effective degree of freedom $\sigma$, which evolves due to an effective potential $V(\sigma)$ \[1\]. Doing so corresponds to the horizon crossing approximation at the perturbed level. Anyhow, this should be a good approximation since we know already that the additional contributions due to the evolution of modes after horizon crossing will hardly ever be observable. Further, assume that one of the fields leaves slow roll and starts to evolve faster. To incorporate the effect of the kink in the multi-field trajectory, introduce a step in the effective potential, followed by a subsequent re-definition of the effective model where the field under consideration is omitted. Comparing the effective model before and after removal of the field will provide the step width $\Delta \sigma$ and height $\Delta V$. Hence, we arrive at a toy model with a single inflaton field, but sharp steps in the inflaton potential. Such steps can cause considerable non-Gaussianities \[57\], in addition to a ringing in the power spectrum, which might actually be easier to observe than $f_{NL}$ itself \[57, 58\]. Subsequently, the fields that left slow roll will settle in a minimum of their potential without influencing the dynamics of the universe any more.

Nevertheless, we do not expect this case to be realized in $\mathcal{N}$-flation: consider a broad mass spectrum (e.g. for $\beta \geq 1/2$) as a concrete model; here, the heavy fields violate indeed the slow roll condition $|\eta_i| < 1$ while inflation still goes on, but they experience an extra damping, not acceleration, since $\eta > 0$. Since they evolve slower than expected from the slow roll approximation, they should not cause any additional non-Gaussianities, but simply hang around up until the light fields start to evolve faster. This will occur shortly before the onset of (p)re-heating, corresponding to the breakdown of slow roll for $\sigma$ in the effective single-field description. Consequently, we expect
(p)re-heating, to be the primary additional source of non-Gaussianity (see e.g. [26]) in $\mathcal{N}$-flation. (P)re-heating $^{13}$ should occur in a slightly stretched-out manner during that last one or two e-folds, since fields do not contribute all at once. Since the universe is evolving throughout, we expect expansion effects to be relevant. Henceforth, simple parametric resonance models ignoring the aforementioned expansion will not be suitable. However, stochastic resonance [59], that is, broad parametric resonance in an expanding universe, seems to be a possibility.

A careful examination of the above mentioned topics, especially (p)re-heating, is in preparation [65].

IV. CONCLUSIONS

In this article we considered $\mathcal{N}$-flation as a concrete realization of assisted inflation motivated by string theory. Interested in distinguishing this multi-field model from simple single-field ones, we computed primordial non-Gaussianities.

After reviewing dynamics of $\mathcal{N}$-flation during slow roll, we evaluated non-linearity parameters characterizing the bi- and trispectrum in the horizon crossing approximation. In this limit $\mathcal{N}$-flation, and other multi-field models, become indistinguishable from their single-field analogs.

As a consequence, we incorporated the evolution of perturbations after horizon crossing. This evolution is due to the presence of isocurvature perturbations and provides a means of discriminating multi-field models from single-field ones. Focusing on the non-linearity parameter $f_{NL}$, which will be heavily constraint by observations in the near future, we evaluated its magnitude for narrow and broad mass distributions. In $\mathcal{N}$-flation, this mass spectrum is described by the Marčenko-Pastur law. We identified additional contributions, which turned out to be only a few percent of the horizon crossing result so that they are unobservable.

The smallness of the additional terms is due to the slow roll approximation employed in this paper. Henceforth, we turn our attention to dynamics beyond slow roll; we argue that large, but possibly transient, contributions to $f_{NL}$ should be expected from fast rolling fields. Such fields are not expected in case of $\mathcal{N}$-flation, but might be present in other multi field inflationary models. We suggest an effective single-field model with steps in its potential to retain the main physical effect of these fields.

A study of (p)re-heating in $\mathcal{N}$-flation, which should provide an additional source of non-

$^{13}$ See e.g. [59, 60, 61, 62, 63, 64] for a sample on the extensive literature on (p)re-heating.
Gaussianity, is in preparation [65].

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APPENDIX A: THE $\mathcal{F}_\alpha$ FUNCTIONS

Here we would like to gather some properties of the functions

$$\mathcal{F}_\alpha(\omega) \equiv \int_{1/\xi}^{1} \sqrt{(1-s)(\xi-s^{-1})} s^\alpha \omega^s ds \, ,$$

where $\xi = (1 + z)^2/(1 - z)^2$ and $z$ is a free parameter. First note that analytic expressions are known if $\omega = 1$: for $\alpha \geq -1$ the functions become [37, 38]

$$\mathcal{F}_\alpha(1) = 2\pi z^2 \frac{(1-z)^2}{(1+z)^2} \sum_{i=1}^{\alpha+1} \frac{1}{\alpha+1} \binom{\alpha+1}{i} \binom{\alpha+1}{i-1} z^{2(i-1)} \, ,$$

by relating $\mathcal{F}$ to the moments of the Marcenko-Pastur mass distribution, as analyzed in [38]. Furthermore, the expectation values $\langle x^{-1} \rangle$ and $\langle x^{-2} \rangle$ were computed in [37], yielding

$$\mathcal{F}_{-2}(1) = \frac{(1-z)^2 \xi}{1-z^2} a \, ,$$

$$\mathcal{F}_{-3}(1) = \frac{(1-z)^4 \xi}{(1-z^2)^3} a \, ,$$

with $a = (1-z)^2/2\pi z^2$.

We can also write down analytic expressions for general $\omega$ in the limit $\xi \to \infty$, which corresponds to the limit $z \to 1$: first note that

$$\tilde{\mathcal{F}}_0(\omega) \equiv \lim_{z \to 1} \mathcal{F}_0(\omega) \, ,$$

$$= \frac{\pi \sqrt{y}}{2 \ln(y)} I_1(\ln(y)/2) \, ,$$

where $I$ is a Bessel function of the first kind. All other $\mathcal{F}_\alpha$ can be computed via recursion since

$$\mathcal{F}_{\alpha+1}(\omega) = \omega \frac{\partial \mathcal{F}_\alpha(\omega)}{\partial \omega} \, ,$$

$$\mathcal{F}_{\alpha-1}(\omega) = \int_{0}^{\omega} \frac{1}{\tilde{\omega}} \mathcal{F}_\alpha(\tilde{\omega}) d\tilde{\omega} \, ,$$

14 It is identified with $\sqrt{\beta}$ in $\mathcal{N}$-flation.
follows directly from the definition \( A_1 \).


[57] X. Chen, R. Easther and E. A. Lim, “Large non-Gaussianities in single field inflation,”


