Relativistic mechanics of Casimir apparatuses in a weak gravitational field

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This paper derives a set of general relativistic Cardinal Equations for the equilibrium of an extended body in a uniform gravitational field. These equations are essential for a proper understanding of the mechanics of suspended relativistic systems. As an example, the prototypical case of a suspended vessel filled with radiation is discussed. The mechanics of Casimir apparatuses at rest in the gravitational field of the Earth is then considered. Starting from an expression for the Casimir energy-momentum tensor in a weak gravitational field recently derived by the authors, it is here shown that, in the case of a rigid cavity supported by a stiff mount, the weight of the Casimir energy \( E_C \) stored in the cavity corresponds to a gravitational mass \( M = E_C/c^2 \), in agreement with the weak Equivalence Principle. The case of a cavity consisting of two disconnected plates supported by separate mounts, where the two measured forces cannot be obtained by straightforward arguments based on the Equivalence Principle, is also discussed.

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I. INTRODUCTION

One of the most intriguing predictions of Quantum Electrodynamics is the existence of irreducible fluctuations of the electromagnetic (e.m.) field in the vacuum. It was Casimir’s fundamental discovery \([1]\) to realize that the effects of this purely quantum phenomenon were not confined to the atomic scale, as in the Lamb shift, but would rather manifest themselves also at the macroscopic scale, in the form of an attractive force between two parallel discharged metal plates at distance \( a \). Under the simplifying assumption of perfectly reflecting mirrors, he obtained a force of magnitude

\[ F_{(C)} = \frac{\pi^2 \hbar c}{240 a^3} A, \tag{1.1} \]

where \( A \) is the area of the plates. By modern experimental techniques the Casimir force has now been measured with an accuracy of a few percent (see Refs. \([2]\) and Refs. therein). For a recent review on both theoretical and experimental aspects of the Casimir effect, see Ref. \([3]\).

Now, the energy associated with the Casimir force in Eq. (1.1) is

\[ E_{(C)} = -\frac{\pi^2 \hbar c}{720 a^3} A, \tag{1.2} \]

and one may wonder if it is possible to measure this vacuum energy directly, rather than the corresponding force. Experiments of this sort would further enhance one’s confidence in the reality of vacuum fluctuations. Recently, we proposed an experiment with superconducting cavities, aiming at measuring the variation of Casimir energy that accompanies the superconducting transition \([4]\). Another line of research concerns the gravitational coupling of the vacuum energy. This problem has been studied by a number of authors, leading to some contradictory conclusions.

In Ref. \([5]\), the authors propose an experiment to measure the Casimir force in the Schwarzschild metric of the galactic center. The experiment is designed to show whether or not virtual quanta follow geodesics. They find gravitational forces that depend on the orientation of the Casimir apparatus with respect to the gravitational field of the earth.

In Ref. \([6]\), the author evaluates the Casimir effect in a weak gravitational field, and obtains corrections to the vacuum energy-momentum tensor and attractive force on the plates, resulting from spacetime curvature. He then points out that, if the cosmological constant arises by virtue of zero-point energy, it is susceptible to fluctuations induced by gravitational sources. He uses a curved line element which describes the weak gravitational field in the vicinity of a mass \( M \) as a perturbation to Minkowski spacetime rather than the flat metric appropriate for a uniform gravitational field, in the Fermi coordinate system attached to the cavity.

In Ref. \([7]\), the author studies the Casimir vacuum energy density for a massless scalar field confined between two nearby parallel plates in a slightly curved, static spacetime background, employing the weak-field approximation, and obtains the gravity-induced correction to Casimir energy. He then finds that the attractive force between the cavity walls is expected to weaken.

In Ref. \([8]\), the local form of the Equivalence Principle (see Eq. (1.3) below) is assumed, and a sketchy computation is presented to show that, if one suspends a
rigid Casimir cavity, the vacuum fluctuations contribute an extra negative weight \( \bar{f}(C) \) which, to leading order in the dimensionless parameter \( \epsilon \equiv 2g a/c^2 \), is equal to

\[
\bar{f}(C) = \frac{E(C)}{\epsilon^2} - \bar{g},
\]

(1.3)

where \( \bar{g} \) is the gravity acceleration. In the same paper, the feasibility of such an experiment is also discussed. An important progress was made in Ref. [9], which contains the first detailed Quantum-Field theoretic computation of the Casimir energy-momentum tensor \( \langle T_{ab}(C) \rangle \) (with angle brackets denoting the vacuum expectation value) in a weak gravitational field. Covariant conservation of \( \langle T_{ab}(C) \rangle \), in that

\[
\nabla_a \langle T_{ab}(C) \rangle = 0,
\]

(1.4)

was explicitly verified therein up to first order in \( \epsilon \). This equation provides a rigorous proof of the nontrivial fact that quantum vacuum fluctuations in a cavity indeed do obey locally the weak Equivalence Principle [15].

Last, in Ref. [10], the validity of the Equivalence Principle for vacuum fluctuations is again assumed from the start, through the assumption that the well-known classical definition of energy momentum-tensor for a classical system coupled to the gravitational field (Eq. (2) of Ref. [10]) extends also to vacuum averages of Quantum Fields propagating in a curved background, which is not obvious as far as we can see. Based on this assumption, variational methods are also used to show that, for the case of parallel conducting plates, the Casimir energy gravitates according to Eq. (1.3). This is in agreement with the early findings of Jaekel and Reynaud, who studied the inertia of Casimir energy in two dimensions [11]. No orientation dependence has been found in Ref. [10], accounting clearly for the discrepancy in this respect as compared with the findings in Ref. [9].

The present paper represents the logical completion of the work in Refs. [8, 9]. Its content can be divided into two main parts. The first part, coinciding with Secs. II and III, discusses the mechanics of an extended body at rest in a uniform gravitational field, within Einstein’s Theory of General Relativity. The analysis starts from the assumption that the body satisfies the local expression of the Equivalence Principle within that Theory, Eq. (2.1) below, and we use this to obtain a simple mathematical proof of the general global conditions, the Cardinal Equations, that ensure mechanical equilibrium of the body. Conditions similar to ours were originally obtained in Refs. [12, 13], in the context of any theory of gravitation satisfying the weak Equivalence Principle, by means of an ingenious gedanken experiment. Using these Cardinal Equations, we show that any system, which obeys the local form of the Equivalence Principle, possesses a passive gravitational mass that is equal to its total inertia. In Sec. III we illustrate the conditions obtained in Sec. II to study the mechanical forces in the prototypical case of a rigid suspended vessel, filled with a fluid. The general conditions derived in Sec. II are indispensable for a correct understanding of the forces in a relativistic system, like a vessel filled with radiation, considered in Sec. III, and even more so in the case of Casimir apparatuses, that are studied in Sec. IV.

In the second part of the paper, coinciding with Sec. IV, we study the relativistic mechanics of Casimir apparatuses at rest in the gravitational field of the Earth. Since from our Ref. [9], we now know that vacuum fluctuations satisfy the local form of the Equivalence Principle, Eq. (1.3), the general theorems derived in the first two sections can be used. In particular, we evaluate the forces exerted by the mounts that hold the apparatus, which represent the actual quantities to be measured in a real experiment. The problem turns out to be rather subtle, because, according to the general Cardinal Equations, the magnitudes of the supporting forces depend on where they are applied, and therefore the answer depends on the setting considered for the mounts. This is a general phenomenon in theories of gravity satisfying the Equivalence Principle, as it was discovered long ago by Nordtvedt [12], and it turns out to be of fundamental importance in the analysis of an essentially relativistic system like a Casimir cavity. We consider two different settings for the mounts. In the first case, we have just one mount, supporting a rigid cavity; for this case, the general theorems in Sec. II immediately lead to Eq. (1.3) for the “weight” of the Casimir apparatus. In the second setup, the plates of the Casimir cavity are disconnected, and they are supported by separate mounts. Using the expression for \( \langle T_{ab}(C) \rangle \) computed in Ref. [9], we obtain the forces exerted by the plates of the respective mounts. Sec. V finally contains our conclusions and a discussion of the results.

II. RELATIVISTIC STATIC CARDINAL EQUATIONS IN A UNIFORM GRAVITATIONAL FIELD

In this Section, we obtain a rigorous proof, within the context of Einstein’s General Theory of Relativity, of the conditions for mechanical equilibrium of an extended body at rest in a uniform gravitational field. In General Relativity, the Equations of motion of a body subject to external forces have the general form

\[
\nabla_a T^{ab} = f^b_{(\text{vol})},
\]

(2.1)

where \( T^{ab} \) is the energy-momentum tensor and \( f^a_{(\text{vol})} \) are the external forces. As is well known, these Equations constitute the local expression of the weak Equivalence Principle in Einstein’s Theory of General Relativity. Therefore, any physical system, whose energy-momentum tensor \( T^{ab} \) satisfies an Equation of the form of Eq. (2.1), obeys the Equivalence Principle.

If the system is also subject to forces \( f^a_{(\text{sur})} \), applied at points of its surface \( \partial \Sigma \), Eqs. (2.1) are supplemented by
the following boundary conditions:

$$T^{ab} n_b = - f^a_{(\text{sur})} \text{ on } \partial \Sigma,$$  \hfill (2.2)

where $n^a$ is the unit normal at $\partial \Sigma$ pointing outwards the body.

Now we consider the case of a body at rest in a uniform gravitational field. As we shall see, in this special case it is possible to derive, from the local conditions Eqs. (2.1) and Eq. (2.2), a set of global conditions ensuring mechanical equilibrium of the body. The global conditions that we shall obtain have a form similar to the familiar Cardinal Equations of Statics in Newtonian Theory, and will provide us with a relativistic concept of weight for an extended body. The weak Equivalence Principle in this context translates into the intuitive statement that the (passive) gravitational mass of a body equals its inertial mass. A striking point of departure from classical theory is however the fact, first discovered in Ref. [12], that the weight of a body, intended as the magnitude of the force that must be applied to hold it, depends on where it is. It is however the fact, first discovered in Ref. [12], that the force of gravity that must be applied to hold a body is equal to its gravitational mass.

We begin by defining a uniform gravitational field as a field for which the acceleration parameter $a$ is constant. Such a field is described by the line element

$$ds^2 = -c^2 (1 + \frac{A z}{c^2})^2 dt^2 + \delta_{ij} dx^i dx^j,$$  \hfill (2.3)

with $A > 0$ the acceleration parameter. This line element describes also the gravitational field of the Earth, in a local Fermi coordinate system, once tidal effects are neglected.

We denote by $u^a$ the normalized velocity field for observers at rest in the metric (2.3):

$$u^a = u^0 \left( \frac{\partial}{\partial t} \right)^a = \frac{c}{|g_{00}|^{1/2}} \left( \frac{\partial}{\partial t} \right)^a.$$  \hfill (2.4)

They possess an acceleration $a^a$ in the upwards $z$-direction, i.e.

$$a^a = \frac{Du^a}{D\tau} = \frac{c^2}{|g_{00}|^{1/2}} \partial_z |g_{00}|^{1/2} z^a = \frac{A}{1 + A z/c^2} z^a.$$  \hfill (2.5)

We define the gravity acceleration $g^a$ as minus the acceleration $a^a$ of stationary observers, i.e.

$$g^a(z) = - \frac{Du^a}{D\tau} = - \frac{A}{1 + A z/c^2} z^a.$$  \hfill (2.6)

We consider an extended body at rest in the coordinate system of Eq. (2.3). The body being at rest, one has

$$T^{ab} = \rho u^a u^b + S^{ab},$$  \hfill (2.7)

where $\rho$ is the inertial mass density and $S^{ab}$ is the stress tensor, satisfying the condition

$$S^b_a u^b = 0.$$  \hfill (2.8)

By virtue of Eq. (2.3), we then have

$$S^{00} = S^{0i} = S^{00} = 0.$$  \hfill (2.9)

and therefore $S^{ab}$ is purely spatial.

If we now insert Eq. (2.7) into the l.h.s. of Eq. (2.1), we obtain

$$\nabla_a (\rho u^a) u^b - \rho g^b = - \nabla_a S^{ab} + f^{(\text{vol})}_b.$$  \hfill (2.10)

Let now $e^a_{(i)}$ be the vector fields for the coordinate spatial axis:

$$e^a_{(i)} = \frac{\partial}{\partial x^i} \quad i = x, y, z.$$  \hfill (2.11)

Upon multiplying Eq. (2.10) by $e_{(j)b}$, we obtain

$$- \rho g_i = - \nabla_a (S^{ab} e_{(i)b}) + S^{ab} \nabla_a (e_{(i)b}) + f^{(\text{vol})}_i,$$  \hfill (2.12)

where we define $g_i \equiv g_a e^a_{(i)}$. A simple computation gives

$$\nabla_a (e_{(i)b}) = - \delta_{i3} A (dt)_a (dt)_b.$$  \hfill (2.13)

and since $S^{ab}(dt)_b = 0$ (see Eq. (2.8)), we see that the second term on the r.h.s. of Eq. (2.12) vanishes. On the other hand, for the first term on the r.h.s. of Eq. (2.12), using well known identities, we find

$$\nabla_a (S^{ab} e_{(i)b}) = \frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|} S^j_i)$$

$$= \frac{1}{\sqrt{|g_{00}|}} \partial_j (\sqrt{|g_{00}|} S^j_i).$$  \hfill (2.14)

Therefore, Eq. (2.12) becomes

$$- \rho g_i = - \frac{1}{\sqrt{|g_{00}|}} \partial_j (\sqrt{|g_{00}|} S^j_i) + f^{(\text{vol})}_i.$$  \hfill (2.15)

Let us now introduce the gravitational red-shift $r_O(P)$ of the point $P$ with coordinates $\{x, y, z\}$ relative to an arbitrary point $O$ with coordinates $\{x(O), y(O), z(O)\}$:

$$r_O(P) = \sqrt{\frac{|g_{00}(P)|}{|g_{00}(O)|}} = \frac{1 + A z/c^2}{1 + A z(O)/c^2}. $$  \hfill (2.16)

Upon multiplying Eq. (2.15) by $r_O(z)$ we obtain

$$- \rho r_O g_i(z) = - \partial_j (r_O S^j_i) + r_O f^{(\text{vol})}_i.$$  \hfill (2.17)

However, in view of Eq. (2.6), see that

$$r_O(z) g_i(z) = g_i(z(O)),$$  \hfill (2.18)

and hence we arrive at the following equation:

$$- \rho g_i(z_O) = - \partial_j (r_O S^j_i) + r_O f^{(\text{vol})}_i.$$  \hfill (2.19)
Upon integrating the above equation over the body’s volume, and in view of Eqs. (2.2), we obtain our First Cardinal Equation:

\[ \vec{P}_O + \vec{F}^{(\text{tot})}_O = \vec{0}. \]  

(2.20)

In this equation, the total external force in Eq. (2.20) is

\[ \vec{F}^{(\text{tot})}_O \equiv \int_{\Sigma} d^3x \rho \vec{f}^{(\text{vol})}_O + \int_{\partial \Sigma} d^2 \sigma \rho \vec{f}^{(\text{sur})}_O, \]  

(2.21)

while the weight \( \vec{P}_O \) is defined as

\[ \vec{P}_O \equiv M \vec{g}_O, \]  

(2.22)

where \( \vec{g}_O \) denotes the gravity acceleration at \( O \) and

\[ M = \int_{\Sigma} d^3x \rho. \]  

(2.23)

Now the quantity \( M \) in Eq. (2.22) is, by definition, the passive gravitational mass of the body, and therefore Eq. (2.23) tells us that \( M \) is equal to the total inertia of the body, as expected from the Equivalence Principle. We stress that the above derivation shows that this identity is a necessary consequence of the covariant equations of motion, Eq. (2.24), which represent the local form of the Equivalence Principle in General Relativity. Therefore, for all physical systems, which satisfy Eq. (2.27), the passive gravitational mass is equal to the total inertia.

In order to derive the Second Cardinal Equation, we now define the center of mass \( \vec{x}_{CM} \) via the equation

\[ \vec{x}_{CM} = \frac{1}{M} \int_{\Sigma} d^3x \rho \vec{x}. \]  

(2.24)

Now we multiply Eq. (2.19), for \( O = CM \), by \( \epsilon_{kli}(x - x_{CM})^l \) and integrate the resulting Equation over the body’s volume. On using the identity

\[ \epsilon_{kli}(x - x_{CM})^l \partial_j(r_{CM} S_i^j) = \partial_j[\epsilon_{kli}(x - x_{CM})^l r_{CM} S_i^j], \]  

(2.25)

implied by the symmetry of \( S^{ij} \), one obtains the Second Cardinal Equation

\[ \vec{\tau}^{(\text{tot})}_{CM} = \vec{0}, \]  

(2.26)

where \( \vec{\tau}^{(\text{tot})}_{CM} \) is the total torque of the external forces, relative to the center of mass:

\[ \vec{\tau}^{(\text{tot})}_{CM} = \int_{\Sigma} d^3x (\vec{x} - \vec{x}_{CM}) \times r_{CM} \vec{f}^{(\text{vol})}_O + \int_{\partial \Sigma} d^2 \sigma (\vec{x} - \vec{x}_{CM}) \times r_{CM} \vec{f}^{(\text{sur})}_O. \]  

(2.27)

Equations similar to Eqs. (2.20-2.21) were first obtained in Ref. [12] within a generic theory of gravitation satisfying the weak Equivalence Principle, by exploiting the phenomenon of gravitational red-shift for photons, via an ingenious gedanken experiment involving an ideal electromechanical device converting into photons the mechanical work done by a heavy body, as it lowers or rises into the gravitational field. By a similar procedure, the authors of Ref. [13] obtained equations similar to our Eqs. (2.20-2.21), ensuring rotational equilibrium of the body.

As we see, Eqs. (2.20-2.27) look remarkably similar to the analogous Equations of Newtonian theory. The striking difference with classical theory is that when forces and torques are added, one has to multiply each of them by the red-shift of the point where they act, relative to the point where they are added. A remarkable consequence of this is that the force that must be applied to a body to hold it still in a gravitational field, depends on where the force is applied [12]. To see this, define the proper weight \( \vec{P} \) of a body as

\[ \vec{P} \equiv M \vec{g}_{CM}, \]  

(2.28)

and suppose that the supporting force \( \vec{f}_Q \) is applied at the point \( Q \). Then, from Eq. (2.20) we obtain

\[ \vec{f}_Q = -M \vec{g}_Q = -r_Q(CM) \vec{P}, \]  

(2.29)

where in the final passage we used Eq. (2.18), for \( O = Q \) and \( P = CM \). Therefore, the magnitude of the applied force is equal to the proper weight, multiplied by the red-shift of the center of mass \( CM \) relative to the point \( Q \).

We would like to comment now on an alternative possible form of the Cardinal Equations, based on a different rearrangement of Eq. (2.20). As we shall see, the alternative form implies a definition of inertia for a stressed body, which explicitly involves the stresses. For this purpose, we have to consider again the last line of Eq. (2.14). If we perform the partial derivatives, and use Eqs. (2.5) and (2.6), we obtain

\[ \frac{1}{\sqrt{|g_{00}|}} \partial_j (\sqrt{|g_{00}|} S_i^j) = \partial_j S_i^j - \frac{1}{c^2} S_i^j \rho \delta_i g_j(z). \]  

(2.30)

If we substitute this expression into Eq. (2.16), and rearrange the terms, we obtain

\[ - \left( \rho \delta_i^j + \frac{1}{c^2} S_i^j \right) g_j = -\partial_j S_i^j + \vec{f}^{(\text{vol})}_i. \]  

(2.31)

This form of the equation suggests that the inertia of a stressed body is not just \( \rho \), but rather is a tensor \( m_i^j \) that depends on the stresses, i.e.,

\[ m_i^j = \rho \delta_i^j + \frac{1}{c^2} S_i^j. \]  

(2.32)

If we integrate this equation over the body’s volume, instead of Eq. (2.20), we obtain

\[ \vec{P} + \vec{F}^{(\text{tot})} = \vec{0}, \]  

(2.33)

where the “weight” \( \vec{P} \) is now defined as

\[ \vec{P} \equiv \int_{\Sigma} d^3x \ m_i^j g_j(z) \hat{e}_i. \]  

(2.34)
and $\mathbf{F}^{(\text{tot})}$ coincides with the classical expression for the total external force:

$$\mathbf{F}^{(\text{tot})} = \int_{\Sigma} d^4x \mathbf{f}_{(\text{vol})} + \int_{\partial \Sigma} d^3 \sigma \mathbf{f}_{(\text{sur})}. \quad (2.35)$$

It is of course possible to derive from Eq. (2.31), by similar steps as those followed earlier, the analogue of the Second Cardinal Equation (2.20), but we shall not write it here. We would like to comment, instead, on the conceptual differences between the two approaches. Indeed, the key difference arises from the fact that the two approaches use different expressions for the sum of forces acting at different points and, in particular, for the sum of the contact forces acting on the faces of an infinitesimal cube inside the body. Consider a small cube with vertex at $\{x, y, z\}$ and sides $\{dx, dy, dz\}$, and consider a pair of opposite faces, say the $yz$ faces at $x$ and $x + dx$. Then, by definition of stress tensor, the forces in the direction $\iota$ acting on these two faces are, respectively, $dy \, dz \, S^{z\iota}(x)$ and $-dy \, dz \, S^{z\iota}(x + dx)$. The question now is: what do we take for the sum of these two forces? If we interpret the picture outlined in Ref. [12] according to which the phenomenon of red-shift for forces represents a genuine physical effect, we believe that in all situations one should say that the sum at $O$ of the elementary forces on the $yz$ faces is

$$dy \, dz \left( r_O(x + dx) S^{z\iota}(x + dx) + r_O(x) S^{z\iota}(x) \right) = -dx \, dy \, dz \, \partial_x \left( r_O(x) S^{z\iota}(x) \right). \quad (2.36)$$

Then, upon summing over the three pairs of faces, we obtain for the total force $dF^\iota_O(x)$ the expression

$$dF^\iota_O = -dx \, dy \, dz \, \partial_x \left( r_O S^{z\iota} \right), \quad (2.37)$$

which is the one used in Eq. (2.17), which eventually leads to Eqs. (2.20) and (2.26). Note that, with this choice, $dF^\iota_O$ depends on the reference point $O$, which is why it has a suffix $O$. On the contrary, in the second approach, forces are not red-shifted as they are translated, and therefore one now writes

$$dF^\iota = -dx \, dy \, dz \, \partial_x S^{z\iota}, \quad (2.38)$$

which is the expression used in classical theory. The extra term that one gets, i.e. the last term on the r.h.s. of Eq. (2.36), is now interpreted as a contribution to the inertia of the matter inside the cube, and is therefore shifted to the l.h.s. of Eq. (2.15), leading to Eq. (2.31), and eventually to the concept of weight in Eq. (2.34).

Of course, both approaches are mathematically equivalent. However, as we think with the author of Ref. [12] that the phenomenon of red-shift for forces represents a genuine physical effect, we believe that in all situations one should accordingly modify the sum of forces acting at different points. Therefore we regard Eq. (2.37) and Eq. (2.21) as the physically correct ones. A further important advantage of this approach is that it leads to a very simple concept of weight that only involves the density of mass $\rho$ of the body, as in Eq. (2.20), in nice agreement with one’s expectations based on the Equivalence Principle. This should be contrasted with the very complicated concept of weight in the second approach, Eq. (2.34), which explicitly involves an average of the stresses. The latter quantity, is not only hard to evaluate in general but, perhaps more importantly, it greatly obscures the interpretation of the weak Equivalence Principle, in the case of stressed bodies.

### III. THE CASE OF A VESSEL FILLED WITH A FLUID

It is instructive to use the previous general formulae to examine a system composed by two subsystems, i.e. a rigid vessel, filled with a fluid. We consider, for simplicity, the case of a rectangular box, hanging by a thread. The box is described by an energy-momentum tensor of the form in Eq. (2.27):

$$T^a_b^{(\text{box})} = \rho^{(\text{box})} u^a u^b + S^a_b^{(\text{box})}, \quad (3.1)$$

while for the fluid one has

$$T^a_b^{(\text{fl})} = \rho^{(\text{fl})} u^a u^b + p^{(\text{fl})} \delta^a_b, \quad (3.2)$$

where $p^{(\text{fl})}$ is the pressure. If we consider the total system formed by the box together with the fluid, the only external force is the force $\vec{f}^{(\text{thr})}(Q)$ applied by the thread, in the suspension point $Q$. We can determine $\vec{f}^{(\text{thr})}$ by using the First Cardinal Equation, Eq. (2.26), with $Q = O$, and we obtain

$$\vec{f}^{(\text{thr})}(Q) = -(M^{(\text{box})} + M^{(\text{fl})}) \vec{g}(Q), \quad (3.3)$$

where

$$M^{(\text{box})} \equiv \int_{(\text{box walls})} d^3x \, \rho^{(\text{box})}, \quad (3.4)$$

and

$$M^{(\text{fl})} \equiv \int_{(\text{fluid})} d^3x \, \rho^{(\text{fl})}. \quad (3.5)$$

A comment on the above equation is now in order. Even though the expression for $\vec{f}^{(\text{thr})}(Q)$ has mathematically the form of the sum of two distinct contributions, one from the box and the other from the fluid, it is wrong to think of the former as the weight of the empty box, i.e. without the fluid in its interior. This is so, because when the box is filled with the fluid, the pressure exerted by the fluid on its internal walls causes a small deformation of the walls, and therefore these pressure forces make some work $W$ on the box. This work causes a small change $\delta M^{(\text{box})}$ in the mass of the box, of magnitude $W/c^2$, and therefore $M^{(\text{box})} \neq M^{(\text{empty box})}$. However, for a very stiff box, $\delta M^{(\text{box})}$ is extremely small and therefore, for all practical purposes, one can identify $M^{(\text{box})}$ with $M^{(\text{empty box})}$. After this identification is made, Eq.
Here, $\Sigma$ can be given the standard classical interpretation, according to which the total weight of the system is the sum of the separate weights of the box and of the fluid.

It is interesting now to repeat the analysis, by considering just the box as the whole system. Now, in addition to $\tilde{f}^{(\text{thr})}(Q)$, the external forces acting on the box include the pressure forces exerted by the fluid filling it. The First Cardinal Equation, again taken for $O = Q$, now gives

$$\tilde{f}^{(\text{thr})}(Q) = -M_{(\text{box})} \tilde{g}(Q) - \tilde{f}_{Q}^{(\text{fl})},$$  \hspace{0.5cm} (3.6)

where

$$\tilde{f}_{Q}^{(\text{fl})} = \int_{\Sigma_{(\text{int})}} d^{2} \sigma \, r_{Q} \, p_{(\text{fl})} \, \hat{n}. \hspace{0.5cm} (3.7)$$

Here, $\Sigma_{(\text{int})}$ is the internal surface of the box, and $\hat{n}$ is the unit normal to $\Sigma_{(\text{int})}$, pointing inside the box, i.e. outward with respect to the cavity filled with fluid. By symmetry, the lateral walls of the cavity give a vanishing net force, and therefore we have

$$\tilde{f}_{Q}^{(\text{fl})} = A \left[ r_{Q}(z_2) \, p_{(\text{fl})}(z_2) - r_{Q}(z_1) \, p_{(\text{fl})}(z_1) \right] \hat{z}, \hspace{0.5cm} (3.8)$$

where $A$ is the area of the base of the box, while $z_2$ and $z_1$ are the heights of the upper and lower sides of the cavity, respectively. Note, again, that the relativistic sum of the pressures is not equal to the algebraic sum of them, as it happens in classical theory, because it involves the respective red-shifts. Of course, Eq. (3.6) should eventually reproduce Eq. (3.3). To see it explicitly, we note that by the same steps leading to Eq. (2.19), one finds that Euler’s Equations for the fluid $\nabla_{a} T_{(\text{fl})}^{a} = 0$ imply

$$-\rho_{(\text{fl})} g_{z}(Q) = - \frac{d}{dz} \left( r_{Q} p_{(\text{fl})} \right). \hspace{0.5cm} (3.9)$$

Upon integrating the above Equation from $z_1$ to $z_2$, we find

$$r_{Q}(z_2) \, p_{(\text{fl})}(z_2) - r_{Q}(z_1) \, p_{(\text{fl})}(z_1) = g_{z}(Q) \int_{z_1}^{z_2} dz \, \rho_{(\text{fl})}(z). \hspace{0.5cm} (3.10)$$

By using this result into Eq. (3.8), we obtain

$$\tilde{f}_{Q}^{(\text{fl})} = \tilde{g}(Q) \, A \int_{z_1}^{z_2} dz \, \rho_{(\text{fl})}(z) = \tilde{g}(Q) \, M_{(\text{fl})}. \hspace{0.5cm} (3.11)$$

Upon inserting this formula into Eq. (3.9), we recover Eq. (3.3).

It is now interesting to consider the same problem from the point of view of the box only, for alternative Eqs. (3.8) and (3.3). The final result, Eq. (3.3) will of course be the same, but it is instructive to see how this comes about if one considers again the problem from the point of view of the box only. Instead of Eq. (3.3), Eqs. (3.8) and (3.3) now give

$$\tilde{f}^{(\text{thr})}(Q) = -\tilde{P}_{(\text{box})} - \tilde{f}^{(\text{fl})}. \hspace{0.5cm} (3.12)$$

Here, according to Eqs. (2.31) and (2.32)

$$\tilde{P}_{(\text{box})} = \int_{(\text{box walls})} d^{3} x \left( \rho_{(\text{box})} \, \delta_{i}^{j} + \frac{1}{c^{2}} S_{(\text{box})}^{j} \right) g_{j}(z) \, \hat{x}^{i}, \hspace{0.5cm} (3.13)$$

while, according to Eq. (2.35), for the total force exerted on the box by the fluid we have the classical formula

$$\tilde{F}^{(\text{fl})} = \int_{\Sigma_{(\text{int})}} d^{2} \sigma \, p_{(\text{fl})} \, \hat{n} = A \left[ p_{(\text{fl})}(z_2) - p_{(\text{fl})}(z_1) \right] \hat{z}. \hspace{0.5cm} (3.14)$$

Now, from Eq. (3.9), we find

$$p_{(\text{fl})}(z_2) - p_{(\text{fl})}(z_1) = g_{z}(Q) \int_{z_1}^{z_2} dz \, \rho_{(\text{fl})} - \int_{z_1}^{z_2} dz \, p_{(\text{fl})} \frac{dr_{Q}}{dz} = g_{z}(Q) \int_{z_1}^{z_2} dz \left( \rho_{(\text{fl})} + \frac{1}{c^{2}} p_{(\text{fl})} \right), \hspace{0.5cm} (3.15)$$

where in the last passage we have used Eq. (2.10), Eq. (2.15) and Eq. (2.7) to write $dr_{Q}/dz = g_{z}(Q)/c^{2}$. If the fluid satisfies an equation of state of the form

$$p_{(\text{fl})} = \gamma \rho_{(\text{fl})} c^{2}, \hspace{0.5cm} (3.16)$$

we then find for $\tilde{F}^{(\text{fl})}$

$$\tilde{F}^{(\text{fl})} = \tilde{g}(Q) \left( 1 + \gamma \right) M_{(\text{fl})}, \hspace{0.5cm} (3.17)$$

a result which, at first sight, seems to be in contradiction with the weak Equivalence Principle. (It is interesting to note that, if the box is filled with thermal radiation, $1 + \gamma = 4/3$. This is the same “anomalous” factor of 4/3 that occurred in the classical models for the electromagnetic mass of the electron, considered by H. A. Lorentz at the end of the nineteenth century). Of course, there is really no contradiction, because what one measures here is not $\tilde{F}^{(\text{fl})}$ by itself, but rather $\tilde{f}^{(\text{thr})}(Q)$, which includes also the “weight” of the box $\tilde{P}_{(\text{box})}$. Now, according to Eq. (3.13), $\tilde{P}_{(\text{box})}$ can be separated in two parts, i.e.

$$\tilde{P}_{(\text{box})} = \tilde{P}_{(\text{box})}^{(1)} + \tilde{P}_{(\text{box})}^{(2)}, \hspace{0.5cm} (3.18)$$

where

$$\tilde{P}_{(\text{box})}^{(1)} = \int_{(\text{box walls})} d^{3} x \, \rho_{(\text{box})} \, \tilde{g}(z), \hspace{0.5cm} (3.19)$$

and

$$\tilde{P}_{(\text{box})}^{(2)} = \frac{1}{c^{2}} \int_{(\text{box walls})} d^{3} x \, S_{(\text{box})}^{j} \, \hat{g}(z) \, \hat{x}^{i}. \hspace{0.5cm} (3.20)$$

Recalling the considerations following Eq. (3.3), for a stiff box $\tilde{P}_{(\text{box})}^{(1)}$ is independent, to a high degree of precision, of the fact the box if filled or empty, and therefore can be interpreted as a feature of the box, by itself. On the contrary, the second contribution $\tilde{P}_{(\text{box})}^{(2)}$ depends
on the stresses in the box walls, and therefore this term is strongly affected by the presence of the fluid, whose pressure on the inner surfaces of the walls leads to additional stresses in the walls. We can see this clearly by explicitly evaluating $F^{(2)}_{(box)}$. We need not perform any extra calculations, because we can exploit the mathematical equivalence of the two formulations of the first Cardinal Equation to obtain $F^{(2)}_{(box)}$. Upon comparing the expression for $f^{(thr)}(Q)$ in Eq. (3.6) with Eq. (3.12), we obtain

$$F^{(2)}_{(box)} = (f^{(fl)} - f^{(thr)}) + (M(\text{box}) \bar{g}(Q) - F^{(1)}_{(box)}). \tag{3.21}$$

Upon using Eqs. (3.11), (3.17), (3.4) and (3.19), we then obtain

$$F^{(2)}_{(box)} = -\gamma \bar{g}(Q) M(\text{fl}) + \int_{\text{box walls}} d^3 x \rho_{(box)} (\bar{g}(Q) - \bar{g}(z)). \tag{3.22}$$

As we see, the first term on the r.h.s cancels the unwanted $\gamma$-dependent contribution in Eq. (3.17). However, the fact that $F^{(2)}_{(box)}$ depends, via this term, on the fluid, shows clearly another deficiency of Eqs. (2.33-2.34): for a system formed by several bodies in contact, Eq. (2.33) leads to a concept of weight for the individual bodies contributing to the system that depends strongly on the other bodies with which it interacts. This should be contrasted with the first formulation based on Eq. (2.20), which on the contrary permits, to a high degree of precision (see comments following Eq. (3.5)), to consider the weights of the individual bodies as independent of each other.

### IV. FORCES ON CASIMIR APPARATUS

We now turn to the central problem of this paper, i.e. determining the forces that act on a Casimir apparatus suspended in the Earth’s gravitational field. For simplicity, we consider the idealized case of a cavity consisting of two perfectly reflecting horizontal plates, with common thickness $D$, separated by an empty gap of width $a$. We let the coordinate system be chosen so that the inner faces of the plates, bounding the cavity, have equations $z = z_1 = 0$ and $z = z_2 = a$, respectively. Then, to leading order $g a/c^2$ ($g = |\bar{g}|$), in Ref. [9] we obtained the following expression for the nonvanishing components of the Casimir energy-momentum tensor $T_{(C)}^{ab}$ in the local coordinate system of Eq. (2.3):

$$\langle T_{(C)}^{00} \rangle(z) = -\frac{\pi^2 h}{c a^2} \left[ \frac{1}{720} + \frac{2 g a}{c^2} \left( \frac{1}{1200} - \frac{z}{3600 a} - \frac{\cot(\pi z/a) \csc^2(\pi z/a)}{240 \pi} \right) \right], \tag{4.1}$$

$$\langle T_{(C)}^{11} \rangle(z) = \langle T_{(C)}^{22} \rangle(z) = \frac{\pi^2 h c}{a^4} \left[ \frac{1}{720} + \frac{2 g a}{c^2} \left( \frac{1}{3600} - \frac{1}{1800 a} - \frac{\cot(\pi z/a) \csc^2(\pi z/a)}{120 \pi} \right) \right], \tag{4.2}$$

$$\langle T_{(C)}^{33} \rangle(z) = -\frac{\pi^2 h c}{a^4} \left[ \frac{1}{240} + \frac{2 g a}{c^2} \frac{1}{720} \left( 1 - \frac{a}{2 z} \right) \right]. \tag{4.3}$$

It can be verified that $\langle T_{(C)}^{ab} \rangle$ is conserved, to the considered accuracy $g a/c^2$:

$$\nabla_a \langle T_{(C)}^{ab} \rangle = 0, \tag{4.4}$$

and therefore vacuum fluctuations satisfy locally the weak Equivalence Principle. This is a very nontrivial result, because a priori there is no general proof that quantum fields conform to this principle. We note also that, since $\langle T_{(C)}^{00} \rangle(z) = \langle T_{(C)}^{11} \rangle(z) = 0$, the Casimir energy-momentum tensor has the form corresponding to a “body” at rest, as in Eq. (2.7). Therefore, all theorems derived in Sec. II automatically apply to a Casimir apparatus. In particular, Eq. (2.28) holds, and therefore the Casimir apparatus has a passive gravitational mass $M_{(C)}$ which is equal to its total inertia:

$$M_{(C)} = \int_{\text{cavity}} d^3 x \rho_{(C)}. \tag{4.5}$$

Recalling that according to Eq. (2.7)

$$\rho_{(C)} = \frac{1}{c^4} \langle T_{(C)}^{ab} \rangle u_a u_b, \tag{4.6}$$

we obtain from Eq. (4.1)

$$M_{(C)} = -\frac{\pi^2 h}{720 c a^3} A + o(g a/c^2) = \frac{E_{(C)}}{c^2} + o(g a/c^2). \tag{4.7}$$

An important remark is now in order: the above value for the passive gravitational mass is trivially obtained by dividing by $c^2$ the standard expression of the Casimir energy in the absence of the gravitational field, Eq. (1.12).

Therefore, it would seem at first sight that the complicated corrections of order $g a/c^2$ to the Casimir energy-momentum tensor in Eqs. (4.1-4.3) are of no use for the purpose of obtaining Eq. (4.7), and that one could have avoided all the pain to compute them. It is clear that this argument is wrong, because Eq. (4.7) assumes the validity of the Equivalence Principles for vacuum fluctuations to order $g a/c^2$, and one does not know whether
this assumption is valid until Eq. (4.9) is proved to be true to that order.

A. A suspended rigid Casimir cavity

In the first setup we consider, the plates are rigidly connected to each other, forming a unique rigid system, supported by a thread. By steps similar to those used in Sec. II, we obtain an Equation analogous to Eq. (4.3) for the force \( \bar{f}^{(\text{thr})}(Q) \) required to support the cavity:

\[
\bar{f}^{(\text{thr})}(Q) = -M_{(\text{box})} \bar{g}(Q) - M_{(C)} \bar{g}(Q),
\]

(4.8)

where \( M_{(\text{box})} \) is defined as in Eq. (4.4). After bearing in mind the observations made after Eq. (4.5) on the interpretation of \( M_{(\text{box})} \), one can think of the second term on the r.h.s. of Eq. (4.8) as the weight \( F(C) \) of the "Casimir mass". Using Eq. (4.10), we obtain to leading order in \( ga/c^2 \)

\[
\bar{F}(C) \approx -\frac{\pi^2 A h}{720 \alpha a^3} \bar{g} = \frac{E_C}{c^2} \bar{g},
\]

(4.9)

in agreement with the weak Equivalence Principle. It is interesting to see how the same result is obtained, if we consider the forces acting only on the rigid walls bounding the cavity, analogously to what was done in Sec. II, for the case of a rigid box filled with a fluid. Again, following the same steps that led to Eqs. (3.6) and (3.7),

we obtain

\[
\bar{f}^{(\text{thr})}(Q) = -M_{(\text{box})} \bar{g}(Q) - \int_{\Sigma_{(\text{int})}} d^2 \sigma \, r_Q \langle T^{ij}_{(C)} \rangle \hat{n}_j \hat{x}_i.
\]

(4.10)

Note again the presence of the red-shift \( r_Q \) multiplying the Casimir stresses in the integral on the r.h.s., which is crucial to obtain the right answer, as we shall now see. By symmetry, the lateral walls of the cavity give a net vanishing contribution to the integral on the r.h.s. of the above Equation, and therefore we have

\[
\int_{\Sigma_{(\text{int})}} d^2 \sigma \, r_Q \langle T^{ij}_{(C)} \rangle \hat{n}_j \hat{x}_i = \mathcal{A} \left[ r_Q(z_2) \langle T^{33}_{(C)} \rangle(z_2) - r_Q(z_1) \langle T^{33}_{(C)} \rangle(z_1) \right] \langle 4.11 \rangle
\]

Now we see from Eq. (2.16) that, to leading order in \( g z / c^2 \), the red-shift \( r_Q \) is

\[
r_Q(z) \approx 1 + \frac{g}{c^2} (z - z_Q).
\]

(4.12)

By using this formula in Eq. (4.11), together with the expression for \( \langle T^{33}_{(C)} \rangle \) in Eq. (4.13), we find that the quantity between square brackets in Eq. (4.11) is equal to

\[
- \frac{\pi^2 h c}{a^4} \left[ \frac{g}{240 \, c^2} (z_2 - z_1) - \frac{4g}{720 \, c^2} (z_2 - z_1) \right] = \frac{\pi^2 h c}{720 \, a^3} \bar{g}.
\]

(4.13)

Upon using this expression into Eq. (4.10), we recover the same result as Eq. (4.9).

B. Disconnected plates with separate mounts

In the second setup that we wish to consider, the two plates are disconnected, and are supported by two separate mounts. If the mounts are connected to the outer faces of plates, the forces \( \bar{f}_1 \) and \( \bar{f}_2 \) that support the plates are applied at points \( Q_1 \) and \( Q_2 \), with heights \( w_2 = a + D \) and \( w_1 = -D \), respectively. It is the case to remark that the forces \( \bar{f}_1 \) and \( \bar{f}_2 \) cannot be obtained by a straightforward use of the Equivalence Principle, contrary to the case of the weight of a rigid cavity studied earlier, and they can be determined only using the explicit expression of \( \langle T^{33}_{(C)} \rangle \) in Eq. (4.3). Applying the first Cardinal Equation to each plate separately, we find for the force \( \bar{f}_1 \) (\( I = 1, 2 \))

\[
\bar{f}_1 = -\bar{P}^{(I)}_{Q_1} - \bar{P}^{(C)}_{Q_1},
\]

(4.14)

where \( \bar{P}^{(I)}_{Q_1} \) is the weight of the \( I \)-th plate, i.e.

\[
\bar{P}^{(I)}_{Q_1} = \bar{g}(w_1) \int_{A_I} d^3 x \rho_I
\]

(4.15)

while \( \bar{P}^{(C)}_{Q_1} \) is the contribution from the Casimir pressure

\[
\bar{f}^{(C)}_{Q_1} = \int dx \, dy \, r_{Q_1}(z_i) \langle T^{33}_{(C)} \rangle(z_i) \hat{n}_i \hat{x}_i.
\]

(4.16)

On using Eq. (4.12) and the expression for \( \langle T^{33}_{(C)} \rangle(z) \) in Eq. (4.13), we obtain for the upper plate, to order \( g / c^2 \):

\[
\bar{f}^{(C)}_{Q_1} \approx -\frac{\pi^2 A h c}{240 \, a^4} \left[ 1 - \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \tilde{z},
\]

(4.17)

while for the lower plate we get

\[
\bar{f}^{(C)}_{Q_2} \approx \frac{\pi^2 A h c}{240 \, a^4} \left[ 1 + \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \tilde{z}.
\]

(4.18)

We note that both forces depend on the thickness of the plates. It is worth noting that in an experiment of this sort, in which the forces \( \bar{f}^{(C)}_{Q_1} \) and \( \bar{f}^{(C)}_{Q_2} \) are measured independently, assuming negligible plates’ thicknesses, the above equations give for the algebraic sum of the two measured forces the value

\[
\bar{f}^{(C)}_{Q_1} + \bar{f}^{(C)}_{Q_2} = \frac{\pi^2 A h c g a}{240 \, a^4} \frac{4}{a^2 c^2} \tilde{z} = 4 \times \frac{E_C}{c^2} \bar{g},
\]

(4.19)

which is four times that in Eq. (4.9). This result shows clearly that what one obtains for the "total gravitational push" on the Casimir apparatus depends crucially on the experimental setup. It is useful to remark also that the previous result for the total force measured by a rigid cavity, Eq. (4.9), is recovered if the forces \( \bar{f}^{(C)}_{Q_1} \) and \( \bar{f}^{(C)}_{Q_2} \) are added, say at \( Q_2 \), using the relativistic law in Sec. II, because then

\[
\bar{f}^{(C)}_{Q_2} + r_{Q_2}(Q_1) \bar{f}^{(C)}_{Q_2} \approx F(C) \left\{ -\left[ 1 - \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] + \cdots \right\}
\]
\[
+ \left[ 1 - \frac{g}{c^2} (2D + a) \right] \left[ 1 + \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \approx \frac{1}{3} \frac{g}{c^2} F(C) \hat{z} \approx \frac{E(C)}{c^2} \hat{g}, \tag{4.20}
\]

which is the same result as Eq. (4.9).

V. CONCLUSIONS AND DISCUSSION

Einstein’s Equivalence Principle, according to which the Laws of Physics in a uniform gravitational field are the same as in a uniformly accelerated frame, is one of the most powerful and general principles of Physics. Initially formulated within the context of Classical Physics, it appears today as a universal Principle, which retains its validity also in the realm of Quantum Physics. Among the Quantum phenomena, one of the most fundamental is that of vacuum fluctuations, with the associated unavoidable content of energy. It would clearly be of great importance to test by experiments whether this Quantum vacuum energy conforms to the Principle of Equivalence. A convincing way to do that would be to verify whether the energy of vacuum fluctuations existing in a cavity with reflecting walls, i.e. the Casimir energy, gravitates as other conventional forms of matter-energy. While the feasibility of such an experiment by current weak-force measurement devices was discussed in Ref. [9], an important theoretical step forward was made in Ref. [8], where it was verified, for the first time, that the Casimir energy-momentum tensor satisfies the local form of the weak Equivalence Principle, Eq. (1.4), in a weak uniform gravitational field. The present paper represents the logical completion of Ref. [8], as it analyzes in detail, from the point of view of General Relativity, the mechanical forces in a Casimir apparatus suspended in the Earth’s gravitational field. This is an essential step, because these forces are the quantities to be confronted with real experiments. For that purpose, we derived a set of Cardinal Equations giving the conditions for mechanical equilibrium for any extended body, satisfying locally the Equivalence Principle, at rest in a uniform gravitational field. The key feature of these Equations is that, in a gravitational field, forces are subject to red-shifts, a phenomenon originally discovered by Nordtvedt [12], using heuristic arguments based on the Equivalence Principle. Consideration of this phenomenon is essential in order to obtain the correct values for the forces occurring in an intrinsically relativistic system, such as a Casimir apparatus. On the basis of these Cardinal Equations, we proved rigorously that, for the case of a rigid cavity, the weight associated with the Casimir energy \( E(C) \) is equal to \( g E(C)/c^2 \), as expected. Moreover, we considered the case of a Casimir cavity consisting of two disconnected plates, supported by separate mounts. The answer to this problem cannot be obtained by simple arguments based on the Equivalence Principle, and requires use of the complete Casimir energy-momentum tensor computed in Ref. [9]. Also for this case the general Cardinal Equations provide the relativistically correct expressions for the forces exerted by the mounts on the plates. In this setting, and for negligible plates’ thickness, the algebraic sum of the two measured forces, is four times the result found for the weight of the rigid cavity. Such an example clarifies that the total push on the cavity depends on the experimental setup, which determines what forces one actually measures.

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[15] Strictly, this implies a nontrivial acceptance of the validity of Quantum Field Theory in curved space-times, so that the fields coupled to gravitation can be quantized in a classical geometry, and all laws of Physics now include all laws of Classical or Quantum Fields within such a framework.
[16] In the context of the classical model for the electron, the importance of this contribution from the stresses of the mechanical system needed to ensure the electron’s...
stability, to correct the anomalous factor of $4/3$, was first recognized by Poincaré.