DIRECT QUANTIZATION OF EQUATIONS OF MOTION:
FROM CLASSICAL DYNAMICS TO TRANSITION AMPLITUDES VIA STRINGS

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New method of quantization is presented. It is based on classical Newton-Lagrange equations of motion (representing the fundamental physical law of mechanics) rather than on their traditional Lagrangian and/or Hamiltonian precursors. It is shown that classical dynamics is governed by canonical two-form $\Omega$, which embodies kinetic energy and forces acting within the system. New type of variational principle employing differential two-form $\Omega$ and “umbilical strings” is introduced. The Feynman path integral over histories of the system is then rearranged to “umbilical world-sheet” functional integral in accordance with the proposed variational principle. In the case of potential-generated forces, world-sheet approach reduces to the standard quantum mechanics. As an example Quantum Mechanics with friction is analyzed in detail.

PACS: 01.70.+w, 02.40.Yy, 03.65.Ca, 45.20.-d

Keywords: quantization of dissipative systems, umbilical strings, path vs. surface integral

Dedicated to my father.

1 Introduction

Description and detailed understanding of classical and quantal phenomena attract attention of many physicists and mathematicians for a long time. Classical and quantum mechanics are the best elaborated, examined and understood parts of physics. Their mathematical setting is concentrated around beautiful and powerful artillery, which includes differential geometry, functional analysis, spectral calculus, group and representation theory, (co)homology techniques and so on.

The aim of the paper is to provide a simple geometrical picture of classical and quantum mechanics for physical systems, where Lagrangian and/or Hamiltonian description is missing. The central object in our approach is a certain canonical two-form $\Omega$, which is defined in an extended tangent bundle. Its main properties are narrowly studied in sections 2 and 3. The differential two-form $\Omega$ serves as a guide for a new type of variational principle. In section 4 we introduce the notion of “umbilical world-sheet.” It generalizes the concept of the history of the system and therefore it becomes important in the context of quantization. Variation uncovers desired classical trajectory and, as a bonus, also some kind of minimal surface. In section 5 we will see how the “umbilical strings” can be used to rearrange the Feynman integral over the histories of the system to the surface functional integral. String formulation embodies a big advantage, it concerns components of the forces rather than their potential. Main message of the “umbilical approach” can be summarized as follows: dissipative time evolution remains unitary, but it does not form a group(oid) according to composition. In section 6 we succeed to compute transition probability amplitude for quantum system with friction performing explicitly the surface functional integration. For the potential-generated forces, the “umbilical world-sheet” approach reduces to the standard quantum mechanics.

This paper is hopefully more readable and elaborated version of the article. There, we focused on rigorous geometrical formulation of the theory. In the actual paper, we prefer to explain the main idea and exhibit its functionality.

*AUTHOR IS LOOKING FOR A POSTDOCTORAL POSITION IN THE AREA OF MATHEMATICAL PHYSICS.
2 Lagrangian mechanics and null-spaces of the distinguished two-form $\Omega$

Physical content of classical mechanics is represented by the second Newton’s law. Its mathematical formulation coincides in general curvilinear coordinates with Lagrange equations\textsuperscript{11-14}

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^i} \right) - \frac{\partial T}{\partial q^i} = Q_i, \quad i = 1, \ldots, n (= \text{the number of degrees of freedom}).$$

(1)

Here, $T(q, \dot{q}, t)$ is the kinetic energy of the system and $Q_i(q, \dot{q}, t)$ is the $i$-th component of a generalized force. In the special case, when forces are potential-generated

$$Q_i = -\frac{\partial U}{\partial q^i} + \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{q}^i} \right),$$

one can introduce the Lagrangian function $L = T - U$ and write down the celebrated Euler-Lagrange equations. Generalized coordinates $\{q^i\}$ cover some open patch of the configuration space ($n$-dimensional manifold) $M$.

If we restrict ourselves to the Lagrangian picture, the space of all physical states is the set of all admissible initial conditions for the differential system $\{1\}$. Geometrically, initial condition specified at the time $t_0$ by the generalized position $q(t_0) = q_0$ and velocity $\dot{q}(t_0) = v_0$, defines a point $(q_0, v_0, t_0)$ in an extended tangent bundle $TM \times \mathbb{R}$. Here an open patch of the extended tangent bundle $\mathcal{P}$ is dressed by the $(2n+1)$-tuple of local coordinates $\{q^1, \ldots, q^n, v^1, \ldots, v^n, t\}$.

Let us destroy the compact form of $\{1\}$ expressing generalized accelerations as functions of the remaining entries:

$$\ddot{q}^i = f^i(q, \dot{q}, q, t), \quad \dot{q}^i = f^i(q, v, t, q), \quad \ddot{v}^i = f^i(q, v, t, q).$$

(2)

The system above can be interpreted as a coordinate expression of a vector field on the extended tangent bundle $TM \times \mathbb{R}$; down-to-earth, according to $\{2\}$ one can assign to any physical state $(q, v, t)$ a tangent vector

$$\Gamma \bigg|_{(q, v, t)} = \partial_i \bigg|_{(q, v, t)} + v^i \partial_{q^i} \bigg|_{(q, v, t)} + f^i(q, v, t, q) \partial_{v^i} \bigg|_{(q, v, t)} \in T_{(q, v, t)}(TM \times \mathbb{R}).$$

(3)

The time evolution with prescribed initial condition $(q_0, v_0, t_0)$ is represented by a curve in the extended tangent bundle

$$\gamma: \mathbb{R}[\tau] \rightarrow TM \times \mathbb{R}, \quad \tau \mapsto \gamma(\tau) \equiv (q = q(\tau), v = v(\tau), t(\tau) = \tau)$$

which passes through this physical state and, moreover, such that the tangent vector $\frac{d}{d\tau} \gamma(\tau) \big|_{\gamma(\tau)}$ at any point of the admissible trajectory equals to $\Gamma \big|_{\gamma(\tau)}$, see Figure $\{1\}$.

We have just observed that classical dynamics is determined by the extended tangent bundle vector field $\Gamma$. Having the function $T(q, v, t)$ and the components of the generalized force $Q(q, v, t)$ we can establish (a priori by hand, but hopefully it will be soon clear that there are some very good reasons for that) the two-form$\textsuperscript{3}$

\textsuperscript{3}To be rigorous, a correct geometrical setting for mechanics in the Lagrangian picture is represented by the line element contact bundle $\mathcal{C}(M \times \mathbb{R})$ of the extended configuration space $M \times \mathbb{R}$. The extended tangent bundle is its open dense submanifold, concisely, point $\{q^1, \ldots, q^n, v^1, \ldots, v^n, t\} \Rightarrow \text{contact point } q = \{q^1, \ldots, q^n\} \in M \times \mathbb{R}$ and line element $t \in \text{span}\{v^i \partial_{q^i} |_{q}, + \partial_t |_{q}\} \subset T_q(M \times \mathbb{R})$. $\text{Importance of the two-form } \Omega \text{ in context of mechanics is emphasized in the Arnold’s classical monograph\textsuperscript{12}. However, I should mention the notice of Tamás Fülöp. He drew my attention to the book\textsuperscript{13} of Tamás Matolcsi, where on the page 35 there is an equivalent expression for the two-form.}
The main properties of $\Omega$ can be summarized as follows:

- It is a differential two-form on the extended tangent bundle $TM \times \mathbb{R}$.
- For any point $(q, v, t)$, it gives the linear map $\iota: T_{(q,v,t)}(TM \times \mathbb{R}) \to T_{(q,v,t)}^*(TM \times \mathbb{R})$, $w \mapsto \alpha := \Omega(w, \cdot) \equiv w \cdot \Omega$.
- If $T$ is regular (i.e., $\partial^2_{v,v} (T)$ is invertible), then the kernel $\ell$ of the contraction $\iota$ is one-dimensional and it is spanned by the vector $\Gamma_{(q,v,t)}$; the subspace $\ell$ is called the null-space of the two-form $\Omega$.
- Whenever $Q$ is potential-generated (i.e., $Q_i = -\partial U/\partial q_i + \frac{1}{2} \left( \partial U / \partial v_i \right)$), then $\Omega$ is exact, i.e., $\Omega = d\theta_L$, where

$$\theta_L := L dt + (\partial v_i, \mathbb{L}) \left\{ dq^i - v^i dt \right\}$$

is the well-known Legendre one-form on the extended tangent bundle that is associated to the Lagrangian function $L = T - U$.

- Performing Legendre transformation: $(q, v, t, L) \mapsto (q, p = \partial H / \partial q, t, H = p_i v^i - L)$, the one-form $\theta_L$ transforms to the canonical Cartan-Poincaré one-form

$$\omega_H := p_i dq^i - H dt$$

over the extended phase space $T^*M \times \mathbb{R}$.

- The two-form $\Omega$, as well as the one-forms $\theta_L$ and $\omega_H$ are invariant with respect to the group of diffeomorphisms of the configuration space $M$.

So we could claim: **Lagrangian mechanics is determined by the null-spaces of the distinguished two-form $\Omega$**. Finding them, it is enough to pick up at each null subspace $\ell$ a vector $w$ for which $w \cdot dt = 1$. Doing this we are point-wise reconstructing the dynamical vector field $\Gamma$. Its integral curves are solutions of the Lagrange equations.

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Footnote:\footnote{One easily verifies that everything remains invariant also with respect to "space-time" transformations: $(q^i, t) \mapsto (Q^i(q, t), T = t)$.}
3  An outline: inverse problem of the calculus of variation. Quantization ambiguity

Before going to variational principle and quantization let us contemplate the inverse problem of the calculus of variation. Roughly speaking the task is the following: assume that there is a system of the Newton-Lagrange equations:

\[ \begin{aligned}
\dot{q}^i &= v^i, \\
\dot{v}^i &= f^i(q, v, t),
\end{aligned} \quad \iff \quad \frac{d}{dt} \left( \frac{\partial T}{\partial v^i} \right) - \frac{\partial T}{\partial q^i} = Q_i, \quad \text{where } T = \frac{1}{2} v^i \delta_{ij} v^j \text{ and } Q_i = f^i(q, v, t).
\]

Does there exist a Lagrangian \( L \), whose variational equations \( \frac{\delta L}{\delta q} = 0 \) are equivalent to the initial differential system? This problem was studied by many authors, for more detail see the papers \( [15-15] \) In what follows we will shortly outline that the answer can be provided in terms of the distinguished two-form. Concisely, first of all we need to construct from \( T = \frac{1}{2} v^i \delta_{ij} v^j \) and \( Q_i = f^i(q, v, t) \) the two-form \( \Omega \). Since the full dynamics is specified by its null-spaces, there is “one degree of freedom” in scaling \( \Omega \rightarrow \Omega' = h(q, v, t) \Omega \), which does not affect them. Here \( h(q, v, t) \) is some (at least locally) non-zero function over the extended tangent bundle. If one succeeds to adjust the function \( h \) in such away that it will act as an integrator, \( d \Omega' = dh \wedge \Omega + h d \Omega = 0 \), then there should exist a local \( \Omega' \)-potential \( \vartheta \) (\( \Omega' = d \vartheta \)). It still remains to verify, whether there exists a function \( L \) that satisfies \( \vartheta = L dt + (\partial_{\vartheta^i} L) \{ dq^i - v^i dt \} \). If yes, then the initial differential system is “derivable” from that Lagrangian.

It is well-known since Darboux that in the one-dimensional “world” there always exists an appropriate Lagrangian. As an example let us analyze a particle moving in \( M = \mathbb{R}[x] \), driven by a force \( Q = Q(x, v, t) \). The integrator \( h(x, v, t) \) of the two-form \( \Omega \) is \( \Omega = Q dx \wedge dt + dv \wedge dx - v dv \wedge dt \), for the system under the consideration, should obey the following partial differential equation:

\[ d \Omega' = 0 \iff \frac{\partial h}{\partial t} + v \frac{\partial h}{\partial x} + Q \frac{\partial h}{\partial v} = -h \frac{\partial Q}{\partial v}. \quad (5) \]

To find all solutions of the above equation, one needs to provide characteristics (integral curves) of the linear operator (vector field): \( \partial_t + v \partial_x + Q \partial_v - \{ h \frac{\partial Q}{\partial v} \} \partial_h \) in \( \mathbb{R}^4[x, v, t, h] \). Suppose that the characteristics are known:

\[ t = t(p), \quad x = x(p), \quad v = v(p), \quad h = h(p), \quad \text{where } p \text{ is some flow parameter}. \]

Then taking any function \( z = z(x, v, t, h) \) such that \( z(p) = z(x(p), v(p), t(p), h(p)) \) is constant on these characteristics, and solving the implicit function problem \( z(x, v, t, h(x, v, t)) = 0 \) with respect to \( h \), we get all solutions of the initial problem \( [3] \).

For concreteness, let \( Q(x, v, t) = -v \) (a particle with friction that is linear in the actual velocity), then the integral curves of the vector field \( \partial_t + v \partial_x - v \partial_v + h \partial_h \) are summarized as follows:

\[ t(p) = p + k_1, \quad x(p) = -k_2 e^{-p} + k_3, \quad v(p) = k_2 e^{-p}, \quad h(p) = k_4 e^p, \]

the numbers \( k_1, k_2, k_3, k_4 \) appear here as unimportant integral constants. For example for any real numbers \( \alpha \) and \( \beta \) the function \( z_{\alpha\beta}(v, t, h) = e^{\alpha t} v^\beta h^{\beta - \alpha - 1} \) is constant along any of the above characteristics. Let us, for simplicity, only consider the special case when \( \beta = \alpha - 1 \); then we can write

\[ \text{integrator: } h(v, t, \alpha) = e^{\alpha t} v^{\alpha - 1} \implies \text{ Lagrangian: } \begin{cases} \alpha \in \mathbb{R}/\{-1, 0\}, & L_\alpha = \frac{e^{\alpha t}}{\alpha+1} e^{\alpha t} \\
L_0 = \frac{e^{\alpha t}}{\alpha+1} e^{\alpha t} \\
L_{-1} = e^{\alpha t} \ln v - v - x \\
L_{-1} = e^{\alpha t} \ln v \end{cases} \]

Another type of solution is provided for example by the function \( h = \frac{v}{\alpha} + 1. \) So one can see that for this relatively simple one-dimensional physical system, there is a rich class of Lagrangians with the property: \( \frac{\delta h}{\delta q} = 0 \implies \{ \dot{x} = v \text{ and } \dot{v} = -v \}. \)

For more than one-dimensional “world” the situation becomes rather more complicated. The above simple partial differential equation \( [5] \) is replaced by a system of several partial differential equations and some difficulties appear also on the ways \( \Omega' \rightarrow \vartheta \) and \( \vartheta \rightarrow L \). But suppose there is a set of Lagrangians that
generates the same classical dynamics. In this class one can introduce an equivalence relation saying that $L_1 \sim L_2 \iff$
\[
\{ \theta L_1 - \theta CL_2 = dF, \text{ where } F \text{ is a function on } TM \times \mathbb{R} \} \iff \{ L_1 = CL_2 + \frac{\partial F}{\partial t} + \psi \frac{\partial F}{\partial \psi} \text{ and } \frac{\partial F}{\partial \sigma} = 0 \}.
\]
The equivalence relation $\sim$ has a very good motivation: whenever $L_1 \sim L_2$, then $\Omega_1 = d\theta L_1$ is equal to $\Omega_2 = d\theta L_2$, but up to non-constant scaling function $h$. This observation immediately implies that Lagrangians $L_\alpha$, listed in the above one-dimensional example, are not equivalent.

The non-equivalence of Lagrangians has strong physical consequences. The non-equivalent Lagrangians lead to non-equivalent quantum mechanics (this is unrelated with the problem of the ordering), i.e. transition amplitudes computed according to them are different, however, their classical limit is the same. This problem is called a quantization ambiguity.

4 Variational principle and “umbilical” string surfaces

Let us shortly remind the reader with the standard variational approach. It is based on the Lepage one-form $\theta L$ (in that case $\Omega = d\theta L$) and variation is carried over the curves in the extended tangent bundle. Down-to-earth, let us choose two points (events) $(q_0, t_0)$ and $(q_1, t_1)$ in the extended configuration space (space-time) $M \times \mathbb{R}$ and consider the following class of admissible extended tangent bundle curves (see Figure 2):
\[
\mathcal{S} := \{ \gamma : \tau \in (t_0, t_1) \mapsto (q = q(\tau), v = v(\tau), t = \tau) \in TM \times \mathbb{R}, \text{ such that } q(\tau = t_0) = q_0 \text{ and } q(\tau = t_1) = q_1 \}.
\]
We shall see that the classical trajectory, specified before with the help of the null-spaces of $\Omega$, belongs to the above class and extremizes the action
\[
S : \gamma \mapsto S[\gamma] := \int_\gamma L \theta L.
\] (6)

Concisely, suppose that $\delta W = (\delta A) \partial_t + (\delta B) \partial_q + (\delta C) \partial_v$, is an infinitesimal variational vector field in some small tubular neighborhood of the curve $\gamma$ (in order not to leave with its flow $\Phi_{\delta W}$ the class $\mathcal{S}$, we must require $\delta W$ to be tangential to the edge submanifolds $(q_0, t_0) = \text{fixed}$ and $(q_1, t_1) = \text{fixed}$). Then, up to the first order in $\delta$, we can write:
\[
\delta S[\gamma] = S[\Phi_{\delta W} \gamma] - S[\gamma] = \int_\gamma L \delta W(\theta L) = \int_\gamma d\theta L(\delta W) \big|_{\gamma(t_0)} - \theta L(\delta W) \big|_{\gamma(t_1)} = \int_{t_0}^{t_1} d\theta L(\delta W) \bigg|_{\gamma(\tau)} \frac{d\gamma}{d\tau}(\tau).
\]
The testing curve $\gamma$ is an extremal of $S$, if for any variation $\delta W$ it holds: $\delta S[\gamma] = 0$. This is only possible if for all $\tau \in (t_0, t_1)$ the tangent vector
\[
\frac{d}{d\tau} \gamma|_{\gamma(\tau)} = \partial_t \big|_{\gamma(\tau)} + \frac{dq}{d\tau} \partial_q \big|_{\gamma(\tau)} + \frac{dv}{d\tau} \partial_v \big|_{\gamma(\tau)}
\]
belongs to the kernel of $d\theta L$, quod erat demonstrandum. Just for the completeness, the boundary terms drop out due to tangentiality of $\delta W$ to the edge submanifolds, where the restriction of the Lepage one-form $\theta L = L dt + (\partial_v, L) \{ dq^2 - v^2 dt \}$ becomes trivial.

In the case when the distinguished two-form $\Omega$ does not possess a potential $\theta L$, we adopt a new type of variational principle. As a byproduct, in the process of variation we get desired classical trajectory plus something more.

Let us fix a reference curve $\gamma_{ref}$ in the class $\mathcal{S}$ of all admissible curves and define the space of its “umbilical world-sheets” (here $\tau$ is the time parameter as before, and $\sigma$ is a new “worldsheet” distance coordinate, see Figure 3):
\[
\mathcal{U}(\gamma_{ref}) := \{ \Sigma : (\tau, \sigma) \in (t_0, t_1) \times (0, 1) \mapsto (q = q(\tau, \sigma), v = v(\tau, \sigma), t = \tau) \in TM \times \mathbb{R}, \text{ such that } \forall \sigma \colon q(\tau = t_0, \sigma) = q_0, q(\tau = t_1, \sigma) = q_1, \text{ and } \Sigma(\tau, \sigma = 0) = \gamma_{ref}(\tau) \}.
\]
In the class of "umbilical world-sheets" a stationary surface of the action

\[ S : \Sigma \mapsto S(\Sigma) := \int_{\Sigma} \Omega \]  

satisfies:

\[ d\Omega \left( \frac{\partial}{\partial \tau} \Sigma(\tau, \sigma), \frac{\partial}{\partial \sigma} \Sigma(\tau, \sigma), . \right) = 0 \quad \text{and} \quad \Omega \left( \frac{\partial}{\partial \sigma} \Sigma(\tau, \sigma = 1), . \right) = 0. \]  

To see this, let us consider in a small sandwich neighborhood of a testing surface \( \Sigma \in U(\gamma_{\text{ref}}) \) an infinitesimal variational vector field \( \delta W \) (similarly as before, in order not to leave with its flow \( \Phi_{\delta W} \) the above "umbilical" class, we need, apart from tangentiality of \( \delta W \) to the edge submanifolds, also \( \delta W|_{\gamma_{\text{ref}}} = 0 \)). Then, up to first order in \( \delta \), we can write:

\[
\delta S[\Sigma] = S[\Phi_{\delta W} \Sigma] - S[\Sigma] = \int_{\Sigma} \mathcal{L}_{\delta W}(\Omega) = \int_{\Sigma} \delta W \cdot d\Omega + d(\delta W \cdot \Omega) = \int_{\Sigma} d\Omega(\delta W, . , \) + \int_{\Sigma} \Omega(\delta W, . , \) = \int_{t_0}^{t_1} \int_0^1 d\tau \left\{ d\Omega(\delta W|_{\gamma_{\text{ref}}}, \frac{\partial \Sigma}{\partial \tau}, \frac{\partial \Sigma}{\partial \sigma}) \right\}(\tau, \sigma) + \int_{\gamma} \Omega(\delta W|_{\gamma_{\text{ref}}}, . , \) - \int_{\gamma} \Omega(\delta W|_{\gamma_{\text{ref}}}, . , \) - \int_{\Sigma} \Omega(\delta W|_{\gamma_{\text{ref}}}, . , \) .
\]

\[ \left\{ \text{each boundary line integral is 0 separately} \right\} \]

The first underbraced integral is zero, because \( \delta W|_{\gamma_{\text{ref}}} = 0 \). The remaining two terms do not contribute due to tangentiality of \( \delta W \) to the edge "D-brane" submanifolds, where the restriction \( \Omega|_{\text{D-branes}} \) equals to zero. If the bulk term in \( \delta S[\Sigma] \) obeys the first and the boundary term the second equation in (8) (we have used the abbreviation \( \gamma(\tau) = \Sigma(\tau, \sigma = 1) \)), then \( \delta S[\Sigma] \) is equal to zero for any variational vector field \( \delta W \), quod erat demonstrandum.

The second equation in (8) says that \( \frac{\partial}{\partial \sigma} \gamma(\tau) \) lies in the kernel of \( \Omega \), therefore it determines the classical trajectory \( \gamma_{\text{class}} \) that connects the space-time events \( (q_0, t_0) \) and \( (q_1, t_1) \). Its genuineness is obvious, it does not depend on the chosen auxiliary reference curve \( \gamma_{\text{ref}} \in \mathcal{S} \).

The complete solution of the first equation in (8) is constrained by the boundary conditions anchoring the stationary surface to the curves \( \gamma_{\text{ref}} \) and \( \gamma_{\text{class}} \). Whether for any admissible reference curve \( \gamma \in \mathcal{S} \) there
Figure 3. Oriented “umbilical surface” $\Sigma$ connects the reference curve $\gamma_{\text{ref}}$ with the “history” $\gamma(\tau) = \Sigma(\tau, \sigma = 1)$. Sideways boundary curves $\lambda_0(\sigma) = \Sigma(t_0, \sigma)$ and $\lambda_1(\sigma) = \Sigma(t_1, \sigma)$ are located within the $n$-dimensional submanifolds $(q_0, t_0) = \text{fixed}$ and $(q_1, t_1) = \text{fixed}$ of the extended tangent bundle. On the figure, these edge submanifolds are schematically represented by two-dimensional “D-branes.”

exists a stationary “umbilical” surface $\Sigma_{\text{stat}} \in \mathcal{U}(\gamma)$ connecting it with the classical trajectory, depends on the physical system under the consideration. For example, for conservative systems, where $d\Omega = d(\phi \theta \lambda) = 0$, any world-sheet $\Sigma$ satisfying the boundary conditions: $\Sigma(\tau, \sigma = 0) = \gamma(\tau)$ and $\Sigma(\tau, \sigma = 1) = \gamma_{\text{class}}(\tau)$ forms the extremal solution of (8). Moreover, the prescription:

$$\gamma \mapsto \Sigma_{\text{stat}} \mapsto S[\gamma] := -\int_{\Sigma_{\text{stat}}} \Omega - \int_{\theta_{\text{stat}}} d\theta_{\lambda} = -\int_{\theta_{\text{stat}}} \theta_{\lambda} = \int_{\theta_{\text{stat}}} \theta_{\lambda} - \int_{\gamma_{\text{class}}} \theta_{\lambda} + \int_{\gamma_{\text{class}}} \lambda_{1} = \int_{\theta_{\text{stat}}} - C \quad (9)$$

is a meaningful definition of the action for the history $\gamma$ (the constant $C$ is the value of the classical action on the trajectory $\gamma_{\text{class}}$).

On the other hand, let us again focus on the one-dimensional particle driven by the force $Q(x, v, t) = -v$. In this case $d\Omega = -dv \wedge dx \wedge dt$ and a world-sheet $\Sigma$ given by the coordinate functions $x = x(\tau, \sigma), v = v(\tau, \sigma)$ and $t = \tau$ is stationary if

$$0 = d\Omega \left( \frac{\partial}{\partial \tau} \Sigma(\tau, \sigma), \frac{\partial}{\partial \sigma} \Sigma(\tau, \sigma), . \right) = \frac{\partial x}{\partial \sigma} dv - \frac{\partial v}{\partial \sigma} dx + \left\{ \frac{\partial x}{\partial \tau} \frac{\partial v}{\partial \sigma} - \frac{\partial v}{\partial \tau} \frac{\partial x}{\partial \sigma} \right\} dt \iff \begin{cases} x(\tau, \sigma) = x(\tau, \sigma = 0) \\ v(\tau, \sigma) = v(\tau, \sigma = 0) \end{cases}$$

Whenever $\Sigma(\tau, \sigma = 0)$ is not equal to $\Sigma(\tau, \sigma = 1)$, there does not exist a non-degenerate solution of (8), i.e. extremal “umbilical” string satisfying the given boundary conditions $\Sigma(\tau, \sigma = 0) = \gamma$ and $\Sigma(\tau, \sigma = 1) = \gamma_{\text{class}}$ is missing. If we accept in the “umbilical” class $\mathcal{U}(\gamma)$ also a degenerate surface (this one is shrunk just to the reference curve), then for the special gauge $\gamma = \gamma_{\text{class}}$ we get the solution of the full system (8) in the relatively simple form $\Sigma_{\text{stat}}(\tau, \sigma) = \gamma_{\text{class}}(\tau)$.
All of this explains in advance why it is impossible to apply the assignment \( \gamma \mapsto \Sigma_{\text{stat}} \mapsto S[\gamma] \) as the universal principle. This is the main difference compared to the result of Marián Fecko’s paper \[19\] in which he analyzed a similar problem in the context of a variational principle for the Nambu mechanics.

We will see that although the value of the classical action \( S[\gamma] \) is missing, it is still possible to perform a quantization of the theory in terms of the two-form \( \Omega \).

5 Quantization: Path versus Surface Integral

In the previous sections we have observed that the classical evolution is completely described when integrating null spaces \( \ell \) of the two-form \( \Omega \). So we could claim: “classical mechanics is only \( \Omega \)-sensitive.” Everything else is a bonus only valid in special cases. We were being impractical not to use the (local) potential \( \theta_L \) or the Lagrangian \( L \), which would enable us to investigate the invariants and/or conserved quantities. But the physical principles are constituted over the equations of motion, not over the Lagrangian or Hamiltonian themselves.

On the other hand, it seems that the standard quantum mechanics is rather \( \theta_L \) (or, if you wish, \( \omega_{\pi} \))-sensitive. The most impressive way how to relate classical and quantal lies in the Feynman path-integral approach.

According to the Feynman prescription\[20,21\] the probability amplitude of the transition of the system from the space-time configuration \((q_0, t_0)\) to \((q_1, t_1)\) is expressed as follows:

\[
A(q_0, t_0; q_1, t_1) \propto \int \mathcal{D}[\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma} \theta_L \right\}.
\]

(10)

The “path-summation” here is taken over the class \( \mathcal{S} \) of all admissible curves in \( TM \times \mathbb{R} \) as it is drawn in Figure 2. The exponent in \( \mathcal{D}[\gamma] \) is the standard integral of the one-form \( \theta_L \) carried over the extended tangent bundle curve \( \gamma \). The question about the “measure” \( \mathcal{D}[\gamma] \) and the proper normalization of the probability amplitude \( A \) are subject to our discussion in the next section. Let us remind the reader that the probability amplitude formula \( \mathcal{D}[\gamma] \) is used less frequently than its phase space version. When expressing generalized velocities in \( \theta_L \) in terms of generalized momenta we get \( A \) as a functional integral in the extended phase space \( T^*M \times \mathbb{R} \):

\[
A(q_0, t_0; q_1, t_1) = \int \mathcal{D}[\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma} \omega_{\pi} \right\}, \quad \text{where one can formally set } \mathcal{D}[\gamma] = \frac{dp_{\theta}}{2\pi} \prod_{\gamma(t_0, t_1)} \frac{dp \, dq}{2\pi}.
\]

The bunches of curves \( \gamma \) and \( \tilde{\gamma} \) that enter the functional integrations are connected by the same Legendre transformation \((q, v, t, L) \mapsto (q, p = \frac{\partial H}{\partial q}, t, \Pi = pv - L)\) as the one-forms \( \theta_L \) and \( \omega_{\pi} \).

The above mentioned sensitiveness of quantum mechanics on the one-form \( \theta_L \) and/or \( \omega_{\pi} \) is evident. In what follows, we propose some modifications leading to the replacement of \( \theta_L \) by the two-form \( \Omega \). This will enable us to “quantize” also dissipative systems. In the special conservative case, our prescription will be equivalent to the Feynman’s.

Our main trick is a simple rearrangement based on the Stokes theorem. Down-to-earth, in the class \( \mathcal{S} \) that enters the “path-summation” in \( \mathcal{D}[\gamma] \), there is one specially distinguished curve, the classical trajectory \( \gamma_{\text{class}} \). Using it, we get for any other \( \gamma \) within this class an oriented 1-cycle:

\[
\partial \Sigma := \gamma + \lambda_1 - \gamma_{\text{class}} - \lambda_0.
\]

Here \( \lambda_0 \) and \( \lambda_1 \) are arbitrarily chosen curves within the submanifolds of \( TM \times \mathbb{R} \), \((q_0, t_0) = \text{fixed}\) and \((q_1, t_1) = \text{fixed}\) that join the initial and final points of \( \gamma \) and \( \gamma_{\text{class}} \), respectively (the choice \( \gamma_{\text{ref}} = \gamma_{\text{class}} \) in Figure 3).

\[\uparrow\]We optimistically propose that solutions of the equations of motion might be “inverted” on relatively broad time interval, i.e. that from given position at the final time we would be able to adjust the initial velocity in such a way that the system will evolve uniquely into the prescribed endpoint.
provides a correct picture of this situation). Since the restriction of \(\theta_L\) (and also of the distinguished two-form \(\Omega\)) to any of these edge “D-brane” submanifolds is trivial, we can write:

\[
\int_{\gamma} \theta_L - \int_{\gamma_{\text{class}}} \theta_L + \int_{\lambda_1} \theta_L - \int_{\lambda_0} \theta_L = \int_{\partial \Sigma} d\theta_L, \quad \text{where} \quad \Sigma \in \mathcal{U}(\gamma_{\text{class}}) \quad \text{and} \quad \partial \Sigma = \gamma + \lambda_1 - \gamma_{\text{class}} - \lambda_0. \tag{11}
\]

Let us remind the reader that

- the second term on the left hand side of \((11)\) is just the value of the classical action on the curve \(\gamma_{\text{class}}\)
- the existence of the “umbilical” string \(\Sigma\) that connects the given curve \(\gamma\) with \(\gamma_{\text{class}}\) is determined by topological properties\(^{24}\) of \(TM \times \mathbb{R}\), e.g. when \(TM \times \mathbb{R}\) is simply-connected, then any 1-cycle \(\partial \Sigma\) is at the same time a 1-boundary of some 2-chain \(\Sigma\).

Motivated by the trick \((11)\), encouraged by the sentence from Richard Feynman’s thesis\(^{24}\) “...the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some sort of least action principle be available ...” and inspired by the variational principle \((1)\), we propose a generalization of the Feynman’s probability amplitude formula in the following way:

\[
A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} S_{\text{class}} \right\} \int_{\mathcal{U}} [D\Sigma] \exp \left\{ \frac{i}{\hbar} \int_{\Sigma} \Omega \right\}. \tag{12}
\]

Here the “surface-summation” is taken over the class \(\mathcal{U} = \mathcal{U}(\gamma_{\text{class}})\) of all admissible “umbilical” world-sheets with the reference curve \(\gamma_{\text{class}}\). Using the formula \((11)\) for the distinguished two-form \(\Omega\) we can write:

\[
\int_{\Sigma} \Omega = \int_{\partial \Sigma} \{ T dt + (\partial_{q^i} T) dq^i - v^i dt \} + \int_{\gamma_{\text{class}}} \{ Q_i dq^i \} \wedge dt \equiv \int_{\partial \Sigma} \theta_T + \int_{\Sigma} \{ Q_i dq^i \} \wedge dt.
\]

The first integral term is obviously independent of the choice of the sideways boundary curves \(\lambda_0\) and \(\lambda_1\) in \(\partial \Sigma = \gamma + \lambda_1 - \gamma_{\text{class}} - \lambda_0\). Moreover, we can split the “surface-summation” carried out in \((12)\) in the following way:

\[
\int_{\mathcal{U}} [D\Sigma] = \int_{\mathcal{G}} \left\{ \int_{\Sigma_\gamma} [D\Sigma_\gamma] \right\},
\]

i.e. first we pick out the boundary curve \(\gamma \in \mathcal{G}\), and then we perform the “summation” over the subset

\[
\{ \Sigma_\gamma \} := \{ \Sigma_\gamma \in \mathcal{U}, \text{ such that } \Sigma_\gamma (\tau, \sigma = 1) = \gamma(\tau) \} \subset \mathcal{U},
\]

which contains all “umbilical” surfaces that are anchored to the fixed curves \(\gamma_{\text{class}}\) and \(\gamma\). After doing this, we get \((12)\) in the equivalent form:

\[
A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} S_{\text{class}} \right\} \int_{\mathcal{G}} [D\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma_{\text{class}}} - \int_{\gamma} \right\} \theta_T \times \left\{ \int_{\{ \Sigma_\gamma \}} [D\Sigma_\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\{ \Sigma_\gamma \}} \{ Q_i dq^i \} \wedge dt \right\} \right\}.
\]

In the case of conservative forces \(\{ Q_i dq^i \} \wedge dt = -d\theta_U = -d \{ U dt + (\partial_{q^i} U) dq^i - v^i dt \} \), the surface integral in the last exponent of the above formula is again only boundary sensitive quantity. Therefore

\[
A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} S_{\text{class}} \right\} \int_{\mathcal{G}} [D\gamma] \exp \left\{ \frac{i}{\hbar} \int_{\gamma_{\text{class}}} - \int_{\gamma} \right\} (\theta_T - \theta_U) \times \text{Vol}_\gamma,
\]

\(^{24}\text{Topological properties we are talking about are “measured” by the fundamental group } \Pi_1(TM \times \mathbb{R}). \text{ For obvious reasons we are cowardly skipping off any discussion of the quantization in topologically nontrivial cases.}\)
where we have adopted the abbreviated notation

$$\text{Vol}_\gamma = \int \{ \mathcal{D} \Sigma \gamma \} = \text{the "number" of the surfaces containing } \gamma \text{ and } \gamma_{\text{class}} \text{ as the subboundaries}.$$  

Suppose there are no topological obstructions on the side of $TM \times \mathbb{R}$, i.e. that all admissible $\gamma$’s are homotopically equivalent. Then the factor $\text{Vol}_\gamma$ is $\gamma$-independent, and it can be dropped as an infinite constant by normalization. Moreover, the factor $S_{\text{class}}$ cancels the integral $\int (\theta_T - \theta_U)$ over the reference curve $\gamma_{\text{class}}$. Thus, in the case of conservative forces the formula (12) reduces precisely to (10), quod erat demonstrandum.

There is still one open point, namely we have to say what exactly we mean by the classical action $S_{\text{class}}$ in (12). The distinguished two-form $\Omega$ specifies classical trajectory $\gamma_{\text{class}}$. It is defined directly in terms of acting forces. Some of them could be potential-generated and therefore $\Omega$ can be split (at least locally) into the two parts:

$$\Omega = \{ Q_i^{\text{diss}} \, dq^i \} \wedge dt + \{ (T - U) \, dt + \partial_v (T - U) \, \{ dq^i - v^i \, dt \} \} =: \Omega_{\text{diss}} + \Omega_{\text{cons}}.$$  

The first part, called dissipative, contains all the non-potential-generated forces and therefore $d\Omega_{\text{diss}} \neq 0$. The remaining closed term $\Omega_{\text{cons}} = d\{ \theta_T - \theta_U \}$ is a conservative part of $\Omega$. The classical action entering the amplitude formula (12) is defined as follows:

$$S_{\text{class}} := \int_{\gamma_{\text{class}}} \left( \theta_T - \theta_U \right).$$

We described above all objects that are necessary for the computation of the probability amplitude. It remains to give some nontrivial example demonstrating the functionality of (12) and then open the discussion. Let us start from backward, with the discussion, and postpone the example to the next paragraph:

$\circ$ to see the correctness of the classical limit and the exceptionality of the classical history in between $(q_0, t_0)$ and $(q_1, t_1)$, one needs to recollect the above “umbilical” variational principle (4); the stationary solution $\Sigma_{\text{stat}}$ of variational equations (3) in the considered “umbilical” class $||$ corresponds to $\gamma_{\text{class}},$

$\circ$ here, to be able to talk about the quantum probability amplitudes, one needs to know the solution of the classical equations of motion with the given initial condition; in the standard approach, the classical solution is not necessary for the quantization, it rather appears as the saddle point dominating the amplitude in the limit $\hbar \to 0$,

$\circ$ the mathematically delicate question of the surface functional measure will not be discussed here; our understanding of this problem will be demonstrated on a special example in the next section,

$\circ$ composition of the transition amplitudes we are accustomed to from the standard quantum mechanics:

$$A(q_0, t_0; q_1, t_1) = \int_M dq \, A(q_0, t_0; q, t) \, A(q, t; q_1, t_1)$$

does not work if $d\Omega \neq 0$, i.e. the dissipative time evolution does not satisfy a groupoid composition rule: $U(t_0, t_1) = U(t_0, t) \circ U(t, t_1)$;

6 An example: quantum mechanics with friction

Let us focus on the quantization of dynamics of a unit mass particle moving in $M = \mathbb{R}[x]$, which is driven by the conservative force $F = -\frac{d}{dx} U(x)$ along with the friction $Q_i^{\text{diss}} = -\kappa v$. The extended tangent bundle for this situation corresponds to an ordinary three-dimensional Cartesian space $\mathbb{R}^3[x, v, t]$, and the distinguished two-form takes the simple form $\Omega = -\kappa v \, dx \wedge dt + d \{ v \, dx - \left\{ \frac{1}{2} v^2 + U(x) \right\} \, dt \}$.

Our aim is to evaluate the transition amplitude as a function of the initial and final events. To keep the better track of the forthcoming calculation and also of its frictionless limit, we maintain the explicit dependence
on the parameter $\kappa$. Suppose we have chosen a solution $\gamma_{\text{class}}(\tau) = (x_{\text{class}}(\tau), v_{\text{class}}(\tau) = \dot{x}_{\text{class}}(\tau), t = \tau)$ of the Newton-Lagrange equation of motion:

$$\ddot{x} = F(x) - \kappa \dot{x}, \quad \text{which satisfies: } x_{\text{class}}(t_0) = q_0 \text{ and } x_{\text{class}}(t_1) = q_1.$$ 

Direct application of the formula (12) leads then to the following expression:

$$A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} S_{\text{class}} \right\} [D\Sigma] \exp \left\{ \frac{i}{\hbar} \int_{\Sigma} \left\{ v \, dx - \frac{1}{2} v^2 + U(x) \right\} dt - \kappa \int_{\Sigma} v \, dx \, dt \right\}. \quad (14)$$ 

The meaning of the first exponent is clear from (13). To evaluate the remaining world-sheet functional integral, let us introduce an auxiliary nodal set:

$$\{(a, b)\}_{\text{nodes}} := \{(t_0+a \Delta, 0+b \varepsilon) \in \mathbb{R}^2[t, \sigma] ; \text{ time index } a \text{ runs from } 0 \text{ to } K \text{ and distance index } b \text{ from } 0 \text{ to } L \},$$

which splits the underlying parametric space $(t_0, t_1) \times (0, 1) \subset \mathbb{R}^2[t, \sigma]$ into infinitesimal rectangular tiles, each of which encloses the area $\Delta \cdot \varepsilon = \frac{t_1-t_0}{K} \cdot \frac{1-0}{L}$ (at the end, the numbers $K$ and $L$ will be send to infinity).

![Figure 4. Schematic picture of the nodal grid, points marked with empty circles are constrained by (16).](image)

After that any “umbilical” string

$$\Sigma : \mathbb{R}^3[\tau, \sigma] \to \mathbb{R}^3[x, v, t], \quad (\tau, \sigma) \mapsto \Sigma(\tau, \sigma) = (x(\tau, \sigma), v(\tau, \sigma), t(\tau, \sigma) = \tau)$$

can be discretized by evaluating its coordinate functions at the nodes of the considered grid $\{(a, b)\}_{\text{nodes}}$, i.e.

$$\Sigma : \{(a, b)\}_{\text{nodes}} \to \{\Sigma(a, b) = (x(a, b), v(a, b), t_0 + a \Delta)\}.$$ 

Here, to keep the ensemble $\{\Sigma(a, b)\} \subset \mathbb{R}^3[x, v, t]$ within the considered “umbilical” class $\mathcal{U}$, we must impose the following simple constraints:

$$\forall a = 0, \ldots, K \quad x(a, 0) = x_{\text{class}}(t_0 + a \Delta), \quad v(a, 0) = v_{\text{class}}(t_0 + a \Delta) \iff \Sigma(\tau, 0) = \gamma_{\text{class}}(\tau),$$

$$\forall b = 1, \ldots, L \quad x(0, b) = q_0, \quad x(K, b) = q_1 \iff \Sigma(t_0, \sigma) \text{ and } \Sigma(t_1, \sigma) \subset \text{ to the “D-branes”}. \quad (15)$$

Therefore, formally, the functional integral over all possible string configurations is a formal limit of the ordinary multiple integral, which is taken over all unconstrained variables in the discretized ensemble $\{\Sigma(a, b)\},$
i.e.

\[ \int [\mathcal{D}\Sigma] := \lim_{K \to \infty} \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{a=1}^{K-1} \prod_{b=1}^{L} dx_{(a,b)} dv_{(a,b)} dv_{(0,b)} dv_{(K,b)} \right\}. \]

When step-wisely discretizing the integrals in the exponent of (14), taking into account the constraints (15), we get for the bulk term (all is done with respect to the chosen orientation of the "umbilical" world-sheet \( \Sigma \):

\[ \int_{\Sigma} v dx \wedge dt = \int_{0}^{t_1} \int_{t_0}^{1} d\sigma \left\{ v(\tau, \sigma) \frac{dx(\tau, \sigma)}{d\sigma} \right\} = \lim_{K \to \infty} \sum_{a=0}^{K-1} \sum_{b=0}^{L-1} \Delta \varepsilon \left\{ v_{(a,b)} \frac{x_{(a,b+1)} - x_{(a,b)}}{\varepsilon} \right\} = \]

\[ = \lim_{K \to \infty} \left\{ \sum_{a=1}^{K-1} \sum_{b=1}^{L-1} \Delta v_{(a,b)} \{ x_{(a,b+1)} - x_{(a,b)} \} + \sum_{a=0}^{K-1} \Delta v_{(a,0)} x_{(a,1)} \right\} - \int_{t_0}^{t_1} d\tau \left\{ v_{\text{class}}(\tau) x_{\text{class}}(\tau) \right\} \]

and similarly for the boundary term:

\[ \int_{\partial \Sigma} \left\{ v dx - \left\{ \frac{1}{2} v^2 + U(x) \right\} dt \right\} = \int_{\Sigma(\tau, \sigma=1)} \left\{ v dx - \left\{ \frac{1}{2} v^2 + U(x) \right\} dt \right\} - \int_{\Sigma(\tau, \sigma=0)} \left\{ v dx - \left\{ \frac{1}{2} v^2 + U(x) \right\} dt \right\} = \]

\[ = \lim_{K \to \infty} \sum_{a=0}^{K-1} v_{(a,L)} \left\{ x_{(a+1,L)} - x_{(a,L)} \right\} - \Delta \left\{ \frac{1}{2} \left( v_{(a,L)} \right)^2 + U(x_{(a,L)}) \right\} \right\} - \int_{t_0}^{t_1} d\tau \left\{ \frac{1}{2} \left( v_{\text{class}}(\tau) \right)^2 - U(x_{\text{class}}(\tau)) \right\}. \]

Putting everything together, integrating over all variables apart \( x_{(a,L)} \) and returning back to the continuum limit, we get the following expression for the transition amplitude\(^{11}\)

\[ A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \int_{t_0}^{t_1} d\tau \left\{ \kappa v_{\text{class}}(\tau) \right\} \right\} \int [\mathcal{D}x(\tau)] \exp \left\{ \int_{t_0}^{t_1} d\tau \left\{ \frac{1}{2} \dot{x}^2 - U(x) - \frac{1}{2} \kappa x v_{\text{class}}(\tau) \right\} \right\}. \quad (16) \]

The phase factor in front of (16) is coming from the definition of the classical action \( S_{\text{class}} \) and from the world-sheet functional integration. The second term is the standard Feynman path integral, which is taken over the histories \( \{ \tau \mapsto (x(\tau) := x(\tau, \sigma = 1), t = \tau) \} \) in the extended configuration space \( M \times \mathbb{R}[\tau] \). However, in comparison with the standard formula, a new term appears here. It is an external source generated by the classical velocity \( v_{\text{class}} \). Its presence has an important consequence, namely, it guarantees that the classical solution \( x_{\text{class}}(\tau) \) is the stationary curve of the considered functional. Moreover, further inspection of (16) shows that if the friction parameter \( \kappa \) tends to zero, the standard propagator in the potential \( U(x) \) is recovered.

Let us remind the reader how to treat ugly infinite constants emerging in the functional integration. If the entering infinities are functionally independent of the coordinates of space-time events, then one can easily neglect them. The only important term inside the functional integral is the phase factor, which depends on coordinates of \((q_0, t_0)\) and \((q_1, t_1)\), i.e. we need to seize the following quantity:

\[ A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} \Phi(q_0, t_0; q_1, t_1) \right\}, \]

anything else is just an inherited rudiment. The proper normalization of the amplitude \( A(q_0, t_0; q_1, t_1) \) is dictated by its physical meaning. The square of its absolute value answers the question about the probability density to observe a particle in a sufficiently small neighborhood of the configuration \((q_1, t_1)\), when before it was observed in a neighborhood of the space-time position \((q_0, t_0)\). This implies desired normalization.

\(^{11}\)To arrive to the above formula, one should handy employ, when performing the routine integrations over the velocities, the spectral form of \( \delta \)-function and the Gauss-Fermi integral. After that, the remaining integrals carried out over the ensemble \( \{x_{(a,b)}; a = 1, \ldots, K - 1, b = 1, \ldots, L - 1\} \) become trivial.
conditions (since we are dealing with the space-time continuum, the normalization to δ-function should be employed):

\[ \{ \text{if } t_1 \rightarrow t_0 \implies A(q_0, t_0; q_1, t_1) \rightarrow \delta(q_1 - q_0) \} \iff \{ \text{at the time } t_0 \text{ system occupies definite position } q_0 \} \]

\[
\int_{-\infty}^{+\infty} dq_1 A(q_0, t_0; q_1, t_1) A^*(q'_0, t_0; q'_1, t_1) = \delta(q'_0 - q_0) \iff \{ \text{total probability is conserved, evolution is unitary} \}
\]

Having everything at hand, let us compute the normalized probability amplitude with the presence of friction in the cases when \( U(x) = 0 \) (free particle) and \( U(x) = \frac{1}{2} \omega x^2 \) (linear harmonic oscillator).

### 6.1 Free particle with damping

The classical trajectory \( x_{\text{class}}(\tau) \) with prescribed endpoints that satisfies the dynamical equation \( \ddot{x} = -\kappa \dot{x} \), is given as follows:

\[
x_{\text{class}}(\tau) = \frac{q_1 e^{\kappa t_1} - q_0 e^{\kappa t_0}}{e^{\kappa t_1} - e^{\kappa t_0}} + \frac{q_1 - q_0}{e^{\kappa t_1} - e^{-\kappa t_0}} e^{-\kappa \tau}, \quad v_{\text{class}}(\tau) = \frac{d}{d\tau} x_{\text{class}}(\tau) \quad \text{and} \quad \begin{cases} x_{\text{class}}(t_0) = q_0, \\ x_{\text{class}}(t_1) = q_1. \end{cases}
\]

The only difficulty that comes from (16), is the path integral with an external source term:

\[
W[v_{\text{class}}] = \int \mathcal{D}x(\tau) \exp \left\{ \int_{t_0}^{t_1} d\tau \left\{ \frac{i}{\hbar} \left( \dot{x}(\tau) \right)^2 - \kappa x(\tau) v_{\text{class}}(\tau) \right\} \right\}.
\]

When one performs a shift transformation

\[ y(\tau) \mapsto x(\tau) = y(\tau) + c(\tau), \quad \text{where} \quad \frac{\dot{c}(\tau)}{v_{\text{class}}(\tau)} = -\kappa v_{\text{class}}(\tau), \]

then the functional measure is unchanged and the external source term gets off. Therefore we can write

\[
W[v_{\text{class}}] = \exp \left\{ \int_{t_0}^{t_1} d\tau \left\{ \frac{i}{\hbar} \left( \frac{1}{2} \left( \dot{c}(\tau) \right)^2 - \kappa c(\tau) v_{\text{class}}(\tau) \right) + (y_1 \dot{c}(t_1) - y_0 \dot{c}(t_0)) \right\} \right\} \int \mathcal{D}y(\tau) \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} d\tau \left\{ \frac{1}{2} \left( \ddot{y}(\tau) \right)^2 \right\} \right\}.
\]

The last functional term is the Schrödinger propagator in the path integral form, which describes the motion of a free quantum particle in between the endpoints \((q_0 - c(t_0), t_0)\) and \((q_1 - c(t_1), t_1)\). When putting together all fragments that enter the formula (15) and taking into account the required normalization conditions, we arrive to the following probability amplitude:

\[
A(q_0, t_0; q_1, t_1) = \frac{1}{\sqrt{2\pi i \hbar \Lambda}} \exp \left\{ \frac{i}{2\hbar \Lambda} (q_1 - q_0)^2 \right\}, \quad \text{where} \quad \Lambda = \frac{2}{\kappa} \tanh \left( \frac{\kappa}{2} (t_1 - t_0) \right). \quad (17)
\]

A short inspection of (17) discloses that in the frictionless limit \( \Lambda = (t_1 - t_0) \) and the amplitude \( A(q_0, t_0; q_1, t_1) \) coincides with the ordinary quantum propagator for a free particle. Moreover, it is now explicitly acknowledged that the standard composition law for the transition amplitudes does not operate.

Let us conclude this subparapraph by performing an analysis of the time evolution in terms of the transition probability amplitude (17). From the point of view of quantum mechanics, the best fit of a unit mass particle with the classical initial condition \((q_0 = 0, v_0 = v_0, t_0 = 0)\) represents the gaussian wave-packet

\[
\Psi(x) \propto \exp \left\{ -\frac{x^2}{2\xi^2} + \frac{i}{\hbar} x v_0 \right\}.
\]
with some initial width $\xi$. At a later time $t$, the system under the consideration will be characterized by the convoluted wave-packet distribution

$$\Psi(x,t) \propto \int_{-\infty}^{+\infty} dq \, \Psi(q) \, A(q,0;x,t).$$

It is clear from the properties of the convolution that the evolved wave-function $\Psi(x,t)$ preserves the gaussian shape (modulo phase factors). Its main characteristics, the mean value of the position $\langle x \rangle$ and the actual width of the wave-packet $\Xi^2$, are varying with the time according to

$$\langle x \rangle = v_0 \Lambda \quad \text{and} \quad \Xi^2 = \xi^2 + \frac{\hbar^2}{\xi^2} \Lambda^2.$$

The velocity of the center of the wave-packed $\frac{dx}{dt}(x) = \frac{A e^{i\kappa t} + B e^{-i\kappa t}}{(1 + e^{-\kappa t})} \propto v_0 e^{-\kappa t}$, i.e. it decreases for $t \gg 1$ exponentially, as one would predict on classical intuition.

### 6.2 Damped harmonic oscillator

The probability amplitude for a damped harmonic oscillator with a unit mass asks for a solution of the Newton’s equation $\ddot{x} = -\omega^2 x - \kappa \dot{x}$:

$$x_{\text{class}}(t) = e^{-\frac{\kappa}{2} t} \left\{ A e^{i\Theta t} + B e^{-i\Theta t} \right\}, \quad \text{where the new frequency } \Theta = \sqrt{\omega^2 - \frac{\kappa^2}{4}}.$$

The dependence of $A$ and $B$ on the initial and final events $(q_0, t_0)$ and $(q_1, t_1)$ is a bit too awkward to be presented here explicitly. When substituting the general $U(x)$ in (16) by the oscillator potential $\frac{1}{2} \omega^2 x^2$ and after performing a similar shift business as in the above example, we arrive to:

$$A(q_0, t_0; q_1, t_1) \propto \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} d\tau \left\{ \frac{1}{2} \frac{x^2_{\text{class}}}{\xi^2} - \frac{1}{4} \omega^2 x_{\text{class}}^2 \right\} \right\}.$$

Integrating the exponent and imposing the normalization conditions provide us with the propagator:

$$A(q_0, t_0; q_1, t_1) = K(t_1-t_0) \exp \left\{ \frac{i}{\hbar} \frac{\Theta}{2 \sin(\Theta(t_1-t_0))} \left[ (q_1^2 + q_0^2) \cos(\Theta(t_1-t_0)) - 2(q_1 q_0) \coth(\frac{\Theta}{2}(t_1-t_0)) \right] \right\} \quad (18)$$
where the normalization factor

$$K(t_1 - t_0) = \sqrt{\Theta \coth \{\frac{\Theta}{2\pi i \hbar} (t_1 - t_0)\}}. \quad (19)$$

It is clear that

- taking the limit $\omega \to 0$; the above amplitude reproduces the free particle result [17],
- in the limit $\kappa \to 0$; the Schrödinger propagator for a harmonic oscillator with the frequency $\omega$ is recovered.

Let us mention that our formula [18] disagrees with a result of the paper [20]. There is presented a different approach to quantum mechanics in the presence of friction (time dependent formalism). Main disagreement consists in the fact that their propagator is not invariant with respect to time translations, however, the classical equations of motion, as well as our formula, remain invariant.

Acknowledgement

Many thanks go to Pavel Bóna, Marián Fecko, Tamás Fulop, Peter Prešnajder, Pavel Ševera and Vladimír Balek for their interest, criticism, fruitful discussions and many useful comments. This research was supported in part by Comenius University Grant UK/359/2006, VEGA Grant 1/3042/06 and ESF project JPD3 BA-2005/1-034. Special thanks go to my wife Tulka for her sympathy and encouragement.

A. M. D. S.

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